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Article

A Direct Road to Majorana Fields

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Abstract: A concise discussion of spin-$1/2$ field equations with a special focus on Majorana spinors is presented. The Majorana formalism which describes massive neutral fermions by the help of two-component or four-component spinors is of fundamental importance for the understanding of mathematical aspects of supersymmetric and other extensions of the Standard Model of particle physics, which may play an increasingly important role at the beginning of the LHC era. The interplay between the two-component and the four-component formalism is highlighted in an introductory way. Majorana particles are predicted both by grand unified theories, in which these particles are neutrinos, and by supersymmetric theories, in which they are photinos, gluinos and other states.

Keywords: symmetry and conservation laws; non-standard-model neutrinos; real and complex spinor representations

Classification: PACS 11.10–Field theory, 11.30.-j–Symmetry and conservation laws, 14.60.St–Non-standard-model neutrinos, right-handed neutrinos, *etc.*

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1. Introduction

Starting from Lorentz covariance as one of the key properties of space-time described in the framework of the special theory of relativity, we derive the free wave equations for the fundamental spin-$1/2$ particles in a concise manner and discuss their properties with a special focus on Majorana spinors [1], based on the representation theory of the proper Lorentz group. The two-component formalism for massive spin-$1/2$ particles is investigated in detail and related to the four-component Dirac and Majorana formalism. This work is intended to give a solid introduction to the basic mathematical aspects of Majorana fermion fields which constitute an important aspect of modern neutrino physics.

One important goal of this work is to show that the most general relativistic field equation for a four-component spinor Ψ can be written in the form

$$i\gamma^\mu \partial_\mu \Psi - \tilde{m}_{_M} \Psi^c - \tilde{m}_{_D} \Psi = 0 \tag{1}$$

with appropriately chosen Majorana and Dirac mass matrices $\tilde{m}_{_M}$ and $\tilde{m}_{_D}$. Such a spinor describes a charged Dirac particle for $\tilde{m}_{_M} = 0$ or a pair of Majorana particles. In the latter case, the theory may contain a physical CP violating phase.

2. Symmetry

2.1. The Lorentz Group

The structure of wave equations is intimately connected with the Lorentz symmetry of space-time. We therefore review the most important properties of the Lorentz group and clarify notational details in this section. Additionally, basic aspects of the representation theory of the Lorentz group, which are inevitable for the comprehension of relativistic wave equations, are derived in detail.

Physical laws are generally assumed to be independent of the observer, and the underlying symmetry which makes it possible to relate the points of view taken by different observers is expressed by the Lorentz group. Gravitational effects will be neglected in the following by the assumption that space-time is flat.

In two different systems of inertia, the coordinates of a point in Minkowski space-time measured by two corresponding observers are related by a proper Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow = SO^+(1,3)$, which does not cause time or space reflections, via $x' = \Lambda x$ or $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, with $\Lambda \in SO^+(1,3)$ defined in further detail below. For the sake of simplicity, the translational symmetry of space-time contained in the Poincaré group is neglected by the assumption that the two coordinate systems share their point of origin.

The indefinite Minkoswki inner product (x,y) of two vectors x, y is preserved by the Lorentz transformation, *i.e.*, for

$$x = (x^0, x^1, x^2, x^3)^{\mathrm{T}} = (x_0, -x_1, -x_2, -x_3)^{\mathrm{T}} \quad \text{and} \quad y = (y^0, y^1, y^2, y^3)^{\mathrm{T}} = (y_0, -y_1, -y_2, -y_3)^{\mathrm{T}}$$

we set

$$(x,y) = x^{\mathrm{T}} g y = x'^{\mathrm{T}} g y' = x^{\mathrm{T}} \Lambda^{\mathrm{T}} g \Lambda y \quad \forall\, x, y\,, \tag{2}$$

where we make use of the usual rules for matrix multiplication and consider x as a column vector and x^{T} as a row vector, or

$$g_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu = x'^\mu y'_\mu = g_{\mu\nu} x'^\mu y'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha y^\beta \tag{3}$$

with the metric tensor g satisfying

$$g_{\mu\nu} = \left(g^{-1}\right)^{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4}$$

$$g^{\mu}_{\ \nu} = g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} \tag{5}$$

Therefore, the proper Lorentz group is defined by

$$SO^{+}(1,3) = \{\Lambda \in GL(4,\mathbb{R}) \,|\, \Lambda^{\mathrm{T}} g \Lambda = g,\ \Lambda^{0}_{\ 0} \geq 1,\ \mathrm{det}\Lambda = +1\} \tag{6}$$

whereas the full Lorentz group $O(1,3)$, consisting of four connected components, is defined by

$$O(1,3) = \{\Lambda \in GL(4,\mathbb{R}) \,|\, \Lambda^{\mathrm{T}} g \Lambda = g\} \tag{7}$$

$SO^{+}(1,3)$ is the identity component in $O(1,3)$, containing the identity Lorentz transformation expressed by the unit matrix diag$(1,1,1,1)$, whereas the other three topologically separated pieces of $O(1,3)$ are the components connected to the time reversal transformation $\Lambda_{T} =$ diag$(-1,1,1,1)$, space inversion $\Lambda_{P} =$ diag$(1,-1,-1,-1)$ and space-time inversion $\Lambda_{PT} =$ diag$(-1,-1,-1,-1)$. Note that from $\Lambda^{\mathrm{T}} g \Lambda = g$ follows $\left(\Lambda^{\mathrm{T}}\right)^{-1} = g \Lambda g^{-1}$ or

$$\left[\left(\Lambda^{\mathrm{T}}\right)^{-1}\right]^{\mu}_{\ \ \nu} = g_{\mu\alpha} \Lambda^{\alpha}_{\ \beta} g^{\beta\nu} = \Lambda_{\mu}^{\ \ \nu} \quad , \quad \Lambda_{\mu}^{\ \alpha} \Lambda^{\nu}_{\ \alpha} = \delta^{\nu}_{\mu} \tag{8}$$

and from $\Lambda^{-1} = g^{-1} \Lambda^{\mathrm{T}} g$ we have correspondingly $\left(\Lambda^{-1}\right)^{\mu}_{\ \ \nu} = \Lambda_{\nu}^{\ \mu}$, keeping in mind that the index closer to the matrix symbol denotes the position of the corresponding matrix element in vertical direction, if the usual rules for matrix manipulations are presumed. The Λ are not tensors, however, formally lowering and raising the indices of $\Lambda^{\mu}_{\ \nu}$ has the effect to generate the matrix (elements) of $\left(\Lambda^{\mathrm{T}}\right)^{-1}$.

2.2. *Two-Dimensional Irreducible Representations of the Lorentz Group*

In order to construct the lowest-dimensional non-trivial representations of the Lorentz group, we introduce the quantities

$$\bar{\sigma}_{\mu} = \sigma^{\mu} = (1, \vec{\sigma})$$

$$\bar{\sigma}^{\mu} = \sigma_{\mu} = (1, -\vec{\sigma}) \tag{9}$$

where the three components of $\vec{\sigma}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

Throughout the paper, we will denote the identity matrix in two dimensions, alternatively, by the symbols 1, I, or σ_0.

From an arbitrary 4-vector x with contravariant components $x^{\mu} = (x^0, x^1, x^2, x^3)$ we construct the following 2×2-matrix X

$$X = \bar{\sigma}_{\mu} x^{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{11}$$

The map $x \to X$ is obviously linear and one-to-one, and X is Hermitian. A further important property of X is the fact that its determinant is equal to the Minkowski norm $x^2 = (x, x)$

$$\text{det}X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x_\mu x^\mu \tag{12}$$

The special linear group $SL(2,\mathbb{C})$ is defined by

$$SL(2,\mathbb{C}) = \{S \in GL(2,\mathbb{C}) \,|\, \text{det}S = +1\} \tag{13}$$

The trick is now to set

$$X' = SXS^+, \quad S \in SL(2,\mathbb{C}) \tag{14}$$

where the $^+$ denotes the Hermitian conjugate matrix. Then one observes that X' is again Hermitian, and

$$\text{det}X' = \text{det}(SXS^+) = \text{det}S\,\text{det}X\,\text{det}S^+ = \text{det}X \tag{15}$$

i.e., X' can again be written as

$$X' = \bar{\sigma}_\mu x'^\mu, \quad \text{where} \qquad x'_\mu x'^\mu = x_\mu x^\mu \tag{16}$$

We conclude that x' and x are related by a Lorentz transformation $x' = \Lambda(S)x$, and since the group $SL(2,\mathbb{C})$ is connected and the map $S \to \Lambda(S)$ is continuous, the map $\lambda : S \to \Lambda(S)$ is a homomorphism of $SL(2,\mathbb{C})$ into the proper Lorentz group $\mathcal{L}_+^\uparrow = SO^+(1,3)$.

We now show that the homomorphism λ is two-to-one. This can be easily understood from the observation that S and $-S$ correspond to the same Lorentz transformation. To see this in more detail, we calculate the kernel of λ, *i.e.*, the set of all $S \in SL(2,\mathbb{C})$ which for any Hermitian matrix X satisfies the equality

$$X = SXS^+ \tag{17}$$

By taking, in particular,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the equality leads to the condition $S = (S^+)^{-1}$, and Equation (17) reduces to $XS - SX = [X, S] = 0$ for any Hermitian X. This implies $S = \alpha I$. From the condition det $S = +1$ follows $S = \pm I$.

Apart from the important group isomorphism

$$\mathcal{L}_+^\uparrow \cong SL(2,\mathbb{C})/\{\pm 1\} \tag{18}$$

just found, matrices in the special linear group $SL(2,\mathbb{C})$ have an additional interesting feature. Defining the antisymmetric matrix ϵ by

$$\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon = -\epsilon^{-1} = -\epsilon^{\mathrm{T}} \tag{19}$$

we can also define the symplectic bilinear form $\langle u, v\rangle = -\langle v, u\rangle$ for two elements (spinors)

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \tag{20}$$

in the two-dimensional complex vector space $\mathbb{C}^2_{\mathrm{C}}$

$$\langle u, v\rangle = u^1 v^2 - u^2 v^1 = u^{\mathrm{T}} \epsilon v \tag{21}$$

This symplectic bilinear form is invariant under $SL(2,\mathbb{C})$

$$\langle u, v\rangle = u^{\mathrm{T}} \epsilon v = \langle Su, Sv\rangle = u^{\mathrm{T}} S^{\mathrm{T}} \epsilon S v \tag{22}$$

Indeed, setting

$$S = \begin{pmatrix} s^1{}_1 & s^1{}_2 \\ s^2{}_1 & s^2{}_2 \end{pmatrix} , \quad \text{where} \quad \det S = s^1{}_1 s^2{}_2 - s^1{}_2 s^2{}_1 \tag{23}$$

a short calculation yields

$$S^{\mathrm{T}} \epsilon S = \begin{pmatrix} s^1{}_1 & s^2{}_1 \\ s^1{}_2 & s^2{}_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s^1{}_1 & s^1{}_2 \\ s^2{}_1 & s^2{}_2 \end{pmatrix} =$$

$$\begin{pmatrix} s^1{}_1 & s^2{}_1 \\ s^1{}_2 & s^2{}_2 \end{pmatrix} \begin{pmatrix} s^2{}_1 & s^2{}_2 \\ -s^1{}_1 & -s^1{}_2 \end{pmatrix} = \begin{pmatrix} 0 & \det S \\ -\det S & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{24}$$

and therefore we have a further group isomorphism

$$\mathcal{L}^{\uparrow}_{+} \cong Sp(2,\mathbb{C})/\{\pm 1\}, \quad SL(2,\mathbb{C}) \cong Sp(2,\mathbb{C}) \tag{25}$$

with the complex symplectic group $Sp(2,\mathbb{C})$ defined by

$$Sp(2,\mathbb{C}) = \{S \in GL(2,\mathbb{C}) \,|\, S^{\mathrm{T}} \epsilon S = \epsilon\} \tag{26}$$

The group isomorphism $SO(3,\mathbb{C}) \cong \mathcal{L}^{\uparrow}_{+}$ is mentioned here as a side remark.

