symmetry

MDPI

Article

Three-Dimensional Lorentz-Invariant Velocities

James M. Hill

STEM, University of South Australia, Adelaide, SA 5006, Australia; jim.hill@unisa.edu.au

Abstract: Lorentz invariance underlies special relativity, and the energy formula and relative velocity formula are well known to be invariant under a Lorentz transformation. Here, we determine the functional forms in terms of four arbitrary functions for those three dimensional velocity fields that are automatically invariant under the most general fully three-dimensional Lorentz transformation. For general three-dimensional motion, using rectangular Cartesian coordinates (x, y, z), we determine the first-order partial differential equations for the three velocity components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ in the $x-$, $y-$ and $z-$directions respectively. These partial differential equations and the associated partial differential relations connecting energy and momentum are fully compatible with the Lorentz-invariant energy–momentum relations and appear not to have been given previously in the literature. We determine the spatial and temporal dependence of the functional forms for those three-dimensional velocity fields that are automatically invariant under three-dimensional Lorentz transformations. An interesting special case gives rise to families of particle paths for which the magnitude of the velocity is the speed of light. This is indicative of the abundant possibilities existing in the "fast lane".

Keywords: special relativity; Lorentz invariance; functional forms; energy and momentum partial differential identities

MSC: 35q75

check for updates

Citation: Hill, J.M. Three-Dimensional Lorentz-Invariant Velocities. *Symmetry* **2024**, *16*, 1133. https://doi.org/10.3390/sym16091133

Academic Editor: Sergei D. Odintsov

Received: 29 July 2024
Revised: 28 August 2024
Accepted: 29 August 2024
Published: 2 September 2024

1. Introduction

In special relativity, the word "special" alludes to invariance under transformations relating constant relative velocity frames of reference, which are known as Lorentz transformations, and a Lorentz-invariant quantity is one that assumes an identical form under a Lorentz transformation. A very attractive notion is that the fundamental structures and mechanisms of the universe are somehow connected with the invariances of the underlying model. A curious fact associated with general relativity is that while spiral galaxies are common in the universe, there appear to be no simple exact solutions of general relativity that reflect these structures. Yet, in both fluid and solid mechanics, logarithmic spirals arise from the invariance of the underlying equations under simple one-parameter stretching and rotations. In this paper, we examine those special relativistic three-dimensional motions for which the three velocity components in the $x-$, $y-$ and $z-$directions are invariant under arbitrary three-dimensional Lorentz transformations. These results might find physical application in cosmological theories with background vector fields, such as proposed in [1,2].

While Lorentz invariance and its consequences are well established in special relativity, it seems to have been overlooked that the imposition of a Lorentz-invariant velocity field $\mathbf{u}(\mathbf{x}, t)$ restricts the functional form of the velocity $\mathbf{u}(\mathbf{x}, t)$ to the solution of a certain partial differential equation. Here, for three-dimensional motion with velocity components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ in the $x-$, $y-$ and $z-$directions, respectively, the requirement that the three velocity equations $dx/dt = u(x, y, z, t)$, $dy/dt = v(x, y, z, t)$ and $dz/dt = w(x, y, z, t)$ remain invariant under an arbitrary three-dimensional Lorentz transformation implies that the three velocity components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$

satisfy certain coupled first-order partial differential Equations (3.1) or (3.3), leading to their functional forms. We note that, by the Lorentz invariance of the three differential equations $dx/dt = u(x,y,z,t)$, $dy/dt = v(x,y,z,t)$ and $dz/dt = w(x,y,z,t)$, we mean that, under an arbitrary Lorentz space-time transformation, the same three differential equations are obtained in the transformed space-time variables. Further, since the Lorentz transformation forms a one-parameter Lie group, we may deduce the governing partial differential equations from an examination of the infinitesimal version of the one-parameter Lie group. Only in the derivation of the partial differential equations do we assume an infinitesimal frame velocity.

We determine general functional forms of the velocity components in terms of four arbitrary functions $\mathscr{F}$, $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$. Assuming that the function $\mathscr{F}$ is determined from the relation $\mathscr{F} = (1 - \mathscr{G}_1^2 - \mathscr{G}_2^2 - \mathscr{G}_3^2)^{1/2}$, we note in particular the singular case $\mathscr{F}^2 + \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2 = 1$ for which the magnitude of the particle velocity is the speed of light, which means that there are infinitely many families of paths for which the particles are moving at the speed of light. The existence of these infinite families of paths with particles travelling at the speed of light indicates the endless possibilities existing at the speed of light.

In [3,4], the author has given corresponding results for the cases of one- and two-dimensional special relativistic motions, respectively. For a single spatial dimension x, the one-dimensional velocity $dx/dt = u(x,t)$ satisfies the first-order partial differential equation

$$t\frac{\partial u}{\partial x} + \frac{x}{c^2}\frac{\partial u}{\partial t} = 1 - \left(\frac{u}{c}\right)^2, \tag{1.1}$$

while, for planar motions, using plane rectangular Cartesian coordinates (x,y), the velocity components $dx/dt = u(x,y,t)$ and $dy/dt = w(x,y,t)$ satisfy the following coupled partial differential equations

$$\begin{aligned} &t\left(\cos\phi\frac{\partial u}{\partial x} + \sin\phi\frac{\partial u}{\partial y}\right) + \frac{(x\cos\phi + y\sin\phi)}{c^2}\frac{\partial u}{\partial t} \\ &\quad = \cos\phi\left(1 - \left(\frac{u}{c}\right)^2\right) - \sin\phi\frac{uw}{c^2}, \\ &t\left(\cos\phi\frac{\partial w}{\partial x} + \sin\phi\frac{\partial w}{\partial y}\right) + \frac{(x\cos\phi + y\sin\phi)}{c^2}\frac{\partial w}{\partial t} \\ &\quad = \sin\phi\left(1 - \left(\frac{w}{c}\right)^2\right) - \cos\phi\frac{uw}{c^2}, \end{aligned} \tag{1.2}$$

where ϕ is the planar angle corresponding to α in the three-dimensional formulation. In terms of solutions, the one-dimensional Equation (1.1) derived in [3] is far more restrictive. Here, we follow closely the development [4], and many of the basic formulae and calculations presented here does not differ significantly from those presented in [4], except, of course, that the results here are fully three-dimensional. Accordingly, here, we present the full formulae, but as concisely as is feasible.

We remind the reader that, for those problems involving partial differential equations and boundary or initial conditions, in order for the present analysis to be useful, it is necessary to ensure the invariance of both the equation and any associated conditions under a one-parameter Lie group of transformations. If this is the case, then, generally speaking, invariance under a one-parameter Lie group of transformations implies the major simplification of the problem (see, for example, [5]). In the present context, any solutions of the coupled partial differential Equations (3.3) will generate solutions of those special relativistic problems provided that any boundary or initial conditions also remain invariant under Lorentz transformation. This means that any associated boundary or initial conditions must be assumed to be expressible in terms of invariants of the full three-dimensional Lorentz group (2.1).

In the following section, we summarise the essential results of special relativity theory that are needed in order to deduce the partial differential Equations (3.1) or (3.3) for the velocity components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$. For fully three-dimensional motion, these partial differential equations are derived in the subsequent section, and the corresponding partial differential equations for energy and momentum are derived in the section thereafter. The calculation details for the solutions (3.6) for $\omega(x,y,z,t)$, $B(x,y,z,t)$, $C(x,y,z,t)$ and $D(x,y,z,t)$ in terms of four arbitrary functions $\mathscr{F}$ and $\mathscr{G}_i$ where $i = 1,2,3$ are presented in Appendix A, and the derived solutions are summarised and illustrated in the final sections of the paper.