We conclude this section by a crucial observation. We have learned above how to construct an action of the Lorentz group on the two-dimensional complex vector space $\mathbb{C}^2_{\mathrm{C}}$, which obviously respects by definition the Lorentz group structure in the sense of representations

$$\Lambda(S_1)\Lambda(S_2) = \Lambda(S_1 S_2) \tag{27}$$

Up to a sign, this relation can be inverted and one may write in a slightly sloppy style $S(\Lambda_1\Lambda_2) = \pm S(\Lambda_1)S(\Lambda_2)$ to express the fact that we have found a double-valued representation of the Lorentz group. One might wonder whether this representation is equivalent to the representation given by the complex conjugate special linear matrices $S^*(\Lambda)$. This is not the case, since it is impossible to relate all $S \in SL(2,\mathbb{C})$ by a basis transform such that

$$S^* = BSB^{-1} \quad \forall\, S \tag{28}$$

However, considering only the subgroup

$$SU(2) = \{U \in GL(2,\mathbb{C}) \,|\, U^+ = U^{-1},\, \det U = 1\} \subset SL(2,\mathbb{C}) \tag{29}$$

the situation is different. A special unitary matrix $U \in SU$ can be written in the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} , \quad \det U = aa^* + bb^* = 1 \tag{30}$$

and a short calculation yields

$$U^* = \epsilon U \epsilon^{-1} \tag{31}$$

with the ϵ defined above. The two-dimensional two-valued representation of the group of rotations $SO(3)$ by special unitary transformations, which can be obtained directly via the trick Equation (14), is equivalent to its complex conjugate representation.

3. Field Equations

3.1. The Scalar Field

3.1.1. Wave Equation

Before we reach our goal to construct wave equations for spinor fields, we shortly review the simplest case of a free (non-interacting) scalar field $\varphi(x)$ satisfying the Klein-Gordon equation ($\hbar = c = 1$)

$$\{\Box + m^2\}\varphi(x) = \{\partial_\mu \partial^\mu + m^2\}\varphi(x) = 0 \tag{32}$$

which is given in the primed coordinate system by

$$\varphi'(x') = \varphi(x) = \varphi(\Lambda^{-1}x') = \varphi'(\Lambda x) \tag{33}$$

The field $\varphi'(x')$ indeed satisfies the Klein-Gordon equation also in the primed coordinate system

$$\{\Box' + m^2\}\varphi'(x') = \{\partial'_\mu \partial'^\mu + m^2\}\varphi'(x') = 0 \tag{34}$$

since the differential operators ∂_μ, ∂^μ transform according to

$$\partial_\nu = \frac{\partial}{\partial^\nu} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial}{\partial x'^\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x'^\mu} = \Lambda^\mu{}_\nu \partial'_\mu \,, \quad \text{or} \tag{35}$$

$$\partial'_\alpha = \delta^\mu_\alpha \partial'_\mu = \Lambda_\alpha{}^\nu \Lambda^\mu{}_\nu \partial'_\mu = \Lambda_\alpha{}^\nu \partial_\nu \tag{36}$$

and is a analogous manner one derives $\partial'^\mu = \partial/\partial x'_\mu = \Lambda^\mu{}_\nu \partial^\nu$. Hence we have

$$\partial'_\mu \partial'^\mu \varphi'(x') = \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta \partial_\alpha \partial^\beta \varphi'(\Lambda x) = g^\alpha_\beta \partial_\alpha \partial^\beta \varphi(x) = \partial_\alpha \partial^\alpha \varphi(x) \tag{37}$$

i.e., Equations (32,34) are equivalent.

Note that a non-trivial Lorentz-invariant first order differential operator $\sim \alpha_\mu \partial^\mu$, $\alpha_\mu \in \mathbb{C}$, acting on a one-component field does not exist, such that the Klein-Gordon equation is necessarily of second order.

3.1.2. Charge Conjugation

Positive and negative energy plane-wave solutions of the Klein-Gordon equation are given by

$$\varphi(x) = e^{\pm ikx}, \quad kx = k_\mu x^\mu, \quad k^0 = \sqrt{\vec{k}^2 + m^2} \tag{38}$$

and the general solution of the Klein-Gordon is a superposition of plane waves which can be expressed by a Fourier integral à la

$$\varphi(x) = \int \frac{d^3k}{2k^0(2\pi)^3}[a(\vec{k})e^{-ikx} + b^*(\vec{k})e^{+ikx}] \tag{39}$$

when the field is complex, or by

$$\varphi(x) = \int \frac{d^3k}{2k^0(2\pi)^3}[a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{+ikx}] \tag{40}$$

when the field is real. Since the positive energy solutions $e^{-ikx} = e^{i\vec{k}\vec{x}-ik^0x_0}$ can be associated with particles and the negative energy solutions $e^{+ikx} = e^{-i\vec{k}\vec{x}+ik^0x_0}$ with antiparticles, it is natural to define charge conjugation for scalar C-number fields by

$$\varphi^c(x) = \eta_c\varphi^*(x) \tag{41}$$

since this transformation does interchange the role of particle and antiparticle solutions and leaves the momentum (and the spin, if particles with spin are considered) of particles unchanged. We set the phase factor η_c, $|\eta_c| = 1$, which is typically chosen to be ± 1 for a neutral field, equal to one in the following.

By quantization, the Fourier coefficients a, a^* and b, b^* become particle creation or annihilation operators a, a^+ and b, b^+, complex conjugation corresponds then to Hermitian conjugation $^+$. Within this formalism, the charge conjugation operator U_c then acts on φ as unitary operator such that $\varphi^c(x) = U_c\varphi(x)U_c^{-1} = \varphi^+(x)$. This can easily be accomplished if we define

$$U_c a^+(\vec{k})U_c^{-1} = b^+(\vec{k}) \tag{42}$$

$$U_c b^+(\vec{k})U_c^{-1} = a^+(\vec{k}) \tag{43}$$

Thus, the charge conjugation operator acts on a one-particle $a^+(\vec{k})|0\rangle$ state with momentum $\vec{k}$

$$U_c a^+(\vec{k})|0\rangle = U_c a^+(\vec{k})U_c^{-1}U_c|0\rangle = U_c a^+(\vec{k})U_c^{-1}|0\rangle = b^+(\vec{k})|0\rangle \tag{44}$$

where we presupposed that the vacuum $|0\rangle = U_c|0\rangle$ is invariant under charge conjugation.

We observe that charge conjugation can be discussed to some extent on a C-number field level, and we will proceed that way when we come to the fermion fields. However, one should always keep in mind that after second quantization, subtleties may arise. Note that quantum mechanical states live in a complex Hilbert space and can be gauged by complex phase factors. From a naive point of view, we may consider a real field as a "wave function" or quantum mechanical state, but then we are not allowed to multiply the field by an imaginary number, since the field is then no longer a real entity. One may also remark that the widespread labeling of antiparticle solutions as "negative energy solutions" is misleading to some extent and should be replaced rather by "negative frequency solutions" in the literature, since the energy of antiparticles is positive.

From two (equal mass) degenerate real fields one may construct a complex field

$$\tilde{\varphi} = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \tag{45}$$

and the corresponding creation and annihilation operators a_1, a_1^+, a_2, and a_2^+ can be combined, e.g., to

$$a = \frac{1}{\sqrt{2}}(a_1 + ia_2), \quad b^+ = \frac{1}{\sqrt{2}}(a_1^+ + ia_2^+) \neq a^+ \tag{46}$$

Vice versa, the two real fields can be recovered from the complex field. One should not consider the real particles as bound states of the charged states, since no interaction is present. The process described above is of a purely formal nature.

Dirac particles can be constructed in an analogous way from two neutral Majorana fields, as we shall see later.

3.2. Weyl Equations

We now turn to wave equations for two-component wave wave functions, and derive the Weyl equations for massless fermions. A linear first-order differential operator acting on a spinor is given by

$$\hat{\sigma} = \bar{\sigma}_\mu \partial^\mu = \begin{pmatrix} \partial^0 + \partial^3 & \partial^1 - i\partial^2 \\ \partial^1 + i\partial^2 & \partial^0 - \partial^3 \end{pmatrix} \tag{47}$$

If we apply a Lorentz transformation to this object, then $\hat{\sigma}$ obviously transforms according to

$$\hat{\sigma}' = \bar{\sigma}_\mu \partial'^\mu = \Lambda^\mu{}_\nu \bar{\sigma}_\mu \partial^\nu = S\hat{\sigma}S^+ \tag{48}$$

according to the trick given by Equation (14). If we apply $\hat{\sigma}$ to a (column) spinor

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{49}$$

we obtain the wave equation

$$\hat{\sigma}\psi = (\sigma_0 \partial_0 - \vec{\sigma}\vec{\nabla})\psi = 0 \tag{50}$$

This is one of the famous Weyl equations [2,3]. However, we have to check whether the differential equation also holds in all systems of inertia. This is true if the spinor ψ transform according to

$$\psi(x) \to \psi'(x') = \epsilon S^* \epsilon^{-1} \psi(\Lambda^{-1}x') \tag{51}$$

i.e., according to a representation which is equivalent to the complex conjugate representation constructed in the previous section. Since all special linear matrices fulfill $S^{\mathrm{T}}\epsilon S = \epsilon$, we also have $(S^{\mathrm{T}})^{-1} = \epsilon S \epsilon^{-1}$ and $(S^+)^{-1} = \epsilon S^* \epsilon^{-1}$, and consequently

$$\hat{\sigma}'\psi' = S\hat{\sigma}S^+(\epsilon S^* \epsilon^{-1}\psi) = S\hat{\sigma}S^+(S^{-1})^+\psi = S\hat{\sigma}\psi = 0 \tag{52}$$

since $\hat{\sigma}\psi = 0$. A similar equation can be obtained if we use the differential operator

$$\check{\sigma} = \epsilon\hat{\sigma}\epsilon^{-1} = (\sigma_0 + \vec{\sigma}^{\mathrm{T}}\vec{\nabla}) = \sigma_\mu^* \partial^\mu \tag{53}$$

and a (row) spinor

$$\tilde{\psi}(x)^{\mathrm{T}} = \big(\tilde{\psi}_1(x) \ \tilde{\psi}_2(x)\big) \tag{54}$$

A wave equation is given by

$$\tilde{\psi}(x)^{\mathrm{T}}\check{\sigma} = 0 \tag{55}$$

equivalent to the transposed equation

$$\check{\sigma}^{\mathrm{T}}\tilde{\psi}(x) = (\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\tilde{\psi}(x) = 0 \tag{56}$$

which is just the Weyl equation with differing sign compared to Equation (50). The spinor transforms according to

$$\tilde{\psi}(x) \rightarrow S\tilde{\psi}(\Lambda^{-1}x')$$

$$\tilde{\psi}(x)^{\mathrm{T}} \rightarrow \tilde{\psi}(\Lambda^{-1}x')^{\mathrm{T}}S^{\mathrm{T}} \tag{57}$$

such that ($\check{\sigma} \rightarrow \epsilon S\hat{\sigma}S^{+}\epsilon^{-1}$, $S^{\mathrm{T}}\epsilon S = \epsilon$)

$$\tilde{\psi}^{\mathrm{T}}S^{\mathrm{T}}\epsilon S\hat{\sigma}S^{+}\epsilon^{-1} = \tilde{\psi}^{\mathrm{T}}\epsilon\hat{\sigma}(\epsilon^{-1}\epsilon)S^{+}\epsilon^{-1} = \tilde{\psi}^{\mathrm{T}}\check{\sigma}\cdot\epsilon S^{+}\epsilon^{-1} = \tilde{\psi}^{\mathrm{T}}\check{\sigma}\cdot(S^{*})^{-1} = 0 \tag{58}$$

Originally, the Weyl equations were considered unphysical since they are not invariant under space reflection. Considering the parity transformation

$$\Lambda_P : (x^0, \vec{x}) \rightarrow (x^0, -\vec{x}), \quad \psi'(x) = \psi(x^0, -\vec{x}) \tag{59}$$

the transformed version of Equation (50) which reads $(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\psi(x) = 0$ would be equivalent if there existed a linear isomorphic transformation S_P of the spinor with

$$S_P(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\psi'(x) = (\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})S_P\psi'(x) \tag{60}$$

for all solutions $\psi(x)$ of the Weyl equation, which implies $\sigma_k S_P = -S_P\sigma_k$, for $k = 1, 2, 3$. But this is only possible for $S_P = 0$

A further apparent defect is the absence of a mass term $\sim m\psi$ or $\sim m\tilde{\psi}$ in the Weyl equations, since these terms behave differently under a Lorentz transformation than the related differential operator part $\hat{\sigma}\psi$ or $\check{\sigma}\tilde{\psi}$. However, this statement is a bit overhasty, as we shall see in the following section.

3.3. Two-Component Majorana Fields

3.3.1. Majorana Wave Equations

We reach now our goal and construct the wave equations for free Majorana fields. We add a mass term to the Weyl equation given by Equation (50) or Equation (52) by observing that the complex conjugate spinor $\psi^{*}(x)$ transforms according to

$$\psi^{*}(x) \rightarrow \epsilon S\epsilon^{-1}\psi^{*}(\Lambda^{-1}x') \tag{61}$$

and therefore the *Majorana mass term* $m\epsilon^{-1}\psi^{*}(x)$ transforms like the differential operator part of the Weyl equation

$$m\epsilon^{-1}\psi^{*}(x) \rightarrow m\epsilon^{-1}\epsilon S\epsilon^{-1}\psi^{*}(\Lambda^{-1}x') = S(m\epsilon^{-1}\psi^{*}(\Lambda^{-1}x')) \tag{62}$$

such that the equation

$$\hat{\sigma}\psi(x) = \pm im\epsilon^{-1}\psi^*(x) \tag{63}$$

or

$$(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) \mp im\epsilon^{-1}\psi^*(x) = 0 \tag{64}$$

is indeed Lorentz invariant. The imaginary unit in front of the mass term has been chosen in order to respect manifestly the *CP invariance* of the theory for the moment, however, more general considerations will be presented below. Note that the sign of the mass term in the present framework is a matter of taste, since from a solution $\psi(x)$ of the wave equation one obtains directly $i\psi(x)$ as a solution of the analogous equation with flipped mass.

The same trick works for the alternative Weyl equation Equation (55). There we have

$$\tilde{\psi}^*(x) \to S^*\tilde{\psi}(\Lambda^{-1}x') \tag{65}$$

and ($\epsilon S^+\epsilon^{-1} = (S^*)^{-1}$)

$$m\tilde{\psi}^{*\mathrm{T}}\epsilon^{-1} \to m\tilde{\psi}^{*\mathrm{T}}S^{*\mathrm{T}}\epsilon^{-1} = m\tilde{\psi}^{*\mathrm{T}}\epsilon^{-1}\epsilon S^+\epsilon^{-1} = m\tilde{\psi}^{*\mathrm{T}}\epsilon^{-1}\cdot(S^*)^{-1} \tag{66}$$

leading to the alternative equation

$$(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\tilde{\psi}(x) \pm im\epsilon^{-1}\tilde{\psi}^*(x) = 0 \tag{67}$$

For both equations, we choose a definite mass sign convention and obtain the left-chiral and right-chiral *two-component Majorana equations*

$$i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) + m\epsilon^{-1}\psi(x)^* = 0 \tag{68}$$

$$i(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\tilde{\psi}(x) - m\epsilon^{-1}\tilde{\psi}(x)^* = 0 \tag{69}$$

which can also be written

$$i\bar{\sigma}^\mu\partial_\mu\psi(x) - m(i\sigma_2)\psi^*(x) = 0 \tag{70}$$

$$i\sigma^\mu\partial_\mu\tilde{\psi}(x) + m(i\sigma_2)\tilde{\psi}^*(x) = 0 \tag{71}$$

Equation (68) may be expressed in matrix form ($x, y, z \sim 1, 2, 3$)

$$i\begin{pmatrix} \partial_0 - \partial_z & -\partial_x + i\partial_y + imK \\ -\partial_x - i\partial_y - imK & \partial_0 + \partial_z \end{pmatrix}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = 0 \tag{72}$$

where the operator K denotes the complex conjugation. Multiplying this equation by

$$-i\begin{pmatrix} \partial_0 + \partial_z & \partial_x - i\partial_y + imK \\ \partial_x + i\partial_y - imK & \partial_0 - \partial_z \end{pmatrix} \tag{73}$$

from the left, one obtains

$$\begin{pmatrix} \Box + m^2 & 0 \\ 0 & \Box + m^2 \end{pmatrix}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = 0 \tag{74}$$

i.e., the components of the Majorana spinor ψ and in an analogous way the components of $\tilde{\psi}$ obey the Klein-Gordon equation, and it is a straightforward task to construct plane wave solutions of the Majorana equation.

A first important remark should be made concerning the existence of two Majorana equations. As we have seen above, the existence of two non-equivalent equations Equation (68) and Equation (69) is related to the fact that a two-dimensional complex spinor can transform in two different ways under Lorentz transformations. This fundamental property of a spinor, whether it transforms according to the representation $S(\Lambda)$ or $S(\Lambda)^*$, is called *chirality*. This phenomenon also exists at a simpler level for the group $U(1) = \{z \in \mathbb{C} \,|\, zz^* = 1\}$. There are two true representations given by $\exp(i\alpha) = z \mapsto z$ and $\exp(i\alpha) \mapsto \exp(-i\alpha)$. Spinor fields are not physical observables, however, their chirality is important when one aims at the construction of observables using spinorial entities.

A further critical remarks should be made about the charge conjugation properties of the Majorana fields constructed above. Starting from

$$(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) - im\epsilon^{-1}\psi^*(x) = 0 \tag{75}$$

inserting $\epsilon^{-1}\epsilon = \sigma_0$ in front of the spinor and multiplying the equation from the left by ϵ, we obtain

$$\epsilon(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\epsilon^{-1}\epsilon\psi(x) - im\epsilon^{-1}\epsilon\psi^*(x) = (\sigma_0\partial_0 + \vec{\sigma}^*\vec{\nabla})\epsilon\psi(x) - im\epsilon^{-1}\epsilon\psi^*(x) = 0 \tag{76}$$

or, equivalently, the complex conjugate equation

$$(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\epsilon\psi^*(x) + im\epsilon^{-1}\epsilon\psi(x) = 0 \tag{77}$$

The new field $\epsilon\Psi^*$ does not fulfill the original Majorana equation, however, charge conjugation acting on wave functions is by definition an antilinear operator which does not alter the direction of the momentum and the spin of a particle. Since there also exists no linear transformation of the field which would remedy this situation, we have to accept that charge conjugation defined here by $\psi \to \epsilon\psi^*$ transforms the field out of its equivalence class.

Although the signs of the spatial derivatives and the mass term differ from the original equation, we may perform a parity transformation including a multiplication of the spinor $\epsilon\psi^*(x^0, -\vec{x})$ by the *CP eigenphase* i. Of course, a CP eigenphase with opposite sign would also suffice. Then we have

$$(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})(i\epsilon\psi^*(x^0, -\vec{x})) - im\epsilon^{-1}(i\epsilon\psi^*(x^0, -\vec{x}))^* = 0 \tag{78}$$

Therefore, the *CP conjugate* spinor $CP[\psi(x^0, \vec{x})] = \pm i\epsilon\psi^*(x^0, -\vec{x})$ is again a solution of the original Majorana equation, and a CP transformed particle can be converted by a Lorentz transformation into its original state. This arguments no longer holds in the massless Weyl case.

Additionally, one should keep in mind that CP conjugation is not an exact symmetry in nature. This implies that the story becomes even more involved when interactions start to play a role. Therefore, the actual *definition of a Majorana particle* is given by the demand that a Majorana particle be an "eigenstate" of the CPT transformation, where T denotes the time reversal operator introduced below. Note, however, that the notion "CPT eigenstate", which is widely found in the literature, should not be misunderstood in the narrow sense that a particle state remains invariant under CPT. e.g., CPT changes the spin of a particle. However, a CPT transformed particle state is again a physical state of the same type of particle. Furthermore, CPT expresses a fundamental symmetry of every local quantum field theory. We shall see below that the definition of a Majorana particle formulated above is contained in a natural manner in the two-component Majorana formalism.

We finally mention that a scalar complex field has *two* two charge degrees of freedom, whereas a Majorana field has *two* polarization degrees of freedom. This observation is a starting point for supersymmetric theories, where fermions and bosons get closely related.

3.3.2. CPT

Based on Equation (75), we find that the field $\psi(-x^0, \vec{x})$ fulfills the equation

$$(-\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(-x^0, \vec{x}) - im\epsilon^{-1}\psi^*(-x^0, \vec{x}) = 0 \tag{79}$$

Using analogous tricks as above, this equation can be converted into

$$(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})(\epsilon\psi^*(-x^0, \vec{x})) - im\epsilon^{-1}(\epsilon\psi^*(-x^0, \vec{x}))^* = 0 \tag{80}$$

Defining the time reversal transformation according to $T[\psi(x^0, \vec{x})] = \epsilon\psi^*(-x^0, \vec{x})$, we obtain

$$CPT[\psi(x^0, \vec{x})] = \pm i\epsilon(\epsilon\psi^*(-x^0, -\vec{x}))^* = \pm i\epsilon^2\psi(-x^0, -\vec{x}) = \mp i\psi(-x^0, -\vec{x}) \tag{81}$$

and the CPT transformed Majorana field again fulfills the original wave equation Equation (68) or Equation (75).

There is a crucial point in this discussion above. While the charge conjugation C and CP are related to symmetries which are violated by some interactions, in local relativistic quantum field theory all particle interactions respect the CPT symmetry [4]. Also, it can be strictly shown that a violation of the CPT symmetry implies a violation of Lorentz invariance[5]. This means that to every particle process in nature there is an associated CPT conjugate process with properties that can be inferred exactly from the original system. Thus, we emphasize again that the basic property which makes a fermion a Majorana particle is the fact that a particle state with definite four-momentum and spin can be transformed by a CPT transformation and a subsequent (space-time) Poincaré transformation into itself. In a world, where neither C, P, T, CP, CT, nor PT are conserved, this is the only way of expressing the fact that a particle is its own antiparticle. A physical Majorana neutrino, subject to maximally C-violating weak interactions, cannot be an eigenstate of C. It may be an "approximate eigenstate" of CP.