2. Results from Special Relativity

Special relativity has become a standard subject such that almost every text on physics or mechanics has a dedicated chapter on special relativity. The older texts are closer to the original motivating issues and the developments that gave birth to the subject. Dingle [6] and McCrea [7] provide student texts, while more comprehensive accounts can be found in Bohm [8], French [9] and Resnick [10]. Both Moller [11] and Tolman [12] provide standard works of reference, and the reader may wish to consult [13], containing the original papers of Einstein, Lorentz, Minkowski and Weyl with notes by Arnold Sommerfeld.

The notion of invariance with respect to frames moving with a constant relative velocity underlies special relativistic mechanics, and particularly those transformations of space and time leaving the wave equation unchanged, referred to as Lorentz transformations. We consider a rectangular Cartesian frame (X,Y,Z) and another rectangular Cartesian frame (x,y,z) moving with constant frame velocity $\mathbf{v}^* = (v^*\cos\alpha, v^*\cos\beta, v^*\cos\gamma)$ relative to the first frame, where α, β and γ are the direction cosines of the frame velocity $\mathbf{v}^*$ such that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and v^* denotes the magnitude of the frame velocity $\mathbf{v}^*$.

We view the magnitude of the relative velocity v^* as a measure of the departure of the current frame (x,y,z) from the rest frame (X,Y,Z), and, throughout, we adopt consistent notation, employing lowercase for variables associated with the moving (x,y,z) frame and uppercase or capitals for those variables associated with the rest (X,Y,Z) frame. Accordingly, time is measured from the (X,Y,Z) frame with the variable T and from the (x,y,z) frame with the variable t so that (X,Y,Z,T) and (x,y,z,t) are the variables of principal interest and we assume that the two frames coincide initially.

For $0 \leqslant v^* < c$, from [11] (p. 42), the standard fully three-dimensional Lorentz transformations with fixed cosine angles α, β and γ are given by

$$\begin{aligned}
x &= X + (\delta-1)(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\cos\alpha - \delta v^* T\cos\alpha,\\
y &= Y + (\delta-1)(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\cos\beta - \delta v^* T\cos\beta,\\
z &= Z + (\delta-1)(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\cos\gamma - \delta v^* T\cos\gamma,\\
t &= \delta T - \frac{\delta v^*}{c^2}(X\cos\alpha + Y\cos\beta + Z\cos\gamma),
\end{aligned} \tag{2.1}$$

where $\delta = [1-(v^*/c)^2]^{-1/2}$, with the inverse and the identity transformations characterised by $-v^*$ and $v^* = 0$, respectively. From the above relations for $0 \leqslant v^* < c$, we may deduce

$$x\cos\alpha + y\cos\beta + z\cos\gamma = \delta(X\cos\alpha + Y\cos\beta + Z\cos\gamma - v^* T), \tag{2.2}$$

and the geometric identity arising as a consequence of zero relative motion perpendicular to the direction of motion

$$\begin{aligned}
&x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2\\
&\quad = X^2 + Y^2 + Z^2 - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)^2.
\end{aligned} \tag{2.3}$$

A second relation arises as a consequence of relative motion in the direction of motion:

$$\begin{aligned}
&(ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2\\
&\quad = (cT)^2 - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)^2.
\end{aligned} \tag{2.4}$$

The invariance (2.4) describes the measurement of the distance perpendicular to the direction of relative motion and confirms the isotropy of space independently of time, and it is most easily proven by writing each of the spatial components of (2.1) in the form

$$x - (x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\alpha \\ = X - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)\cos\alpha,$$

and squaring and adding. The invariance (2.4) describes the time-dependent coupling in the direction of travel and is easily established using the working variables $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$ and $\zeta = X\cos\alpha + Y\cos\beta + Z\cos\gamma$ and evaluating $(ct)^2 - \xi^2$ from the two relations $ct = \delta(cT - v^*\zeta/c)$ and $\xi = \delta(\zeta - v^*T)$. Together, the relations (2.3) and (2.4) yield the well-known special relativistic identity equivalent to the Minkowski line element, namely

$$(ct)^2 - (x^2 + y^2 + z^2) = (cT)^2 - (X^2 + Y^2 + Z^2). \tag{2.5}$$

The three invariances (2.3), (2.4) and (2.5) subsequently arise in the solution of the coupled partial differential Equations (3.6) given in Appendix A.

With velocity components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ in the $x-$, $y-$ and $z-$directions, respectively, defined by

$$u(x,y,z,t) = \frac{dx}{dt}, \quad v(x,y,z,t) = \frac{dy}{dt}, \quad w(x,y,z,t) = \frac{dz}{dt}, \\ U(X,Y,Z,T) = \frac{dX}{dT}, \quad V(X,Y,Z,T) = \frac{dY}{dT}, \quad W(X,Y,Z,T) = \frac{dZ}{dT},$$

we may deduce, by the division of the differential versions of (2.1), the Einstein addition of velocity laws in the $x-$ and $y-$directions, respectively; thus,

$$u = \frac{U + (\delta - 1)(U\cos\alpha + V\cos\beta + W\cos\gamma)\cos\alpha - \delta v^*\cos\alpha}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)}, \tag{2.6}$$

$$v = \frac{V + (\delta - 1)(U\cos\alpha + V\cos\beta + W\cos\gamma)\cos\beta - \delta v^*\cos\beta}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)}, \\ w = \frac{W + (\delta - 1)(U\cos\alpha + V\cos\beta + W\cos\gamma)\cos\gamma - \delta v^*\cos\gamma}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)},$$

noting that we have the relations

$$u\cos\alpha + v\cos\beta + w\cos\gamma = \frac{U\cos\alpha + V\cos\beta + W\cos\gamma - v^*}{1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2}, \tag{2.7}$$

$$v\cos\gamma - w\cos\beta = \frac{V\cos\gamma - W\cos\beta}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)}, \\ w\cos\alpha - u\cos\gamma = \frac{W\cos\alpha - U\cos\gamma}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)}, \\ u\cos\beta - v\cos\alpha = \frac{U\cos\beta - V\cos\alpha}{\delta(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)}.$$

By squaring and adding the above relations (2.7), we may show that

$$\frac{u^2 + v^2 + w^2}{c^2} = 1 + \frac{((U^2 + V^2 + W^2)/c^2 - 1)}{\delta^2(1 - v^*(U\cos\alpha + V\cos\beta + W\cos\gamma)/c^2)^2},$$

and, from this equation, it is clear that $u^2 + v^2 + w^2 = c^2$ if and only if $U^2 + V^2 + W^2 = c^2$. Further, we have the important relation

$$\frac{1}{\left(1-(u^2+v^2+w^2)/c^2\right)^{1/2}} = \frac{\delta\left(1-v^*(U\cos\alpha+V\cos\beta+W\cos\gamma)/c^2\right)}{\left(1-(U^2+V^2+W^2)/c^2\right)^{1/2}}. \tag{2.8}$$