3.4. Plane Wave Solutions

In the previous section, we tacitly chose phase conventions and the mass term in the Majorana equations such that the resulting fields displayed most simple transformations properties under CP conjugation. We consider now the most general mass term for one type of spin-$1/2$ Majorana particles and calculate explicit solutions of the non-interacting left-chiral Majorana equation

$$i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) = \eta m\epsilon\psi(x)^* \tag{82}$$

where $\eta = e^{i\delta}$ is a phase $|\eta| = 1$. In the non-interacting case, the complex mass term ηm has no physical impact and could be rendered real in Equation (82) by globally gauging the wave function $\psi(x) \to \psi'(x) = \psi(x)e^{-i\delta/2}$, such that

$$i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) - \eta m\epsilon\psi(x)^* = i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})e^{i\frac{\delta}{2}}\psi'(x) - e^{i\delta}m\epsilon e^{-i\frac{\delta}{2}}\psi'(x)^*$$

$$= e^{i\frac{\delta}{2}}[i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi'(x) - m\epsilon\psi'(x)^*] = 0 \tag{83}$$

When interactions are involved, the discussion of Majorana phases becomes important [6].

Equation (82) follows from the Lagrangian ($\psi^+ = \psi^{\mathrm{T}*}$)

$$\mathcal{L} = \psi^+ i\bar{\sigma}^\mu\partial_\mu\psi - \eta\frac{m}{2}\psi^+(i\sigma_2)\psi^* + \eta^*\frac{m}{2}\psi^{\mathrm{T}}(i\sigma_2)\psi \tag{84}$$

which contains so-called Majorana mass terms. Note that the mass terms above equals zero, if the components of the Majorana field are assumed to be ordinary numbers. However, in the present fermionic case, we adopt the rule that the classical fermionic fields *anticommute*.

For particles at rest, Equation (82) can be written as

$$i\dot{\psi}_1 = \eta m\psi_2^*, \quad i\dot{\psi}_2 = -\eta m\psi_1^* \tag{85}$$

Differentiating the equation on the left and using the equation on the right $\dot{\psi}_2^* = -i\eta^* m\psi_1$, one obtains

$$\ddot{\psi}_1 = -i\eta m\dot{\psi}_2^* = -|\eta|^2m^2\psi_1 = -m^2\psi_1 \tag{86}$$

ψ_1 is therefore a linear combination of e^{-imx^0} and e^{+imx^0}, and solutions describing a particle with spin parallel or antiparallel to the 3- or z-direction are given by ($\psi_2 = -\frac{i}{m}\eta\dot{\psi}_1^*$)

$$\psi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-imx^0} + \eta \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{+imx^0} \tag{87}$$

$$\psi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-imx^0} + \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{+imx^0} \tag{88}$$

Plane wave solutions always contain a positive and negative energy (or, rather, *frequency*) part, which combine to describe an uncharged particle. As a hand-waving argument, one may argue that the "negative energy" part in the solutions displayed above corresponds to a particle hole and must therefore be equipped with an opposing spinor.

The phase η has no physical meaning as long as weak interactions or mixings of different particles are absent, field operators related to the negative energy part $\sim e^{+imx^0}$ of the wave functions above can then be redefined such that the phase disappears. Otherwise, this phase will appear as one of the so-called Majorana phases, potentially in conjunction with other phases related to charged Dirac fermions.

Solutions for moving particles can be generated by boosting the solutions given above according to the transformation law Equation (51). The corresponding matrices are given here without derivation

$$S = \sqrt{\frac{E+m}{2m}}\left(1 + \frac{\vec{\sigma}\vec{k}}{E+m}\right) \tag{89}$$

$$\epsilon S^* \epsilon^{-1} = \sqrt{\frac{E+m}{2m}}\left(1 - \frac{\vec{\sigma}\vec{k}}{E+m}\right) \tag{90}$$

A full derivation of the results in Equations (89,90) can be found, e.g., in [7]. A Majorana particle with momentum $\vec{k}$ and spin in z-direction is described, e.g., by

$$\psi_{+\frac{1}{2}}(\vec{k}, x) = \epsilon S^* \epsilon^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} + \eta \epsilon S^* \epsilon^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{+ikx}, \quad kx = k_\mu x^\mu \tag{91}$$

Helicity

In the following, we focus on an alternative representation of two-component Majorana spinors. Given a momentum $\vec{k}$ in a direction specified by the polar coordinates θ and ϕ

$$\vec{k} = |\vec{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{92}$$

the Pauli spin states [8]

$$h_+ = \begin{pmatrix} +\cos(\theta/2)e^{-i\phi/2} \\ +\sin(\theta/2)e^{+i\phi/2} \end{pmatrix}, \quad h_- = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ +\cos(\theta/2)e^{+i\phi/2} \end{pmatrix} \tag{93}$$

are helicity eigenstates

$$\vec{\sigma}\vec{k}h_\pm = \pm|\vec{k}||h_\pm \tag{94}$$

where

$$\vec{\sigma}\vec{k} = |\vec{k}| \begin{pmatrix} \cos\theta & \sin\theta e^{-\phi} \\ \sin\theta e^{+i\phi} & -\cos\theta \end{pmatrix} \tag{95}$$

The Pauli spinors are obviously related by ($\epsilon = i\sigma_2$)

$$h_+^c = \epsilon h_+^* = -h_-, \quad h_-^c = \epsilon h_-^* = +h_+ \tag{96}$$

In order to solve the Majorana equation Equation (68)

$$i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\psi(x) = m\epsilon\psi^*(x) \tag{97}$$

we make the Ansatz

$$\psi(x) = [\alpha_+ h_+ e^{-ikx} + \tilde{\alpha}_+ h_- e^{+ikx}] + [\alpha_- h_- e^{-ikx} + \tilde{\alpha}_- h_+ e^{+ikx}] \tag{98}$$

Note that a negative frequency solution $\sim e^{+ikx}$ with a negative helicity spinor h_- corresponds to a particle with positive helicity, which clarifies the meaning of the indices of the coefficients above. Inserting the Ansatz Equation (98) in Equation (97) leads to the relations (with $E = k^0 = (\vec{k}^2 + m^2)^{1/2}$, and $K = |\vec{k}|$ in this section)

$$+\alpha_+(E+K) = +\tilde{\alpha}_+^* m, \quad \tilde{\alpha}_+(-E+K) = -\alpha_+^* m \tag{99}$$

$$+\alpha_-(E-K) = -\tilde{\alpha}_-^* m, \quad -\tilde{\alpha}_-(E+K) = +\alpha_-^* m, \tag{100}$$

where $\vec{\sigma}(-i\vec{\nabla})e^{\mp ikx} = \pm\vec{\sigma}\vec{k}e^{\mp ikx}$ was also used. Up to normalization factors, these conditions imply

$$\alpha_+ = +\sqrt{E-K}e^{+i\delta}, \quad \tilde{\alpha}_+ = +\sqrt{E+K}e^{-i\delta} \tag{101}$$

$$\alpha_- = +\sqrt{E+K}e^{+i\delta'}, \quad \tilde{\alpha}_- = -\sqrt{E-K}e^{-i\delta'} \tag{102}$$

with $\delta, \delta' \in \mathbb{R}$.

Hence, a general Fourier representation of a free Majorana field can be written as

$$\psi(x) = \int \frac{d^3k}{2k^0(2\pi)^3}\Big\{[+\sqrt{E-K}\alpha_+(\vec{k})h_+e^{-ikx} + \sqrt{E+K}\alpha_+^*(\vec{k})h_-e^{+ikx}]+$$

$$[+\sqrt{E+K}\alpha_-(\vec{k})h_-e^{-ikx} - \sqrt{E-K}\alpha_-^*(\vec{k})h_+e^{+ikx}]\Big\} \tag{103}$$

After second quantization, α_+, α_-, α_+^*, and α_-^* become the creation and annihilation operators for the Majorana particle in the $\pm-$helicity states.

Note that the square root terms above reappear in the matrix elements describing, e.g., the beta decay of the neutron. In this process, the right-handed antineutrino is produced predominantly along with the electron, the amplitude being of the order $\sqrt{E_\nu + K_\nu} \simeq \sqrt{2E_\nu}$. The left-handed neutrino has a much smaller amplitude $\sqrt{E_\nu - K_\nu} \simeq m_\nu/\sqrt{2E_\nu}$. In the massless case, only positive-helicity particles would be produced. Similarly, in the high-energy limit only the terms containing α_+^* and α_- in Equation (103) survive, which correspond to positive-helicity (right-handed) negative energy particles and left-handed positive energy particles, respectively. These two possibilities correspond to two helicity states of a neutral Majorana neutrino, or to the right-handed antineutrino and left-handed neutrino, respectively. Both cases can only be distinguished if interactions are present.

One should note that helicity and chirality of particles have to be distinguished. The helicity of a particle is right-handed if the direction of its spin is the same as the direction of its momentum. It is left-handed if the directions of spin and motion are opposite. The chirality of a particle is more abstract. It is determined by whether the particle transforms in a "right" or "left-handed" representation of the Poincaré group, or, to be more precise, whether its spinor transforms according to a given two-dimensional representation of the Lorentz group or the corresponding complex conjugate and, therefore, non-equivalent representation. The chirality of a particle refers to the mathematical formalism used for the description of the particle and has no physical meaning as an observable. For massive particles, helicity is not conserved, since the direction of the momentum depends on the frame of reference of the observer. In the case of massless particles, helicity is a conserved quantity and can be related directly to the chirality of the particle.

4. The Dirac Equation

4.1. Representations

The Dirac equation is the well-known partial differential equation of first order, which has been used very successfully for the description, e.g., of electrons and positrons since its formulation by Paul Dirac in 1928 [9]. The equation for the 4-component spinor Ψ describing interaction-free Dirac particles reads ($\hbar = c = 1$)

$$\{i\gamma^\mu \partial_\mu - m\}\Psi(x) \tag{104}$$

where $x = (x^0, x^1, x^2, x^3) = (x_0, -x_1, -x_2, -x_3)$ are the space-time coordinates and the γ^μ the famous Dirac matrices $\gamma_{Dirac} =: \gamma^\mu$, satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{105}$$

There is a standard choice for the Dirac matrices in the literature, given by ($i = 1, 2, 3$)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{106}$$

where the 1 stands for the 2×2 identity matrix.

An important result from the theory of Clifford algebras states that every set of matrices $\tilde{\gamma}^\mu$ satisfying the anti-commutation relations Equation (105) is equivalent to the standard choice defined above in the sense that

$$\tilde{\gamma}^\mu = U\gamma^\mu U^{-1} \tag{107}$$

for some suitable invertible matrix U. This is a nice feature of the gamma matrices, since it makes sure that two physics communities living in different solar systems can easily compare their calculations by a simple transformation. In this sense, the Dirac equation is unique.

For the so-called chiral (or *Weyl*) representation, one has

$$\gamma^0_{ch(iral)} = \gamma^0_{ch} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k_{ch} = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \gamma^5_{ch} = \gamma^{ch}_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{108}$$

with

$$\gamma^\mu_{chiral} = U\gamma^\mu_{Dirac}U^{-1}, \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U^{-1} = U^+ \tag{109}$$

where the $^+$ denotes the Hermitian conjugate. The chiral Dirac matrices can also be written as

$$\gamma^\mu_{ch} = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \tag{110}$$

The charge conjugation of a Dirac spinor is given in the chiral representation by $\psi^c = \eta_c \tilde{C}\bar{\psi}^{\mathrm{T}}$, $\bar{\psi}^{\mathrm{T}} = \gamma^{0\mathrm{T}}_{ch}\psi^*$, with η_c an arbitrary unobservable phase, generally taken a being equal to unity and the matrix $\tilde{C}$ is given by

$$\tilde{C} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \tag{111}$$

or

$$C[\psi] = \psi^c = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\psi^* \tag{112}$$

The definition Equation (112) of charge conjugation is equally valid in the standard representation.

The Majorana representation of Dirac matrices is given by

$$i\gamma^0_M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i\gamma^1_M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$i\gamma^2_M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i\gamma^3_M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{113}$$

where

$$\gamma^\mu_M = \gamma^\mu_{Majorana} = V\gamma^\mu_{Dirac}V^+, \quad V = V^{-1} = V^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{pmatrix} \tag{114}$$

Note that the Majorana-Dirac matrices are purely imaginary, such that the Dirac equation contains only real coefficients. Therefore, it is possible to enforce that the solutions of the equation are purely real. Such solutions correspond to a Majorana field, since the charge conjugation operator in the Majorana representation is given simply by the complex conjugation of all spinor components. This basic definition of charge conjugation, which is motivated by the fact that both a given solution and its complex conjugate fulfill the same Dirac equation with real coefficients, can still be modified by a phase.

The definition of charge conjugation by Equation (112) conforms to the charge conjugation defined in a rather heuristic manner in Section 3.3 for two-component spinors in an obvious way. This correspondence can also be achieved for the parity transformation. Defining the parity transformed Dirac spinor in an arbitrary representation by

$$\Psi^p(x'^0, \vec{x}') = \gamma^0\Psi(x^0, \vec{x}), \quad x'^0 = x^0, \quad \vec{x}' = -\vec{x} \tag{115}$$

one finds that the new spinor satisfies the Dirac equation as well, since

$$\gamma^\mu\partial'_\mu\Psi^p(x') = (\gamma^0\partial'_0 - \gamma^k\partial'_k)\gamma^0\Psi(x^0, \vec{x}) = (\gamma^0\partial_0 + \gamma^k\partial_k)\gamma^0\Psi(x^0, \vec{x}) = \gamma^\mu\partial_\mu\Psi(x) \tag{116}$$

where we used that $\gamma^k\gamma^0 = -\gamma^0\gamma^k$ for $k = 1, 2, 3$.

For a detailed discussion of the time reversal operator we refer to the literature. Apart from an admissible phase factor, it is given in the Dirac representation for a four-component spinor by

$$\Psi(x^0, \vec{x}) \rightarrow T[\Psi(x^0, \vec{x})] = i\gamma^3\gamma^5\Psi(-x^0, \vec{x})^* = -\gamma^2\gamma^5\gamma^0\Psi(-x^0, \vec{x})^* \tag{117}$$

Note that both the replacement $x^0 \rightarrow -x^0$ and the complex conjugation in Equation (117) flip the positive and negative frequency part of a plane wave solution of the Dirac equation. As a consequence, in the case of a C-symmetric theory based on, e.g., the Dirac equation describing non-interacting charged fermions like electrons and positrons, the charge of the particles is invariant under a time reversal transformation. However, spin and momentum change sign. The reader should also keep in mind that the definitions used in the present work for the charge, parity and time reversal transformations change their formal structure when they are considered in the framework of (second quantized) quantum field theories. It should also be pointed out that the definitions presented in this section are constructed in a manner that is compatible with the corresponding definitions originally given for two-component spinors.

4.2. Chiral Decomposition of the Dirac and Majorana Equation

Decomposing the Dirac spinor in the chiral representation into two two-component spinors

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} \tag{118}$$

the Dirac equation becomes

$$i \begin{pmatrix} 0 & \sigma_0 \partial_0 - \vec{\sigma}\vec{\nabla} \\ \sigma_0 \partial_0 + \vec{\sigma}\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = 0 \tag{119}$$

or

$$(\sigma_0 \partial_0 - \vec{\sigma}\vec{\nabla})\Psi_L + im\Psi_R = 0 \tag{120}$$

$$(\sigma_0 \partial_0 + \vec{\sigma}\vec{\nabla})\Psi_R + im\Psi_L = 0 \tag{121}$$

The Dirac equation describes two fields with different chirality, which are coupled by mass terms. In the Standard Model, only Ψ_L-fields take part in the electroweak interaction. Without mass term in the Dirac equation, Ψ_L would describe a Dirac neutrino field in the (obsolete version of the) Standard Model, where neutrinos are assumed to be massless.

Defining the four-component Majorana spinor by

$$\Psi_M = \begin{pmatrix} \tilde{\psi} \\ \psi \end{pmatrix} \tag{122}$$

the two-component Majorana equations Equation (68) and Equation (69) can be cast into the four-component *Majorana equation*

$$i\gamma^\mu \partial_\mu \Psi_M - m\Psi_M^c = 0 \tag{123}$$

This way we absorb both the left- and the right-chiral two-component Majorana fields in one four-spinor, describing indeed 4 degrees of freedom. This way one may construct a new type of Majorana field which describes indeed charge neutral particles, a strategy, which will be used below. It is, however, always possible to extract left- and right-chiral fields from the Majorana spinor above by using the corresponding projection operators. Actually, the mass term in the Majorana equation can be generalized such that it describes neutral particles with two different masses m_1 and m_2, as discussed in Section 6.

4.3. CP Violating Phase

The observation that there are different types of spin-$1/2$ particles mights raise the question whether the Dirac equation is really unique up to different choices of the Dirac matrices. In fact, it is not. The modified Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m e^{i\Theta' \gamma^5} \psi(x) = 0 \tag{124}$$

is not in conflict with relativistic covariance. However, a non-trivial so-called chiral phase Θ' may lead to CP violating effects when interactions are present, due to its properties under parity transformation or charge conjugation which involves complex conjugation. Note that the phase discussed here is in close analogy to the phase which appeared already above in Equation (82).

It is straightforward to verify that the mass of the mass eigenstates following from Equation (124) is given by the parameter m, also in close analogy to the discussion presented in Section 3.4, where a phase appeared in a mass term. The chiral mass term $\tilde{m} = m e^{i\Theta'\gamma^5}$ should not be confused with the effective complex mass of a decaying particle $M + i\Gamma/2$. The γ^5-part of the chiral mass term appearing in the Hamiltonian corresponding to Equation (124) is indeed Hermitian and respects unitary time evolution of the Dirac wave function. Defining a chirally transformed wave function

$$\psi'(x) = e^{+i\frac{\Theta'}{2}\gamma^5} \psi(x) \tag{125}$$

one first observes that

$$e^{+i\frac{\Theta'}{2}\gamma^5} \gamma^\mu = \gamma^\mu e^{-i\frac{\Theta'}{2}\gamma^5} \tag{126}$$

since γ^5 anticommutes with the Dirac matrices: $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ for $\mu = 0, ..., 3$. Consequently, $\psi'(x)$ fulfills the ordinary Dirac equation with a real mass, since

$$i\gamma^\mu \partial_\mu \psi'(x) - m\psi'(x) = i\gamma^\mu \partial_\mu e^{+i\frac{\Theta'}{2}\gamma^5} \psi(x) - m e^{-i\frac{\Theta'}{2}\gamma^5} e^{+i\Theta'\gamma^5} \psi(x)$$

$$= e^{-i\frac{\Theta'}{2}\gamma^5} \left[i\gamma^\mu \partial_\mu \psi(x) - m e^{+i\Theta'\gamma^5} \psi(x) \right] = 0 \tag{127}$$

and $\psi(x)$ fulfills Equation (124). From this observation, one should not conclude that the chiral phase can be simply rotated away in theories with interactions like the Standard Model. There, the chiral phase of the quarks is linked with the so-called Θ-Term, which might be present in the gluonic part of the QCD Lagrangian, via the so-called triangle anomaly [10–12]. Furthermore, chiral rotations are not a symmetry of the full theory. The combined Θ- and Θ' terms are related to a potential electric dipole moment of the neutron in a highly non-trivial way [13].

4.4. Four-Component Majorana Fields

In the following, we will rename the field ψ obeying Equation (68) by χ_L and $\tilde{\psi}$ obeying Equation (69) by χ_R, in order to stress their transformation properties under $SL(2,\mathbb{C})$. It is a bit unfortunate that the symbols L and R, which denote the chirality of the fields, might suggest a connection to the helicity (or handedness) of particles, which is not a conserved quantity in the massive case.

We construct additionally a four-component spinor

$$\nu_{_L} = \begin{pmatrix} 0 \\ \chi_{_L} \end{pmatrix} \tag{128}$$

such that we can write Equation (68) in the *chiral* representation as follows

$$i \begin{pmatrix} 0 & \partial_0 - \vec{\sigma}\vec{\nabla} \\ \partial_0 + \vec{\sigma}\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi_{_L} \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \epsilon\chi^*_{_L} \\ 0 \end{pmatrix} = 0 \tag{129}$$

or

$$i\gamma^\mu_{ch}\partial_\mu\nu_{_L} - m\nu^c_{_L} = i\gamma^\mu_{ch}\partial_\mu\nu_{_L} - m(\nu_{_L})^c = 0, \quad \frac{1}{2}(1-\gamma^5_{ch})\nu_{_L} = \nu_{_L} \tag{130}$$

where the symbol c denotes charge conjugation as it is defined in the chiral representation. We proceed one step further and define a *neutral* four-component field

$$\nu^M_{_1} = \nu_{_L} + \nu^c_{_L} = \begin{pmatrix} \epsilon\chi^*_{_L} \\ \chi_{_L} \end{pmatrix}, \quad \nu^M_{_1} = \nu^{M\,c}_{_1} \tag{131}$$

We have seen in the section above that $\epsilon\psi^*$ fulfills Equation (77), hence we have

$$i\gamma^\mu_{ch}\partial_\mu\nu^M_{_1} - m\nu^M_{_1} = 0, \quad \nu^M_{_1} = \nu^{M\,c}_{_1} \tag{132}$$

This four-component Majorana field is often used in supersymmetric theories, and the same construction naturally works for $\nu_2 = \chi_{_R} + \chi^c_{_R}$. A formalism using ν_1's only must be equivalent to one using ν_2's only. The fields ν_1 and ν_2 are charge conjugation invariant by construction, and it is common practice to denote this type of fields as the actual Majorana fields. The free chiral Majorana fields can readily be recovered by from the charge self-conjugate Majorana fields by projections $\nu_{R,L} = \frac{1}{2}(1 \pm \gamma^5_{ch})\nu_{1,2}$. One notational advantage of the four-component formalism relies on the fact that the description of interactions of Majorana neutrinos with Dirac particles naturally involves expressions based on Dirac four-component spinors.

Equation (130) is equivalent to

$$i\gamma^\mu_{ch}\partial_\mu\nu^c_{_L} - m\nu_{_L} = 0 \tag{133}$$

hence we have

$$i\gamma^\mu_{ch}\partial_\mu(\nu_{_L} - \nu^c_{_L}) - m(\nu^c_{_L} - \nu_{_L}) = 0 \tag{134}$$

or

$$i\gamma^\mu_{ch}\partial_\mu[i(\nu_{_L} - \nu^c_{_L})] - m[i(\nu_{_L} - \nu^c_{_L})]^c = i\gamma^\mu_{ch}\partial_\mu[i(\nu_{_L} - \nu^c_{_L})] - m[i(\nu_{_L} - \nu^c_{_L})] = 0 \tag{135}$$

The same construction works for $\nu_{_R}$, and we may also construct the following combinations of Majorana (quantum) fields

$$\psi_1 = \frac{1}{\sqrt{2}}(\nu_{_L} + \nu^c_{_L} + \nu_{_R} + \nu^c_{_R}), \quad \psi_1 = \psi^c_1 \tag{136}$$

$$\psi_2 = \frac{i}{\sqrt{2}}(\nu_{_L} - \nu^c_{_L} + \nu_{_R} - \nu^c_{_R}), \quad \psi_2 = \psi^c_2 \tag{137}$$

We do not discuss normalization factors in the present qualitative discussion on a first-quantized level. Combining the two physically distinct fields given above leads to a Dirac field $\psi_{_D}$

$$\psi_{_D} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \sim \nu_{_L} + \nu_{_R} \tag{138}$$

which is no longer neutral, since $\psi^c_{_D} \neq \psi^c_{_D}$.