With energy and momentum in the three spatial directions in the two frames defined, respectively, by

$$\begin{aligned}
e &= \frac{e_0}{\left(1-(u^2+v^2+w^2)/c^2\right)^{1/2}}, & E &= \frac{e_0}{\left(1-(U^2+V^2+W^2)/c^2\right)^{1/2}}, \\
p &= \frac{m_0 u}{\left(1-(u^2+v^2+w^2)/c^2\right)^{1/2}}, & P &= \frac{m_0 U}{\left(1-(U^2+V^2+W^2)/c^2\right)^{1/2}}, \\
q &= \frac{m_0 v}{\left(1-(u^2+v^2+w^2)/c^2\right)^{1/2}}, & Q &= \frac{m_0 V}{\left(1-(U^2+V^2+W^2)/c^2\right)^{1/2}}, \\
r &= \frac{m_0 w}{\left(1-(u^2+v^2+w^2)/c^2\right)^{1/2}}, & R &= \frac{m_0 W}{\left(1-(U^2+V^2+W^2)/c^2\right)^{1/2}},
\end{aligned} \tag{2.9}$$

where m_0 is the rest mass and $e_0 = m_0c^2$, we may deduce from (2.8) and the velocity relations (2.6) the Lorentz-invariant energy momentum relations

$$\begin{aligned}
e &= \delta(E - v^*(P\cos\alpha + Q\cos\beta + R\cos\gamma)), \\
p &= P + (\delta-1)(P\cos\alpha + Q\cos\beta + R\cos\gamma)\cos\alpha - \delta\frac{v^*}{c^2}E\cos\alpha, \\
q &= Q + (\delta-1)(P\cos\alpha + Q\cos\beta + R\cos\gamma)\cos\beta - \delta\frac{v^*}{c^2}E\cos\beta, \\
r &= R + (\delta-1)(P\cos\alpha + Q\cos\beta + R\cos\gamma)\cos\gamma - \delta\frac{v^*}{c^2}E\cos\gamma,
\end{aligned} \tag{2.10}$$

and the identities

$$e^2 - c^2(p^2+q^2+r^2) = E^2 - c^2(P^2+Q^2+R^2) = e_0^2. \tag{2.11}$$

3. Lorentz-Invariant Velocity Components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$

In this section, we determine the most general three-dimensional velocity field $\mathbf{u}(\mathbf{x},t)$ with velocity components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ in the $x-$, $y-$ and $z-$directions, respectively, that remain invariant under the Lorentz transformation (2.1) . Equivalently, we determine the velocity components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ such that the three differential problems $dx/dt = u(x,y,z,t)$, $dy/dt = v(x,y,z,t)$ and $dz/dt = w(x,y,z,t)$ transform into $dX/dT = u(X,Y,Z,T)$, $dY/dT = v(X,Y,Z,T)$ and $dZ/dT = w(X,Y,Z,T)$, respectively, under the general three-dimensional Lorentz transformation (2.1). Since the Lorentz transformation (2.1) forms a one-parameter group of transformations in the frame velocity v^*, we need only to expand (2.1) to the first order in v^* and equate the corresponding infinitesimals to obtain the first-order partial differential equations for $u(x,y,z,t)$, $u(x,y,z,t)$ and $w(x,y,z,t)$. For infinitesimally small velocities v^*, on retaining only the linear terms involving v^*, the Lorentz transformation (2.1) becomes simply

$$x = X - v^*T\cos\alpha, \qquad y = Y - v^*T\cos\beta, \qquad z = Z - v^*T\cos\gamma,$$
$$t = T - \frac{v^*}{c^2}(X\cos\alpha + Y\cos\beta + Z\cos\gamma),$$

so that, for example, on expanding $dx/dt = u(x,y,z,t)$, we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{dX - v^* dT \cos\alpha}{dT - v^*(dX\cos\alpha + dY\cos\beta + dZ\cos\gamma)/c^2} \\ &= \frac{u(X,Y,Z,T) - v^*\cos\alpha}{1 - v^*(u(X,Y,Z,T)\cos\alpha + v(X,Y,Z,T)\cos\beta + w(X,Y,Z,T)\cos\gamma)/c^2} \\ &= u\left(X - v^*T\cos\alpha, Y - v^*T\cos\beta, Z - v^*T\cos\gamma, T - \frac{v^*}{c^2}(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\right),\end{aligned}$$

which, on expanding and equating the first-order terms in v^* and then reverting to the (x,y,z,t) variables, and with similar calculations for $dy/dt = v(x,y,z,t)$ and $dz/dt = w(x,y,z,t)$, we may deduce the following coupled partial differential equations for $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$:

$$\begin{aligned}& t\left(\cos\alpha\frac{\partial u}{\partial x} + \cos\beta\frac{\partial u}{\partial y} + \cos\gamma\frac{\partial u}{\partial z}\right) + \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c^2}\frac{\partial u}{\partial t} \\ &\quad = \cos\alpha\left(1 - \left(\frac{u}{c}\right)^2\right) - \cos\beta\frac{uv}{c^2} - \cos\gamma\frac{uw}{c^2}, \\ & t\left(\cos\alpha\frac{\partial v}{\partial x} + \cos\beta\frac{\partial v}{\partial y} + \cos\gamma\frac{\partial v}{\partial z}\right) + \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c^2}\frac{\partial v}{\partial t} \\ &\quad = \cos\beta\left(1 - \left(\frac{v}{c}\right)^2\right) - \cos\alpha\frac{uv}{c^2} - \cos\gamma\frac{vw}{c^2}, \\ & t\left(\cos\alpha\frac{\partial w}{\partial x} + \cos\beta\frac{\partial w}{\partial y} + \cos\gamma\frac{\partial w}{\partial z}\right) + \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c^2}\frac{\partial w}{\partial t} \\ &\quad = \cos\gamma\left(1 - \left(\frac{w}{c}\right)^2\right) - \cos\alpha\frac{uw}{c^2} - \cos\beta\frac{vw}{c^2}.\end{aligned} \tag{3.1}$$

On making use of the inverse relations to (2.1), namely

$$\begin{aligned}X &= x + (\delta - 1)(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\alpha + \delta v^* t\cos\alpha, \\ Y &= y + (\delta - 1)(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\beta + \delta v^* t\cos\beta, \\ Z &= z + (\delta - 1)(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\gamma + \delta v^* t\cos\gamma, \\ T &= \delta t + \frac{\delta v^*}{c^2}(X\cos\alpha + Y\cos\beta + Z\cos\gamma),\end{aligned}$$

and (2.2), we may show, using the chain rule and by direct substitution, that the linear partial differential operator L is Lorentz-invariant, namely

$$L = ct\left(\cos\alpha\frac{\partial}{\partial x} + \cos\beta\frac{\partial}{\partial y} + \cos\gamma\frac{\partial}{\partial z}\right) + \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c}\frac{\partial}{\partial t}, \tag{3.2}$$

$$L = cT\left(\cos\alpha\frac{\partial}{\partial X} + \cos\beta\frac{\partial}{\partial Y} + \cos\gamma\frac{\partial}{\partial Z}\right) + \frac{(X\cos\alpha + Y\cos\beta + Z\cos\gamma)}{c}\frac{\partial}{\partial T},$$

and the coupled partial differential Equation (3.1) becomes simply

$$L(u) = c\cos\alpha\left(1-\left(\frac{u}{c}\right)^2\right) - \cos\beta\frac{uv}{c} - \cos\gamma\frac{uw}{c},$$
$$L(v) = c\cos\beta\left(1-\left(\frac{v}{c}\right)^2\right) - \cos\alpha\frac{uv}{c} - \cos\gamma\frac{vw}{c},$$
$$L(w) = c\cos\gamma\left(1-\left(\frac{w}{c}\right)^2\right) - \cos\alpha\frac{uw}{c} - \cos\beta\frac{vw}{c}. \quad (3.3)$$

In terms of a vector and matrix notation **u**, **a** and **Q** defined, respectively, by

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} 1-(u/c)^2 & -uv/c^2 & -uw/c^2 \\ -uv/c^2 & 1-(v/c)^2 & -vw/c^2 \\ -uw/c^2 & -vw/c^2 & 1-(w/c)^2 \end{pmatrix},$$

Equation (3.3) becomes $L(\mathbf{u}) = \mathbf{Qa}$ and it is a simple matter to show that $\det\mathbf{Q} = 1-(u^2+v^2+w^2)/c^2$, so that $\det\mathbf{Q} = 0$ if and only if the particle is travelling at the speed of light. This particular characteristic is shared by both the corresponding one- and two-dimensional Equations (1.1) and (1.2) derived, respectively, in [3] and [4].