Note that each Majorana field $\nu^M_{_1}$ or $\nu^M_{_2}$ from Equation (132) is completely determined by two complex quantities

$$\nu^M_{_1} = \begin{pmatrix} \epsilon\chi^*_L \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi^*_{L_2} \\ -\chi^*_{L_1} \\ \chi_{L_1} \\ \chi_{L_2} \end{pmatrix}, \qquad \nu^M_{_2} = \begin{pmatrix} \chi_R \\ -\epsilon\chi^*_R \end{pmatrix} = \begin{pmatrix} \chi^1_R \\ \chi^2_R \\ -\chi^{2*}_R \\ \chi^{1*}_R \end{pmatrix} \tag{139}$$

whereas a Dirac spinor has 4 complex entries, such that the four-component Majorana fields constructed above have only two physical degrees of freedom instead of four. We will see below that we can represent Majorana spinors in a purely real form, where the two complex spinor components χ_{L_1} and χ_{L_2} correspond to four real numbers.

The fact that it is possible to construct charge conjugation eigenstates in the four-component formalism should be considered as a convenient mathematical trick. We emphasize again that CPT is the fundamental symmetry of local relativistic quantum field theories, whereas C and CP are not. To give a rather intuitive picture of the meaning of the CPT symmetry, we point out that in an even-dimensional Euclidean space, a point reflection of the space can also be obtained from a continuous rotation of the space itself. This is impossible for the space-time reflection PT in Minkowski space. Therefore, we have no reason to assume that for every physical process, there exists a space-time reflected process. However, the requirement of locality *and* relativistic invariance in quantum field theory is strong enough to ensure that an additional transformation C can always be found such that to every physical process, a C-PT mirrored process exists. The C-conjugate state of a physical particle does not necessarily exist, it may be sterile in the sense of non-interacting. The "true" antiparticle is related to its partner via the CPT transformation, and both the particle and the antiparticle have the same mass. All these observations necessitate a generalization of the considerations made so far.

4.5. A Comment on Gauge Invariance

A very important property of a charged Dirac field describing, e.g., electrons and positrons, which is coupled to a vector potential A_μ describing the electromagnetic field, is the gauge invariance of the Dirac equation including the coupling to the gauge field

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - m\Psi = 0 \tag{140}$$

where e is the negative charge of the electron. The gauge transformed fields

$$\Psi'(x) = e^{-ie\eta(x)}\Psi(x) \tag{141}$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\eta(x) \tag{142}$$

still fulfill the equation of motion Equation (140), and also the Maxwell equations with the electric current source term remains invariant.

A glimpse at Equations (68) and (69) clearly shows that a corresponding gauge transformation is impossible for Majorana fields, since the complex conjugate spinor in the mass term acquires the wrong phase such that the gauge transformed spinor no longer fulfills the Majorana equation. Therefore, a Majorana particle is truly neutral. This situation changes abruptly, when the Majorana mass term vanishes.

It is common practice to define charge conjugation for the vector potential by

$$A^c_\mu = -A_\mu \tag{143}$$

since then electric and magnetic field also change sign as one expects when positive and negative charges in a physical system are exchanged. To complicate the situation, one could additionally invoke a gauge transformation of the vector potential which does not change the physically observable electric and magnetic fields. A full discussion of charge conjugation (and other discrete transformations as space reflection and time reversal) is indeed much more involved, since the quantization of gauge potentials is not trivial. An introduction on some aspects of symmetry transformations in quantum field theory is given by [14].

Accepting the simple convention Equation (143), the charge conjugate version of the interacting Dirac equation

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - m\Psi = 0 \tag{144}$$

is

$$[i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - m\Psi]^c = i\gamma^\mu(\partial_\mu - ieA_\mu)\Psi^c - m\Psi^c = 0 \tag{145}$$

It is therefore clearly forbidden that $\Psi = \Psi^c$ for a Dirac field.

5. Real Forms of Majorana Fields

5.1. Generators of the Lorentz Group and the Unimodular Group

A pure Lorentz boost in x^1-direction with velocity $v = \beta c$ is expressed by the matrix

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \gamma^2 - \gamma^2\beta^2 = 1 \tag{146}$$

For $\beta \ll 1$, we can write to first order in β

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1 + \beta K_1 \tag{147}$$

where K_1 is a generator for boosts in x^1-direction, and from the theory of continuous groups it is well-known that the original Lorentz boost can be recovered by exponentiating the generator

$$\exp(\xi K_1) = \begin{pmatrix} +\cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & +\cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \cosh(\xi) = \gamma \tag{148}$$

with an appropriately chosen boost parameter ξ. For boosts in x^2- and x^3-direction, one has

$$K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{149}$$

and the generators for rotations around the x^1-, x^2-, and x^3-axis are given by

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{150}$$

The six generators of the proper Lorentz group given above satisfy the commutation relations

$$[S_l, S_m] = +\varepsilon_{lmn} S_n, \quad [K_l, K_m] = -\varepsilon_{lmn} S_n, \quad [K_l, S_m] = +\varepsilon_{lmn} K_n \tag{151}$$

with $\varepsilon_{123} = 1$, $\varepsilon_{lmn} = -\varepsilon_{mln} = -\varepsilon_{lnm}$ the totally antisymmetric tensor in three dimensions. Note that the generators above get often multiplied with the imaginary unit i in the physics literature in order to get Hermitian matrices.

The Lie algebra $so(1,3)$ of the proper Lorentz group is isomorphic to the Lie algebra $sl(2,\mathbb{C})$, and it is not difficult to find a basis of generators in $sl(2,\mathbb{C})$ which span the real six-dimensional vector space such that an infinitesimal Lorentz transformation

$$1 + \nu_l S_l + \beta_l K_l \tag{152}$$

corresponds one-to-one to an infinitesimal spinor transformation

$$1 + \nu_l \tilde{S}_l + \beta_l \tilde{K}_l \tag{153}$$

in accordance with Equation (14). The generators in $sl(2,\mathbb{C})$ are given by

$$\tilde{S}_k = -i\sigma_k, \quad \tilde{K}_k = \sigma_k \tag{154}$$

and since the Pauli matrices fulfill the anti-commutation relation

$$[\sigma_l, \sigma_m] = i\varepsilon_{lmn}\sigma_n \tag{155}$$

one readily verifies that

$$[\tilde{S}_l, \tilde{S}_m] = +\varepsilon_{lmn}\tilde{S}_n, \quad [\tilde{K}_l, \tilde{K}_m] = -\varepsilon_{lmn}\tilde{S}_n, \quad [\tilde{K}_l, \tilde{S}_m] = +\varepsilon_{lmn}\tilde{K}_n \tag{156}$$

5.2. The Real Spinor Representation of the Lorentz Group in Four Dimensions

The $sl(2,\mathbb{C})$-generators presented above are matrices with complex elements, and it is clearly impossible so satisfy the algebraic relations Equation (156) with matrices containing only real elements. Interestingly, there exists a real representation of the group $SL(2,\mathbb{C})$ in four dimensions. Instead of giving a detailed derivation, we directly list a set of real generators with all demanded properties. We denote the boost generators by K_1^r, K_2^r, K_3^r, which are given by

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{157}$$

respectively, and the rotations are generated by S_1^r, S_2^r, S_3^r, given by

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{158}$$

respectively. An explicit calculation illustrates that the matrices above indeed generate a double-valued spinor representation of the proper Lorentz group, since

$$\exp(2\pi\nu_l S_l^r) = -1 \quad \text{for } \nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{159}$$

i.e., one full physical rotation of a spinor around an arbitrary axis $\vec{\nu}$ changes the sign of the spinor.

Although the *irreducible* real double-valued representation of the proper Lorentz group constructed above acts on the four-dimensional real space like the proper Lorentz group itself, one should keep in mind that it clearly has completely different geometrical properties.

5.3. Real Majorana Spinors

We finally construct real Majorana spinors in a concrete manner, starting from the solutions of the Dirac equation in the standard representation.

Using for the moment the standard representation of Dirac matrices, plane wave solutions describing electrons (particles) with their spin in $z-$direction and momentum $\vec{k}$ are given, up to normalization factors, by

$$u_s(\vec{k},x) = \begin{pmatrix} \xi_s \\ \frac{\vec{\sigma}\vec{k}}{E+m}\xi_s \end{pmatrix} e^{-ikx}, \quad s = \pm\frac{1}{2}, \quad \xi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{160}$$

whereas positrons (antiparticles) with their spin parallel or antiparallel to the $z-$axis and momentum $\vec{k}$ are described by the charge conjugate plane wave spinors ($\epsilon\vec{\sigma}^* = -\vec{\sigma}\epsilon$)

$$v_s(\vec{k},x) = i\gamma^2 u_s(\vec{k},x)^* = u_s(\vec{k},x)^c = \begin{pmatrix} \frac{\vec{\sigma}\vec{k}}{E+m}\epsilon\xi_s \\ \epsilon\xi_s \end{pmatrix} e^{+ikx} \tag{161}$$

where charge conjugation is defined by Equation (112) and $E = k^0 = (\vec{k}^2 + m^2)^{1/2}$ is the (positive) energy of the particles.

If we combine particle and antiparticle solutions corresponding to the same spin and momentum according to

$$w_s(\vec{k},x) = \frac{1}{\sqrt{2}}(u_s(\vec{k},x) + iv_s(\vec{k},x)) \tag{162}$$

we obtain *neutral* charge conjugation eigenstates with eigenvalue $-i$, since

$$w_s(\vec{k},x)^c = \frac{1}{\sqrt{2}}(u_s(\vec{k},x)^c - iv_s(\vec{k},x)^c) = -iw_s(\vec{k},x) \tag{163}$$

Why the $v-$spinor has been equipped with an imaginary prefactor will become clear below. From w_s, we construct new spinors m_s which are solutions of the Dirac equation in the Majorana representation. This is achieved by a linear transformation

$$m_s(\vec{k},x) = Vw_s(\vec{k},x) \tag{164}$$

using the matrix V from Equation (114), which connects the standard and Majorana representation. An explicit calculation yields ($x \sim x^\mu$, $(x^1,x^2,x^3) \sim (x,y,z)$)

$$m_{+\frac{1}{2}}(\vec{k},x) = \begin{pmatrix} +\cos(kx) - \frac{1}{E+m}(k_x\sin(kx) - k_y\cos(kx)) \\ +\frac{k_z}{E+m}\sin(kx) \\ -\frac{k_z}{E+m}\cos(kx) \\ +\sin(kx) - \frac{1}{E+m}(k_x\cos(kx) + k_y\sin(kx)) \end{pmatrix} \tag{165}$$

$$m_{-\frac{1}{2}}(\vec{k},x) = \begin{pmatrix} +\frac{k_z}{E+m}\sin(kx) \\ +\cos(kx) + \frac{1}{E+m}(k_x\sin(kx) + k_y\cos(kx)) \\ -\sin(kx) - \frac{1}{E+m}(k_x\cos(kx) - k_y\sin(kx)) \\ +\frac{k_z}{E+m}\cos(kx) \end{pmatrix} \tag{166}$$

It is possible to simply define charge conjugation in the Majorana representation by the complex conjugation of all spinor components. In this case, the plane wave solutions $m_{\pm\frac{1}{2}}(\vec{k},x)$ are obviously charge conjugation eigenstates with eigenvalue $+1$. By construction, the eigenvalue within the standard picture was chosen to be $-i$. This apparent discrepancy is due to the fact that defining charge conjugation in the Majorana representation by complex conjugation of the spinor components implies a phase convention which is not in accordance with the convention usually taken within the standard representation. A complete analogy would be obtained by defining the charge conjugation of a spinor ψ_M in the Majorana representation via the standard charge conjugation

$$\psi_S^c = i\gamma^2\psi_S^* \tag{167}$$

by

$$\psi_M^c = V\psi_S^c = V(i\gamma^2(V^{-1}\psi_M)^*) = i(V\gamma^2V^{*-1})\psi_M^* = -i\psi_M^* \tag{168}$$

which explains the differing eigenvalues obtained above. Note that the charge conjugation matrix is not obtained by a basis transform à la $\gamma_M^\mu = V\gamma^\mu V^{-1}$ due to the antilinear nature of charge conjugation.

An alternative combination of particle and antiparticle solutions would be

$$\tilde{w}_s(\vec{k}, x) = \frac{1}{\sqrt{2}}(u_s(\vec{k}, x) - iv_s(\vec{k}, x)) \tag{169}$$

The corresponding Majorana spinors $\tilde{m}_{\pm\frac{1}{2}}(\vec{k}, x)$ are then given by

$$\tilde{m}_{+\frac{1}{2}}(\vec{k}, x) = \begin{pmatrix} \sin(kx) + \frac{1}{E+m}(k_x\cos(kx) + k_y\sin(kx)) \\ -\frac{k_z}{E+m}\cos(kx) \\ -\frac{k_z}{E+m}\sin(kx) \\ -\cos(kx) - \frac{1}{E+m}(k_x\sin(kx) - k_y\cos(kx)) \end{pmatrix} \tag{170}$$

$$\tilde{m}_{-\frac{1}{2}}(\vec{k}, x) = \begin{pmatrix} -\frac{k_z}{E+m}\cos(kx) \\ +\sin(kx) - \frac{1}{E+m}(k_x\cos(kx) - k_y\sin(kx)) \\ +\cos(kx) - \frac{1}{E+m}(k_x\sin(kx) + k_y\cos(kx)) \\ +\frac{k_z}{E+m}\sin(kx) \end{pmatrix} \tag{171}$$

In the standard representation, the plane wave solutions $w_s(\vec{k}, x)$ and $\tilde{w}_s(\vec{k}, x)$ are inequivalent. At first sight, also $\tilde{m}_{\pm\frac{1}{2}}(\vec{k}, x)$ seem to be of a different quality when compared to $m_{\pm\frac{1}{2}}(\vec{k}, x)$. However, the $\tilde{m}_{\pm\frac{1}{2}}(\vec{k}, x)$ are obtained from the $m_{\pm\frac{1}{2}}(\vec{k}, x)$ by the replacements $\cos(kx) \to \sin(kx)$ and $\sin(kx) \to -\cos(kx)$, or

$$\begin{aligned} \cos(kx) &\to \cos(kx - \tfrac{\pi}{2}) \\ \sin(kx) &\to \sin(kx - \tfrac{\pi}{2}) \end{aligned} \tag{172}$$

and therefore m and $\tilde{m}$ are physically equivalent in the sense that they can be related by a space-time translation, which is not physically observable for states with sharply defined momentum.

As a first conclusion, we observe that the real Majorana fields discussed above transform according to a real representation of the Lorentz group, whereas Dirac fields are related to both inequivalent two-dimensional complex representations of the Lorentz group. If we allow the spinors in the Majorana representation to become complex again, all four spin-charge degrees of freedom of a Dirac spinor which can be constructed as linear combinations from, e.g., $m_{\pm\frac{1}{2}}(\vec{k},x)$ and $i\tilde{m}_{\pm\frac{1}{2}}(\vec{k},x)$ (the latter two states with negative C-parity) are introduced again, such that the Dirac spinor describes charged particles again.

In is worthwhile to elaborate the Majorana case in further detail. The well-known Dirac case requires no further discussion, since the four independent solutions describing a particle with well-defined charge, spin and momentum are readily constructed and given above, and a (plane wave) solution of the Dirac equation

$$i\gamma_\mu \partial^\mu \Psi_{_D} - m\Psi_{_D} = 0 \tag{173}$$

(like e.g., $u_s(\vec{k},x)$ or $v_s(\vec{k},x)$) multiplied by a complex phase remains a solution of the same equation. This is not true in the Majorana case. One might be surprised why we only found two physically distinct solutions of the Majorana equation from the four plane wave solutions given above in the Dirac representation.

To resolve this situation, we first realize that multiplying $m_{\pm\frac{1}{2}}(\vec{k},x)$ by the imaginary unit leads to charge eigenstates with negative C-parity

$$[im_{\pm\frac{1}{2}}(\vec{k},x)]^* = [im_{\pm\frac{1}{2}}(\vec{k},x)]^c = -im_{\pm\frac{1}{2}}(\vec{k},x) \tag{174}$$

Yet, the plane wave solutions $im_{\pm\frac{1}{2}}(\vec{k},x)$ are no longer solutions of the Majorana equation

$$i\gamma_\mu \partial^\mu \Psi_{_M} - m\Psi^c_{_M} = 0 \tag{175}$$

but of

$$i\gamma_\mu \partial^\mu \Psi_{_M} + m\Psi^c_{_M} = 0 \tag{176}$$

since charge conjugation changes the sign of a purely imaginary spinor. This defect can be easily resolved. Since γ^5 anticommutes with the Dirac matrices $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, given an arbitrary solution $\Psi_{_D}$ of the Dirac equation Equation (173), $\gamma^5\Psi_{_D}$ is a solution of the Dirac equation with flipped mass

$$i\gamma_\mu \partial^\mu (\gamma^5\Psi_{_D}) + m(\gamma^5\Psi_{_D}) = 0 \tag{177}$$

and the same observation applies to the Majorana equation. Modifying the spinors $w_s(\vec{k},x)$ given by Equation (162) according to

$$w_s(\vec{k},x) \rightarrow \hat{w}_s(\vec{k},x) = \gamma^5 w_s(\vec{k},x) \tag{178}$$

and transforming $\hat{w}_s(\vec{k},x) \rightarrow V\hat{w}_s(\vec{k},x)$ into the Majorana representation yields two imaginary solutions of the Majorana equation, since the multiplication of the original solutions of the Dirac equation with γ^5 in conjunction with the fact that the spinors in the Majorana representation are imaginary leads to the correct sign of the mass term in the Majorana equation. An explicit calculation yields

$$\hat{m}_s(\vec{k},x) = V\gamma^5 w_s(\vec{k},x) \tag{179}$$

with

$$\hat{m}_{\frac{1}{2}}(\vec{k},x) = i\begin{pmatrix} -\frac{k_z}{E+m}\sin(kx) \\ +\cos(kx) - \frac{1}{E+m}(k_x\sin(kx) - k_y\cos(kx)) \\ +\sin(kx) - \frac{1}{E+m}(k_x\cos(kx) + k_y\sin(kx)) \\ +\frac{k_z}{E+m}\cos(kx) \end{pmatrix} \tag{180}$$

and

$$\hat{m}_{+\frac{1}{2}}(\vec{k},x) = i \begin{pmatrix} -\cos(kx) - \frac{1}{E+m}(k_x \sin(kx) + k_y \cos(kx)) \\ \frac{k_z}{E+m}\sin(kx) \\ \frac{k_z}{E+m}\cos(kx) \\ \sin(kx) + \frac{1}{E+m}(k_x \cos(kx) - k_y \sin(kx)) \end{pmatrix} \tag{181}$$

We finally conclude that the Majorana equation Equation (175) actually describes four mass degenerate particle degrees of freedom with definite four-momentum. These can be told apart, e.g., by their charge conjugation eigenvalues, *i.e.*, by requiring $\Psi_M = \pm\Psi_M$. According to the transformation properties of the Majorana spinor under the *real* spinor representation of the Lorentz group, solutions of the Majorana equation can be combined linearly with *real* coefficients. The Majorana equation itself does not force its solutions to be charge conjugation eigenstates. It will become clear below, that it is even possible that a four-component Majorana spinor describes Majorana particles with two different masses.

The discussion presented so far is not the most general one. In the case of the Dirac equation, charge conjugation (or rather CPT conjugation) symmetry ensures that only one physical mass enters the equation. In the Majorana case, a four-component spinor may represent a collection of two two-component Majorana spinors with different masses. Furthermore, complex phases are possible in the most general form of the Majorana equation, which then respects only the fundamental CPT symmetry.

6. Majorana and Dirac Mass Terms

6.1. Lagrangian Formulation

In the foregoing sections, we introduced different fields like the free (non-interacting) right- and left-chiral two-component Majorana spinors χ_R and χ_L, which fulfill in the most general case the massive wave equations

$$i(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})\chi_R(x) - \eta_R m_R \epsilon^{-1}\chi_R^*(x) = 0 \tag{182}$$

$$i(\sigma_0\partial_0 - \vec{\sigma}\vec{\nabla})\chi_L(x) + \eta_L m_L \epsilon^{-1}\chi_L^*(x) = 0 \tag{183}$$

with real mass terms $m_{R,L}$ and $|\eta_{R,L}| = 1$, and, in the absence of complex mass phases, four-component Majorana spinors like

$$\nu_R = \begin{pmatrix} \chi_R \\ 0 \end{pmatrix}, \quad \nu_L = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} \tag{184}$$

or

$$\nu_1^M = \begin{pmatrix} \epsilon\chi_L^* \\ \chi_L \end{pmatrix} = \begin{pmatrix} \chi_{L_2}^* \\ -\chi_{L_1}^* \\ \chi_{L_1} \\ \chi_{L_2} \end{pmatrix}, \quad \nu_2^M = \begin{pmatrix} \chi_R \\ -\epsilon\chi_R^* \end{pmatrix} = \begin{pmatrix} \chi_R^1 \\ \chi_R^2 \\ -\chi_R^{2*} \\ \chi_R^{1*} \end{pmatrix} \tag{185}$$

We have also seen that the fields above are basically equivalent, *i.e.*, from a physical point of view, it is irrelevant whether one uses left- or right-chiral fields in a mathematical formalism to describe a non-interacting Majorana field. e.g., from the left-chiral field χ_L, which can be projected out from ν_1^M, one readily obtains a right-chiral field $\epsilon\chi_L^*$. If χ_L obeys Equation (183), then $\epsilon\chi_L^*$ obeys the equation

$$i(\sigma_0\partial_0 + \vec{\sigma}\vec{\nabla})(\epsilon\chi_L^*(x)) - \eta_L^* m_L \epsilon^{-1}(\epsilon\chi_L^*(x))^* = 0 \tag{186}$$

of course with the corresponding complex conjugate mass term, as one easily derives by the help of the identity $\epsilon\vec{\sigma}\epsilon^{-1} = -\vec{\sigma}^*$. Furthermore, how ν_1^M and ν_2^M are connected also becomes obvious by relating $\chi_{L_1} \leftrightarrow -\chi_R^{2*}$ and $\chi_{L_2} \leftrightarrow \chi_R^{1*}$.