On introducing A B, C and D through the relations

$$A = u\cos\alpha + v\cos\beta + w\cos\gamma, \qquad B = v\cos\gamma - w\cos\beta,$$
$$C = w\cos\alpha - u\cos\gamma, \qquad D = u\cos\beta - v\cos\alpha, \quad (3.4)$$

we may show that Equation (3.3) takes on the remarkably simple form

$$L(A) = c\left(1-\left(\frac{A}{c}\right)^2\right), \quad L(B) = -\frac{AB}{c}, \quad L(C) = -\frac{AC}{c}, \quad L(D) = -\frac{AD}{c}. \quad (3.5)$$

With the substitution $A = c\tanh\omega$ for some function $\omega(x,y,t)$, these equations become

$$L(\omega) = 1, \quad L(B) = -B\tanh\omega, \quad L(C) = -C\tanh\omega, \quad L(D) = -D\tanh\omega, \quad (3.6)$$

and the final three equations may all be re-written since, for example,

$$L(B)\cosh\omega + B\sinh\omega = L(B)\cosh\omega + B\sinh\omega L(\omega) = L(B\cosh\omega) = 0,$$

so that Equation (3.3) is finally simplified to become

$$L(\omega) = 1, \quad L(B\cosh\omega) = 0, \quad L(C\cosh\omega) = 0, \quad L(D\cosh\omega) = 0,$$

where $A = c\tanh\omega$. In the following section, we present the corresponding partial differential relations for energy and momentum. The calculation details for the solutions of the coupled partial differential Equation (3.6) for $\omega(x,y,z,t)$, $B(x,y,z,t)$, $C(x,y,z,t)$ and $D(x,y,z,t)$ are presented in Appendix A, and the solutions are summarised in the subsequent section.

4. Partial Differential Relations for Energy and Momentum

From the relations (2.9) for energy and momentum, we may deduce the following expressions for the partial derivatives

$$\frac{\partial e}{\partial x}=\frac{m_0}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right),$$
$$\frac{\partial p}{\partial x}=\frac{m_0}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right)\frac{\partial u}{\partial x}+\frac{uv}{c^2}\frac{\partial v}{\partial x}+\frac{uw}{c^2}\frac{\partial w}{\partial x}\right),$$
$$\frac{\partial q}{\partial x}=\frac{m_0}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{w}{c}\right)^2\right)\frac{\partial u}{\partial x}+\frac{uv}{c^2}\frac{\partial u}{\partial x}+\frac{vw}{c^2}\frac{\partial w}{\partial x}\right),$$
$$\frac{\partial r}{\partial x}=\frac{m_0}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2\right)\frac{\partial w}{\partial x}+\frac{uw}{c^2}\frac{\partial u}{\partial x}+\frac{vw}{c^2}\frac{\partial v}{\partial x}\right),$$

with similar expressions for the partial derivatives with respect to y, z and t. On making use of these relations, and with some rearrangement and division of (3.1) by $(1-(u^2+v^2+w^2)/c^2)^{3/2}$, we may deduce the partial differential relations connecting the partial derivatives of the momentum and energy p, q and e; thus,

$$t\left(\cos\alpha\frac{\partial p}{\partial x}+\cos\beta\frac{\partial p}{\partial y}+\cos\gamma\frac{\partial p}{\partial z}\right)+\frac{\xi}{c^2}\frac{\partial p}{\partial t}=\frac{e}{c^2}\cos\alpha, \tag{4.1}$$
$$t\left(\cos\alpha\frac{\partial q}{\partial x}+\cos\beta\frac{\partial q}{\partial y}+\cos\gamma\frac{\partial q}{\partial z}\right)+\frac{\xi}{c^2}\frac{\partial q}{\partial t}=\frac{e}{c^2}\cos\beta,$$
$$t\left(\cos\alpha\frac{\partial r}{\partial x}+\cos\beta\frac{\partial r}{\partial y}+\cos\gamma\frac{\partial r}{\partial z}\right)+\frac{\xi}{c^2}\frac{\partial r}{\partial t}=\frac{e}{c^2}\cos\gamma,$$
$$t\left(\cos\alpha\frac{\partial e}{\partial x}+\cos\beta\frac{\partial e}{\partial y}+\cos\gamma\frac{\partial e}{\partial z}\right)+\frac{\xi}{c^2}\frac{\partial e}{\partial t}=p\cos\alpha+q\cos\beta+r\cos\gamma,$$

where $\xi=x\cos\alpha+y\cos\beta+z\cos\gamma$.

In terms of the invariant operator L defined by (3.2) , these partial differential relations become more transparent and are simply

$$cL(p)=e\cos\alpha,\qquad cL(q)=e\cos\beta,\qquad cL(r)=e\cos\gamma, \tag{4.2}$$
$$L(e)=pc\cos\alpha+qc\cos\beta+rc\cos\gamma,$$

which arise using the definitions (2.9) and the relations (3.3) as follows:

$$\begin{aligned}cL(p)&=m_0c\left(\frac{L(u)}{(1-(u^2+v^2+w^2)/c^2)^{1/2}}+\frac{u}{c^2}\frac{(uL(u)+vL(v)+wL(w))}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\right)\\&=\frac{m_0c}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right)L(u)+\frac{uv}{c^2}L(v)+\frac{uw}{c^2}L(w)\right)\\&=\frac{m_0c^2\cos\alpha}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right)\\&=\frac{e_0\cos\alpha}{(1-(u^2+v^2+w^2)/c^2)^{1/2}}\\&=e\cos\alpha,\end{aligned}$$

$$
\begin{aligned}
cL(q) &= m_0c\left(\frac{L(v)}{(1-(u^2+v^2+w^2)/c^2)^{1/2}} + \frac{v}{c^2}\frac{(uL(u)+vL(v)+wL(w))}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\right) \\
&= \frac{m_0c}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{w}{c}\right)^2\right)L(w)+\frac{uv}{c^2}L(u)+\frac{wv}{c^2}L(w)\right) \\
&= \frac{m_0c^2\cos\beta}{(1-((u^2+v^2+w^2)/c^2)^{3/2}}\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right) \\
&= \frac{e_0\cos\beta}{(1-(u^2+v^2+w^2)/c^2)^{1/2}} \\
&= e\cos\beta,
\end{aligned}
$$

$$
\begin{aligned}
cL(r) &= m_0c\left(\frac{L(w)}{(1-(u^2+v^2+w^2)/c^2)^{1/2}} + \frac{w}{c^2}\frac{(uL(u)+vL(v)+wL(w))}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\right) \\
&= \frac{m_0c}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2\right)L(w)+\frac{uw}{c^2}L(u)+\frac{wv}{c^2}L(v)\right) \\
&= \frac{m_0c^2\cos\gamma}{(1-((u^2+v^2+w^2)/c^2)^{3/2}}\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right) \\
&= \frac{e_0\cos\gamma}{(1-(u^2+v^2+w^2)/c^2)^{1/2}} \\
&= e\cos\gamma,
\end{aligned}
$$

$$
\begin{aligned}
L(e) &= \frac{e_0}{c^2}\frac{(uL(u)+vL(v)+wL(w))}{(1-(u^2+v^2+w^2)/c^2)^{3/2}} \\
&= \frac{m_0(uc\cos\alpha+vc\cos\beta+wc\cos\gamma)}{(1-(u^2+v^2+w^2)/c^2)^{3/2}}\left(1-\left(\frac{u}{c}\right)^2-\left(\frac{v}{c}\right)^2-\left(\frac{w}{c}\right)^2\right) \\
&= pc\cos\alpha+qc\cos\beta+rc\cos\gamma.
\end{aligned}
$$

Formally, we may use the results (4.2) to obtain $L^2(e)=e$ and

$$
\begin{aligned}
L^2(p) &= (p\cos\alpha+q\cos\beta+r\cos\gamma)\cos\alpha, \\
L^2(q) &= (p\cos\alpha+q\cos\beta+r\cos\gamma)\cos\beta, \\
L^2(r) &= (p\cos\alpha+q\cos\beta+r\cos\gamma)\cos\gamma.
\end{aligned}
$$

Further, for (2.11), we may apply the operator L to $e^2=e_0^2+c^2(p^2+q^2+r^2)$ to confirm the validity of the equation $eL(e)=c^2(pL(p)+qL(q)+rL(r))$, as might be anticipated. The partial differential relations (3.1) are also fully compatible with the Lorentz-invariant energy–momentum relations (2.10). Since L is a Lorentz-invariant operator, the application of L to the first equation of (2.10) yields a linear combination of the second and third equations of (2.10), while its application to both the second and third yields the first relation.

5. Summary of Solutions of Coupled Partial Differential Equations (3.3)

Equations (3.3) constitute first-order partial differential equations, which are formally solved in Appendix A using Lagrange's characteristic method, leading to (3.6). The final details are as follows:

$$
\begin{aligned}
\omega(x,y,z,t) &= \sinh^{-1}\left(\frac{\xi}{((ct)^2-\xi^2)^{1/2}}\right)+\Phi(C_1,C_2,C_3,C_4), \qquad (5.1)\\
B(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_1(C_1,C_2,C_3,C_4), \\
C(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_2(C_1,C_2,C_3,C_4), \\
D(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_3(C_1,C_2,C_3,C_4),
\end{aligned}
$$

where $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$, and Φ, Ψ_1, Ψ_2 and Ψ_3 all denote arbitrary functions of the indicated arguments C_1, C_2, C_3, C_4 as given below,

$$C_1 = x\cos\beta - y\cos\alpha, \quad C_2 = x\cos\gamma - z\cos\alpha, \quad C_3 = y\cos\gamma - z\cos\beta, \\ C_4 = ((ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2)^{1/2}, \tag{5.2}$$

noting that $C_1^2 + C_2^2 + C_3^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$. As described in Appendix A, the final solutions of (3.5) for A, B, C and D as defined by (3.4) are given by

$$A(x,y,z,t) = c\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right), \qquad B(x,y,z,t) = c\mathscr{G}_1\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \\ C(x,y,z,t) = c\mathscr{G}_2\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \qquad D(x,y,z,t) = c\mathscr{G}_3\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \tag{5.3}$$

where $\mathscr{F}(C_1, C_2, C_3, C_4)$ and $\mathscr{G}_i(C_1, C_2, C_3, C_4)$ denote arbitrary functions of the indicated arguments, for $i = 1, 2, 3$ and $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$.

From the definitions (3.4) of A, B, C and D, the general Lorentz-invariant velocity components $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ inherit the particular functional forms determined from the relations

$$u\cos\alpha + v\cos\beta + w\cos\gamma = c\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right), \\ v\cos\gamma - w\cos\beta = c\mathscr{G}_1\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \\ w\cos\alpha - u\cos\gamma = c\mathscr{G}_2\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \\ u\cos\beta - v\cos\alpha = c\mathscr{G}_3\frac{((ct)^2 - \xi^2)^{1/2}}{ct + \xi\mathscr{F}}, \tag{5.4}$$

where $\mathscr{F}(C_1, C_2, C_3, C_4)$ and $\mathscr{G}_i(C_1, C_2, C_3, C_4)$ denote four arbitrary functions of the indicated arguments, for $i = 1, 2, 3$ and $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$. From the above equations, we may readily deduce

$$u = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\alpha + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_3\cos\beta - \mathscr{G}_2\cos\gamma)\right\}, \\ v = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\beta + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_1\cos\gamma - \mathscr{G}_3\cos\alpha)\right\}, \\ w = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\gamma + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_2\cos\alpha - \mathscr{G}_1\cos\beta)\right\}. \tag{5.5}$$

By squaring and adding the four relations in either (5.4) or (5.5), we obtain

$$u^2 + v^2 + w^2 = c^2\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)^2 + (\mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2)\frac{(ct)^2 - \xi^2}{(ct + \xi\mathscr{F})^2}\right\}, \tag{5.6}$$

from which it is clear that if $\mathscr{F}$ is determined from the relation $\mathscr{F} = (1 - \mathscr{G}_1^2 - \mathscr{G}_2^2 - \mathscr{G}_3^2)^{1/2}$, then $u^2 + v^2 + w^2 = c^2$, and there exist infinitely many families of singular paths with particles moving at the speed of light $u^2 + v^2 + w^2 = c^2$ that arise from the special case $\mathscr{F}^2 + \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2 = 1$.

From Equations (2.9) and (5.6) and with the abbreviation $\mathscr{G}^2 = \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2$, we obtain the following expression for the particle energy e,

$$e = \frac{e_0}{(1-(u^2+v^2+w^2)/c^2)^{1/2}} = \frac{e_0(ct+\xi\mathscr{F})}{((ct)^2-\xi^2)^{1/2}(1-(\mathscr{F}^2+\mathscr{G}^2))^{1/2}}, \tag{5.7}$$

while, from the expressions for the momenta p, q and r

$$p = \frac{m_0 u}{(1-(u^2+v^2+w^2)/c^2)^{1/2}}, \quad q = \frac{m_0 v}{(1-(u^2+v^2+w^2)/c^2)^{1/2}}, \quad r = \frac{m_0 w}{(1-(u^2+v^2+w^2)/c^2)^{1/2}}, \tag{5.8}$$

we obtain

$$\begin{aligned} pc &= \frac{e_0}{(1-(\mathscr{F}^2+\mathscr{G}^2))^{1/2}}\left\{\frac{(ct\mathscr{F}+\xi)\cos\alpha}{((ct)^2-\xi^2)^{1/2}} + (\mathscr{G}_3\cos\beta - \mathscr{G}_2\cos\gamma)\right\}, \\ qc &= \frac{e_0}{(1-(\mathscr{F}^2+\mathscr{G}^2))^{1/2}}\left\{\frac{(ct\mathscr{F}+\xi)\cos\beta}{((ct)^2-\xi^2)^{1/2}} + (\mathscr{G}_1\cos\gamma - \mathscr{G}_3\cos\alpha)\right\}, \\ rc &= \frac{e_0}{(1-(\mathscr{F}^2+\mathscr{G}^2))^{1/2}}\left\{\frac{(ct\mathscr{F}+\xi)\cos\gamma}{((ct)^2-\xi^2)^{1/2}} + (\mathscr{G}_2\cos\alpha - \mathscr{G}_1\cos\beta)\right\}. \end{aligned} \tag{5.9}$$

In the final section of the paper, we provide some illustrations of these formulae assuming a specific dependence on the arbitrary functions $\mathscr{F}$ and $\mathscr{G}_i$ for $i = 1,2,3$.

Finally, in this section, we note an interesting connection with the covariant curvature tensor R_{ijkm}. For general $\mathscr{F}$, on using the expressions (5.5) with $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$ and $\eta = ((ct)^2-\xi^2)^{1/2}$, we have the differential relations

$$\frac{d\xi}{dt} = c\left(\frac{ct\mathscr{F}+\xi}{ct+\xi\mathscr{F}}\right), \qquad \frac{d\eta}{dt} = c\frac{((ct)^2-\xi^2)^{1/2}}{(ct+\xi\mathscr{F})},$$

noting that we have used $d\xi/dt = u\cos\alpha + v\cos\beta + w\cos\gamma$, and, in the derivation of the second equation, we have used the first. Thus, from (5.6), we have

$$\left(\frac{ds}{dt}\right)^2 = u^2+v^2+w^2 = \left(\frac{d\xi}{dt}\right)^2 + (\mathscr{G}_1^2+\mathscr{G}_2^2+\mathscr{G}_3^2)\left(\frac{d\eta}{dt}\right)^2,$$

and, therefore, with the abbreviation $\mathscr{G}^2 = \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2$, we may deduce the metric $(ds)^2 = (d\xi)^2 + \mathscr{G}^2(d\eta)^2$. For this metric in the two independent variables $(x^1, x^2) = (\xi, \eta)$, from [14] (p. 56), there is only one non-zero component of the covariant curvature tensor R_{ijkm}, namely R_{1212}, which is given by

$$R_{1212} = -\mathscr{G}\frac{\partial^2\mathscr{G}}{\partial\xi^2}.$$

For a flat space, we expect $R_{1212} = 0$, so that $\mathscr{G}$ necessarily has the structure $\mathscr{G} = I(\eta)\xi + J(\eta)$, where $I(\eta)$ and $J(\eta)$ denote arbitrary functions of η.

6. Some Illustrations of the Solutions (5.1) and the Momenta Expressions (5.9)

In this section, for the purposes of illustration and to verify the analysis at least for a special case, we assume a particular dependence on the four arbitrary functions $\mathscr{F}$ and $\mathscr{G}_i$, where $i = 1,2,3$. Specifically, we examine the case when these functions depend only on $C_4 = ((ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2)^{1/2}$ and we use the notation $\eta = ((ct)^2-\xi^2)^{1/2}$, where $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$. We first illustrate the solutions (5.1) for $A(x,y,z,t)$ and $B(x,y,z,t)$ and then give an application of the momenta expressions (5.9) . We consider the development of special relativity formulated in [15], which predicts that the momenta $p(x,y,z,t)$, $q(x,y,z,t)$ and $r(x,y,z,t)$ and the wave energy $\mathscr{E}(x,y,z,t) = -e(x,y,z,t) - V(x,y,z,t)$ each satisfy the planar classical wave equation, namely

$$\frac{\partial^2 \mathscr{E}}{\partial t^2} = c^2\left(\frac{\partial^2 \mathscr{E}}{\partial x^2} + \frac{\partial^2 \mathscr{E}}{\partial y^2} + \frac{\partial^2 \mathscr{E}}{\partial z^2}\right), \tag{6.1}$$

where $V(x,y,t)$ denotes an applied external potential that is generating conventional spatial forces $f_x(x,y,z,t)$, $f_y(x,y,z,t)$ and $f_z(x,y,z,t)$ and a non-conventional force $g(x,y,z,t)$ in the direction of time, such that

$$f_x = -\frac{\partial V}{\partial x}, \quad f_y = -\frac{\partial V}{\partial y}, \quad f_z = -\frac{\partial V}{\partial z},$$
$$gc^2 = -\frac{\partial V}{\partial t},$$

and $g(x,y,z,t)$ is more commonly recognised as the mass or energy production term. We refer the reader to [15] for further details of this particular extension of special relativity.

We first assume that the arbitrary functions $\mathscr{F}$ and $\mathscr{G}_1$ are functions of $\eta = ((ct)^2 - \xi^2)^{1/2}$ only, so that, with A and B defined by

$$A(x,y,z,t) = c\left(\frac{ct\mathscr{F}(\eta) + \xi}{ct + \xi\mathscr{F}(\eta)}\right), \qquad B(x,y,z,t) = \frac{c\mathscr{H}(\eta)}{ct + \xi\mathscr{F}},$$

where $\mathscr{H}(\eta) = \eta\mathscr{G}_1(\eta)$, we might deduce the following expressions for the partial derivatives

$$\frac{\partial A}{\partial x} = \frac{c\cos\alpha}{(ct+\xi\mathscr{F})^2}\left(ct(1-\mathscr{F}^2) - \xi\eta\frac{d\mathscr{F}}{d\eta}\right),$$
$$\frac{\partial A}{\partial t} = \frac{c^2}{(ct+\xi\mathscr{F})^2}\left(-\xi(1-\mathscr{F}^2) + ct\eta\frac{d\mathscr{F}}{d\eta}\right),$$
$$\frac{\partial B}{\partial x} = -\frac{\xi\cos\alpha}{\eta(ct+\xi\mathscr{F})^2}\left((ct+\xi\mathscr{F})\frac{d\mathscr{H}}{d\eta} - \xi\mathscr{H}\frac{d\mathscr{F}}{d\eta}\right) - \frac{\mathscr{H}\mathscr{F}\cos\alpha}{(ct+\xi\mathscr{F})^2},$$
$$\frac{\partial B}{\partial t} = \frac{c^2 t}{\eta(ct+\xi\mathscr{F})^2}\left((ct+\xi\mathscr{F})\frac{d\mathscr{H}}{d\eta} - \xi\mathscr{H}\frac{d\mathscr{F}}{d\eta}\right) - \frac{c\mathscr{H}}{(ct+\xi\mathscr{F})^2},$$

with similar expressions for the partial derivatives with respect to y and z and for the functions C and D. On making use of these expressions for the partial derivatives, together with the definition (3.2) of the operator L, it is easy to verify that the equations $L(A) = c(1-(A/c)^2)$ and $L(B) = -AB/c$ are correctly satisfied.

The special relativity theory formulated in [15] predicts that the components of momentum as given by (5.8) satisfy the classical three-dimensional wave equation (6.1). Assuming that the arbitrary functions $\mathscr{F}$ and $\mathscr{G}_1$ are functions of $\eta = ((ct)^2 - \xi^2)^{1/2}$ only and that $\mathscr{G}_2 = \mathscr{G}_3 = 0$, we see from (5.8) that the momentum p in the $x-$direction has the structure $pc = ct\phi(\eta) + \xi\psi(\eta)$, where the functions $\phi(\eta)$ and $\psi(\eta)$ here are defined by

$$\phi(\eta) = \frac{e_0\cos\alpha\mathscr{F}(\eta)}{\eta(1-(\mathscr{F}(\eta)^2 + \mathscr{G}(\eta)^2))^{1/2}}, \quad \psi(\eta) = \frac{e_0\cos\alpha}{\eta(1-(\mathscr{F}(\eta)^2 + \mathscr{G}(\eta)^2))^{1/2}},$$

and where $\mathscr{G}^2$ denotes $\mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2 = \mathscr{G}_1^2$. The question therefore arises as to whether it is possible to choose the functions $\phi(\eta)$ and $\psi(\eta)$ (or equivalently the functions $\mathscr{F}(\eta)$ and $\mathscr{G}(\eta)$) such that $pc = ct\phi(\eta) + \xi\psi(\eta)$ satisfies the wave equation. On making use of partial differential expressions such as

$$\frac{\partial \phi}{\partial x} = -\frac{\xi \cos \alpha}{\eta} \frac{d\phi}{d\eta}, \qquad \frac{\partial \phi}{\partial t} = \frac{c^2 t}{\eta} \frac{d\phi}{d\eta},$$
$$\frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\xi^2}{\eta^2} \frac{d^2\phi}{d\eta^2} - \frac{(ct)^2}{\eta^3} \frac{d\phi}{d\eta} \right) \cos^2 \alpha, \qquad \frac{\partial^2 \phi}{\partial t^2} = c^2 \left(\frac{(ct)^2}{\eta^2} \frac{d^2\phi}{d\eta^2} - \frac{\xi^2}{\eta^3} \frac{d\phi}{d\eta} \right),$$

with similar expressions for $\psi(\eta)$ and the partial derivatives in the y and z directions, the condition that p satisfies the wave equation becomes

$$ct \left(\frac{d^2\phi}{d\eta^2} + \frac{3}{\eta} \frac{d\phi}{d\eta} \right) + \xi \left(\frac{d^2\psi}{d\eta^2} + \frac{3}{\eta} \frac{d\psi}{d\eta} \right) = 0.$$

Thus, the functions $\phi(\eta)$ and $\psi(\eta)$ must be such that

$$\frac{d^2\phi}{d\eta^2} + \frac{3}{\eta} \frac{d\phi}{d\eta} = \frac{d^2\psi}{d\eta^2} + \frac{3}{\eta} \frac{d\psi}{d\eta} = 0,$$

and therefore any expressions of the form $\phi(\eta) = \alpha_1 + \alpha_2/\eta^2$ and $\psi(\eta) = \alpha_3 + \alpha_4/\eta^2$ where α_j for $j = 1, 2, 3, 4$ denote four arbitrary constants will ensure that the wave equation is correctly satisfied.

As a simple illustration of this result, we consider the case $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = \lambda - \mu$ and $\alpha_4 = \lambda + \mu$ for certain constants λ and μ. In this case, the momentum is given by

$$pc = \frac{\lambda}{ct - \xi} - \frac{\mu}{ct + \xi},$$

which clearly satisfies the classical wave equation. The functions $\mathscr{F}$ and $\mathscr{G}$ are obtained by solving the two equations

$$e_0 \cos \alpha \mathscr{F} \eta = (\lambda - \mu)(1 - (\mathscr{F}(\eta)^2 + \mathscr{G}(\eta)^2))^{1/2},$$
$$e_0 \cos \alpha \eta = (\lambda + \mu)(1 - (\mathscr{F}(\eta)^2 + \mathscr{G}(\eta)^2))^{1/2},$$

from which we may readily deduce

$$\mathscr{F}(\eta) = \frac{\lambda - \mu}{\lambda + \mu}, \quad \mathscr{G}(\eta) = \frac{(4\lambda\mu - (e_0 \eta \cos \alpha)^2)^{1/2}}{\lambda + \mu},$$

so that $\mathscr{F}$ is a constant, and, evidently, the constants λ and μ must be of the same sign in order to ensure that $\mathscr{G}$ is well defined.

7. Conclusions

For fully three-dimensional motion with rectangular Cartesian coordinates (x, y, z), we have shown that the requirement that the three velocity equations $dx/dt = u(x, y, z, t)$, $dy/dt = v(x, y, z, t)$ and $dz/dt = w(x, y, z, t)$ remain invariant under the general three-dimensional Lorentz transformation (2.1) gives rise to the three coupled partial differential Equations (3.1) for the three velocity components $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ in the $x-$, $y-$ and $z-$directions, respectively. These first-order partial differential equations are solved using Lagrange's characteristic method to deduce the solutions

$$u = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\alpha + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_3\cos\beta - \mathscr{G}_2\cos\gamma)\right\}, \tag{7.1}$$
$$v = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\beta + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_1\cos\gamma - \mathscr{G}_3\cos\alpha)\right\},$$
$$w = c\left\{\left(\frac{ct\mathscr{F} + \xi}{ct + \xi\mathscr{F}}\right)\cos\gamma + \frac{((ct)^2 - \xi^2)^{1/2}}{(ct + \xi\mathscr{F})}(\mathscr{G}_2\cos\alpha - \mathscr{G}_1\cos\beta)\right\},$$

in terms of four arbitrary functions $\mathscr{F}(C_1, C_2, C_3, C_4)$ and $\mathscr{G}_i(C_1, C_2, C_3, C_4)$ for $i = 1, 2, 3$, $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$ and C_j for $j = 1, 2, 3, 4$ denote four independent integrals of the solution procedure that are defined by (5.2), namely

$$C_1 = x\cos\beta - y\cos\alpha, \quad C_2 = x\cos\gamma - z\cos\alpha, \quad C_3 = y\cos\gamma - z\cos\beta,$$
$$C_4 = ((ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2)^{1/2}.$$

With the velocity components given by (7.1) , the corresponding particle energy e and momenta p, q and r are given, respectively, by (5.7) and (5.8).

We observe the very curious fact that with A, B, C and D given by (A.3) involving the four arbitrary functions $\mathscr{F}$, $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$, the singular case $\mathscr{F}^2 + \mathscr{G}_1^2 + \mathscr{G}_2^2 + \mathscr{G}_3^2 = 1$ is such that $u^2 + v^2 + w^2 = A^2 + B^2 + C^2 + D^2 = c^2$ for all arbitrary functions $\mathscr{F}$, $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$ satisfying the constraint. The existence of these infinitely many singular families of paths with particles moving at the speed of light is indicative of the abundant possibilities that might exist in the "fast lane". To the author's knowledge, neither the coupled partial differential Equations (3.1) and (3.3), nor the particular functional forms determined from either (5.3) or (5.5), nor the fact that the energy e and momenta p, q and r satisfy the partial differential relations (4.1) or in operator form (4.2) have been previously given in the literature.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest: The author states that he has no conflicting interests, financial or otherwise.

Appendix A

Appendix A.1. Derivation of Solutions of Partial Differential Equation (3.6)

In this appendix, we use Lagrange's characteristic method to determine solutions of the partial differential Equation (3.6) for $\omega(x, y, z, t)$, $B(x, y, z, t)$, $C(x, y, z, t)$ and $D(x, y, z, t)$ in terms of four arbitrary functions $\mathscr{F}$, $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$. Lagrange's method introduces a characteristic parameter s through the eight equations

$$\frac{dx}{ds} = ct\cos\alpha, \qquad \frac{dy}{ds} = ct\cos\beta, \qquad \frac{dz}{ds} = ct\cos\gamma, \tag{A1}$$
$$\frac{dt}{ds} = \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c}, \qquad \frac{d\omega}{ds} = 1,$$
$$\frac{d(B\cosh\omega)}{ds} = 0, \qquad \frac{d(C\cosh\omega)}{ds} = 0, \qquad \frac{d(D\cosh\omega)}{ds} = 0,$$

to deduce eight independent integrals. Explicit solutions for ω, $B\cosh\omega$, $C\cosh\omega$ and $D\cosh\omega$ are then obtained by taking one integral to be an arbitrary function of the space- and time-independent integrals. The solutions so obtained are the most useful in the sense that the dependent variables are direct functions of the space and time coordinates. These solutions are not necessarily the most general since other solutions can be constructed by taking any of the integrals to be an arbitrary function of any of the remaining integrals. For the above system, there will be other implicitly defined solutions involving more than one of the dependent variables.

On division by the first equation of (A1) to eliminate the characteristic parameter s, we have

$$\frac{dy}{dx} = \frac{\cos\beta}{\cos\alpha}, \qquad \frac{dz}{dx} = \frac{\cos\gamma}{\cos\alpha}, \tag{A2}$$

$$\frac{dt}{dx} = \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c^2 t\cos\alpha}, \qquad \frac{d\omega}{dx} = \frac{1}{ct\cos\alpha},$$
$$\frac{d(B\cosh\omega)}{dx} = 0, \qquad \frac{d(C\cosh\omega)}{dx} = 0, \qquad \frac{d(D\cosh\omega)}{dx} = 0.$$

The first and second equations readily integrate to yield $x\cos\beta - y\cos\alpha = C_1$ and $x\cos\gamma - z\cos\alpha = C_2$, and, by symmetry, we have $y\cos\gamma - z\cos\beta = C_3$. The third equation becomes

$$\begin{aligned}\frac{dt}{dx} &= \frac{(x\cos\alpha + y\cos\beta + z\cos\gamma)}{c^2 t\cos\alpha}\\ &= \frac{(x\cos^2\alpha + \cos\beta(x\cos\beta - C_1) + \cos\gamma(x\cos\gamma - C_2))}{c^2 t\cos^2\alpha}\\ &= \frac{(x - C_1\cos\beta - C_2\cos\gamma)}{c^2 t\cos^2\alpha},\end{aligned}$$

which, on integration and simplification, yields $(ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 = C_4^2$. The fourth equation of (A1) yields

$$\begin{aligned}\frac{d\omega}{dx} = \frac{1}{ct\cos\alpha} &= \frac{1}{\cos\alpha\left(C_4^2 + (x\cos\alpha + y\cos\beta + z\cos\gamma)^2\right)^{1/2}}\\ &= \frac{1}{\left(C_4^2\cos^2\alpha + (x - C_1\cos\beta - C_2\cos\gamma)^2\right)^{1/2}}.\end{aligned}$$

With the substitution $x - C_1\cos\beta - C_2\cos\gamma = C_4\cos\alpha\sinh\Omega$, this equation may be immediately integrated to yield

$$\begin{aligned}\omega = \Omega + C_5 &= \sinh^{-1}\left(\frac{x - C_1\cos\beta - C_2\cos\gamma}{C_2^{1/2}\cos\alpha}\right) + C_5\\ &= \sinh^{-1}\left(\frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{((ct)^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2)^{1/2}}\right) + C_5,\end{aligned}$$

and the remaining equations of (A1) integrate trivially to give $B\cosh\omega = C_6$, $C\cosh\omega = C_7$ and $D\cosh\omega = C_8$, so that the solutions of (3.6) for $\omega(x,y,z,t)$, $B(x,y,z,t)$, $C(x,y,z,t)$ and $D(x,y,z,t)$ are obtained from

$$\begin{aligned}\omega(x,y,z,t) &= \sinh^{-1}\left(\frac{\xi}{((ct)^2 - \xi^2)^{1/2}}\right) + \Phi(C_1, C_2, C_3, C_4),\\ B(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_1(C_1, C_2, C_3, C_4),\\ C(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_2(C_1, C_2, C_3, C_4),\\ D(x,y,z,t) &= c\,\mathrm{sech}\,\omega\Psi_3(C_1, C_2, C_3, C_4),\end{aligned}$$

where $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$ and Φ, Ψ_1, Ψ_2 and Ψ_3 all denote arbitrary functions of the indicated arguments.

These expressions may be simplified using $\sinh^{-1}(z) = \log(z + (z^2+1)^{1/2})$ and using $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$ as the working variable; then,

$$\sinh^{-1}\left(\frac{\xi}{((ct)^2-\xi^2)^{1/2}}\right) = \frac{1}{2}\log\left(\frac{ct+\xi}{ct-\xi}\right),$$

so that if we redefine the arbitrary function $\Phi = (\log F)/2$, we have

$$\frac{A(x,y,z,t)}{c} = \tanh\omega = \frac{e^{2\omega}-1}{e^{2\omega}+1}$$
$$= \left(\frac{(ct+\xi)F-(ct-\xi)}{(ct+\xi)F+(ct-\xi)}\right) = \left(\frac{ct\mathscr{F}+\xi}{ct+\xi\mathscr{F}}\right),$$

where $\mathscr{F} = (F-1)/(F+1)$ is yet another redefinition of the arbitrary function. Similarly, for $B(x,y,z,t)$, $C(x,y,z,t)$ and $D(x,y,z,t)$, we have

$$\frac{B(x,y,z,t)}{c} = \frac{2\Psi_1}{(e^{\omega}+e^{-\omega})} = \frac{2F^{1/2}((ct)^2-\xi^2)^{1/2}\Psi_1}{(ct+\xi)F+(ct-\xi)}$$
$$= \mathscr{G}_1\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}},$$
$$\frac{C(x,y,z,t)}{c} = \mathscr{G}_2\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}}, \qquad \frac{D(x,y,z,t)}{c} = \mathscr{G}_3\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}},$$

where $\mathscr{G}_i = 2F^{1/2}\Psi_i/(F+1) = (1-\mathscr{F}^2)^{1/2}\Psi_i$ for $i = 1,2,3$. Thus, with A, B, C and D defined by (3.4), the general solutions of (3.5) are given by

$$A(x,y,z,t) = c\left(\frac{ct\mathscr{F}+\xi}{ct+\xi\mathscr{F}}\right), \qquad B(x,y,z,t) = c\mathscr{G}_1\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}},$$
$$C(x,y,z,t) = c\mathscr{G}_2\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}}, \qquad D(x,y,z,t) = c\mathscr{G}_3\frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}},$$

where $\mathscr{F}(C_1,C_2,C_3,C_4)$ and $\mathscr{G}_i(C_1,C_2,C_3,C_4)$ denote arbitrary functions of the indicated arguments, for $i = 1,2,3$ and $\xi = x\cos\alpha + y\cos\beta + z\cos\gamma$.

Finally, we comment that the combined transformations

$$\mathscr{F} = \frac{F-1}{F+1} = \frac{e^{2\Phi}-1}{e^{2\Phi}+1} = \tanh\Phi,$$

show that $\Phi = \tanh^{-1}\mathscr{F}$ and therefore a Lorentz invariance appears through a translation in Φ, while $\mathscr{F}$ transforms like a velocity; thus, $\mathscr{F}^* = (\mathscr{F} - v^*/c)/(1-\mathscr{F}v^*/c)$. Accordingly, with $\zeta = X\cos\alpha + Y\cos\beta + Z\cos\gamma$, using the transformational formulae $\xi = \delta(\zeta - v^*T)$ and $ct = \delta(cT - \zeta v^*/c)$, the key quantities in (A1) transform under a Lorentz transformation as

$$\frac{ct\mathscr{F}+\xi}{ct+\xi\mathscr{F}} = \frac{cT\mathscr{F}^*+\zeta}{cT+\zeta\mathscr{F}^*}, \qquad \frac{((ct)^2-\xi^2)^{1/2}}{ct+\xi\mathscr{F}} = \frac{(1-(v^*/c)^2)^{1/2}}{(1-v^*\mathscr{F}/c)}\frac{((cT)^2-\zeta^2)^{1/2}}{cT+\zeta\mathscr{F}^*}.$$

References

1. Esposito-Farese, G.; Pitrou, C.; Uzan, J.P. Vector theories in cosmology. *Phys. Rev. D* **2010**, *81*, 063519. [CrossRef]
2. Meierovich, B.E. Vector fields in multi-dimensional cosmology. *Phys. Rev. D* **2011**, *84*, 064037. [CrossRef]
3. Hill, J.M. Functional forms for Lorentz invariant velocities. *Mathematics* **2024**, *12*, 1609. [CrossRef]
4. Hill, J.M. Planar Lorentz invariant velocities with a wave equation application. *Wave Motion* **2024**, *130*, 103368. [CrossRef]
5. Hill, J.M. *Differential Equations and Group Methods for Scientists and Engineers*; Studies in Advanced Mathematics; CRC Press: Boca Raton, FL, USA, 1992; ISBN 0-8493-4442-5.
6. Dingle, H. *The Special Theory of Relativity*; Methuen and Co., Ltd.: London, UK, 1961.
7. McCrea, W.H. *Relativity Physics*; Methuen and Co., Ltd.: London, UK, 1947.

8. Bohm, D. *The Special Theory of Relativity*; W. A. Benjamin, Inc.: New York, NY, USA, 1965.
9. French, A.P. *Special Relativity*; Thomas Nelson and Sons Ltd.: London, UK, 1968.
10. Resnick, R. *Introduction to Special Relativity*; John Wiley and Sons Inc.: New York, NY, USA, 1968.
11. Moller, C. *The Theory of Relativity*; Clarendon Press: Oxford, UK, 1966.
12. Tolman, R.C. *Relativity, Thermodynamics and Cosmology*; Clarendon Press: Oxford, UK, 1946.
13. Perrett, W.; Jeffery, G.B. *The Principle of Relativity*; Dover Publications Inc.: New York, NY, USA 2017.
14. Spain, B. *Tensor Calculus*; Oliver and Boyd: Edinburgh/London, UK, 1960.
15. Hill, J.M. *Mathematics of Particle-Wave Mechanical Systems*; Springer: Berlin/Heidelberg, Germany, 2022; ISBN 978-3-031-19792-5, ISBN 978-3-031-19793-2. [CrossRef]