We consider now two physically *different* Majorana fields χ_R and χ_L, which are described by one left-chiral and a right-chiral two-component spinor field for the sake of convenience. The most general Lagrangian $\mathcal{L}$ which describes the free dynamics of these field is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$, where ($i\sigma_2 = \epsilon$)

$$\mathcal{L}_0 = \chi_L^+ i\bar{\sigma}^\mu\partial_\mu\chi_L + \chi_R^+ i\sigma^\mu\partial_\mu\chi_R \tag{187}$$

and

$$\begin{aligned} \mathcal{L}_m \;=\; & -\eta_L\frac{m_L}{2}\chi_L^+ i\sigma_2\chi_L^* + \eta_L^*\frac{m_L}{2}\chi_L^{\mathrm{T}} i\sigma_2\chi_L \\ & +\eta_R\frac{m_R}{2}\chi_R^+ i\sigma_2\chi_R^* - \eta_R^*\frac{m_R}{2}\chi_R^{\mathrm{T}} i\sigma_2\chi_R \\ & -\eta_D m_D\chi_L^+\chi_R - \eta_D^* m_D\chi_R^+\chi_L \end{aligned} \tag{188}$$

where the $m_{D(irac)}$-terms couple the left- and right-chiral fields, as it is the case in the Dirac equation. Note that all the terms above transform as scalars. e.g., having a glimpse at Equations (51) and (57) and neglecting space-time arguments for the moment, one finds

$$\chi_R \rightarrow S\chi_R, \quad \chi_L \rightarrow \epsilon S^*\epsilon^{-1}\chi_L \tag{189}$$

and thus indeed

$$\chi_L^+\chi_R \rightarrow \chi_L^+\epsilon S^{\mathrm{T}}\epsilon^{-1}S\chi_R = \chi_L^+\epsilon\epsilon^{-1}S^{-1}\epsilon\epsilon^{-1}S\chi_R = \chi_L^+\chi_R \tag{190}$$

For the following discussion, we require first that $m_L \neq 0$ or $m_R \neq 0$. Before coming back to the Lagrangian itself, we analyze the wave equation which follows from the Lagrangian. One has

$$i\begin{pmatrix} 0 & \sigma_0\partial_0 - \vec{\sigma}\vec{\nabla} \\ \sigma_0\partial_0 + \vec{\sigma}\vec{\nabla} & 0 \end{pmatrix}\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} - \begin{pmatrix} \eta_D m_D & +\eta_L m_L\epsilon K \\ -\eta_R m_R\epsilon K & \eta_D^* m_D \end{pmatrix}\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = 0 \tag{191}$$

where K denotes complex conjugation. Introducing the four-spinor

$$\Psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} \tag{192}$$

Equation (191) can now be written

$$i\gamma^\mu\partial_\mu\Psi - \tilde{m}_M\Psi^c - \tilde{m}_D\Psi = 0 \tag{193}$$

with appropriately chosen mass matrices $\tilde{m}_{_M}$ and $\tilde{m}_{_D}$

$$\tilde{m}_{_M} = \begin{pmatrix} \eta_{_L} m_{_L} & 0 \\ 0 & \eta_{_R} m_{_R} \end{pmatrix}, \quad \tilde{m}_{_D} = \begin{pmatrix} \eta_{_D} m_{_D} & 0 \\ 0 & \eta_{_D}^* m_{_D} \end{pmatrix} \tag{194}$$

since charge conjugation is given in the chiral representation by

$$C[\Psi] = \Psi^c = \tilde{\epsilon}\Psi^* = i\gamma^2\Psi^* = i\gamma^2\gamma^0\bar{\Psi}^{\mathrm{T}}, \quad \tilde{\epsilon} = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \tag{195}$$

The present two-neutrino theory contains three phases $\eta_{_D}$ and $\eta_{_{L,R}}$. Though, two phases can be eliminated (*i.e.*, set equal to one) by multiplying $\chi_{_{L,R}}$ with appropriate phases, respectively, without changing the physical content of the theory. e.g., as we have seen above, it is possible to remove the Dirac phase $\eta_{_D}$ by a chiral transformation of the spinor according to Equation (125). A subsequent multiplication of both $\chi_{_{L,R}}$ by the same phase can then be used modify $\eta_{_L}$ or $\eta_{_R}$, but in general, a CP violating phase will survive. Vice versa, it is possible to remove the phases $\eta_{_L}$ and $\eta_{_R}$, then possibly a non-trivial phase $\eta'_{_D}$ survives. How the respective phases are intertwined can be most easily inferred from the invariance of the mass Lagrangian $\mathcal{L}_m$ under a phase redefinition of the fields. In the special case where $m_{_D} = 0$, all phases can be removed and the wave equation describes to Majorana particles, and the case where $\tilde{m}_{_M} = 0$ leads to the usual Dirac theory, as follows directly from Equation (193). We now examine two different scenarios in further detail.

6.2. CP Symmetric Theory with Majorana Masses

In the case $\tilde{m}_{_M} \neq 0$, we may introduce new four-component fields inspired by Equation (185)

$$\nu_{_1} = \begin{pmatrix} \eta_{_L}\epsilon\chi_{_L}^* \\ \chi_{_L} \end{pmatrix}, \qquad \nu_{_2} = \begin{pmatrix} \chi_{_R} \\ -\eta_{_R}\epsilon\chi_{_R}^* \end{pmatrix} \tag{196}$$

such that we obtain in the chiral representation ($\epsilon^2 = -1$)

$$\nu_{_1}^c = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} \eta_{_L}\epsilon\chi_{_L}^* \\ \chi_{_L} \end{pmatrix}^* = \begin{pmatrix} \epsilon\chi_{_L}^* \\ \eta_{_L}^*\chi_{_L} \end{pmatrix} = \eta_{_L}^* \begin{pmatrix} \eta_{_L}\epsilon\chi_{_L}^* \\ \chi_{_L} \end{pmatrix} \tag{197}$$

$$\nu_{_2}^c = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} \chi_{_R} \\ -\eta_{_R}\epsilon\chi_{_R}^* \end{pmatrix}^* = \begin{pmatrix} \eta_{_R}^*\chi_{_R} \\ -\epsilon\chi_{_R}^* \end{pmatrix} = \eta_{_R}^* \begin{pmatrix} \chi_{_R} \\ -\eta_{_R}\epsilon\chi_{_R}^* \end{pmatrix} \tag{198}$$

i.e., $\nu_{_1}$ and $\nu_{_2}$ are charge conjugation eigenstates. In addition, we introduce the Dirac adjoint spinors ($\epsilon^{\mathrm{T}} = -\epsilon$)

$$\bar{\nu}_{_1} = \nu_{_1}^+\gamma_{ch}^0 = \begin{pmatrix} \eta_{_L}^*\chi_{_L}^{\mathrm{T}}\epsilon^{\mathrm{T}} & \chi_{_L}^+ \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi_{_L}^+ & -\eta_{_L}^*\chi_{_L}^{\mathrm{T}}\epsilon \end{pmatrix} \tag{199}$$

and

$$\bar{\nu}_{_2} = \nu_{_2}^+\gamma_{ch}^0 = \begin{pmatrix} \chi_{_R}^+ & -\eta_{_R}^*\chi_{_R}^{\mathrm{T}}\epsilon^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \eta_{_R}^*\chi_{_R}^{\mathrm{T}}\epsilon & \chi_{_R}^+ \end{pmatrix} \tag{200}$$

Next, we observe that

$$\bar{\nu}_1\nu_1 = +\eta_L\chi_L^+\epsilon\chi_L^* - \eta_L^*\chi_L^{\mathrm{T}}\epsilon\chi_L \tag{201}$$

$$\bar{\nu}_2\nu_2 = -\eta_R\chi_R^+\epsilon\chi_R^* + \eta_R^*\chi_R^{\mathrm{T}}\epsilon\chi_R \tag{202}$$

and

$$\bar{\nu}_1\nu_2 = \chi_L^+\chi_R - \eta_L^*\eta_R\chi_L^{\mathrm{T}}\chi_R^* = \chi_L^+\chi_R + \eta_L^*\eta_R\chi_R^+\chi_L \tag{203}$$

$$\bar{\nu}_2\nu_1 = (\bar{\nu}_1\nu_2)^+ = \chi_R^+\chi_L + \eta_R^*\eta_L\chi_L^+\chi_R \tag{204}$$

where we used the fact that the fields are anticommuting (Grassmann) numbers.

As discussed above, we are free to gauge $\chi_{L,R}$ with appropriate phases so that $\eta_L = \eta_R = 1$. Generally, a (modified) Dirac phase η_D' will survive. *Only* when this phase is trivial, *i.e.*, for $\eta_D' = 1$ the mass term $\mathcal{L}_m$ can be written by the help of self-charge conjugate fields $\bar{\nu}_{1,2}$

$$\mathcal{L}_m = -\frac{1}{2}\begin{pmatrix} \bar{\nu}_1 & \bar{\nu}_2 \end{pmatrix}\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{205}$$

where now $C[\nu_{1,2}] = \nu_{1,2}^c = +\nu_{1,2}$. To be more specific, the trick above works if $\eta_L^*\eta_R\eta_D^2 = 1$.

Since the mass matrix above is real and symmetric, a real linear combination $\nu_{1,2}'$ of the fields $\nu_{1,2}$ can be found, which is again self-charge conjugate, so that the mass matrix becomes diagonal. Of course, the mass of the neutral fields $\nu_{1,2}'$ is given by the eigenvalues $m_{1,2}$ of the mass matrix (note the normalization factor). From

$$(\lambda - m_1)(\lambda - m_2) = (\lambda - m_L)(\lambda - m_R) - m_D^2 \tag{206}$$

we obtain

$$m_{1,2} = \frac{m_L + m_R}{2} \pm \sqrt{\left(\frac{m_L - m_R}{2}\right)^2 + m_D^2} \tag{207}$$

We conclude that in the presence of Dirac *and* Majorana mass terms and in the absence of CP violating phases, the Lagrangian $\mathcal{L}_m$ describes two charge neutral Majorana particles. This special case corresponds to the discussion originally given in [15], which is now widely found in the literature. For a discussion in the Lagrangian framework of the Dirac field as the degenerate limit of $m_L = m_R = 0$ in the more general case of two Majorana particles, we also refer to [15].

6.3. CP Violating Theory with Majorana Masses

The case where a phase remains in the theory is more involved. In order to motivate the discussion below, we point out that that in the general Lagrangian $\mathcal{L}$ defined above, two Majorana fields which transform according to the left- and right-chiral representation of the Lorentz group were used. However, in extensions of the Standard Model, it is usually assumed that the Higgs mechanism generates mass terms for leptons like the electron and muon neutrino, which can be written within a simplified model in several equivalent forms, e.g.,

$$-\mathcal{L}_m^{e\mu} \sim \left(\begin{array}{cc} \nu_e^{\mathrm{T}} & \nu_\mu^{\mathrm{T}} \end{array} \right)_L \tilde{C}\tilde{M}_{e\mu} \left(\begin{array}{c} \nu_e \\ \nu_\mu \end{array} \right)_L + h.c. \tag{208}$$

where $\tilde{C} = i\gamma^2\gamma^0$ is a charge conjugation matrix ($-\tilde{C} = \tilde{C}^{\mathrm{T}} = \tilde{C}^{+} = \tilde{C}^{-1}$) and $\tilde{M}_{e\mu}$ is a complex and symmetric 2×2 matrix in flavor space

$$\tilde{M} = \left(\begin{array}{cc} m_{ee}e^{i\gamma_e} & m_{e\mu}e^{i\gamma_{e\mu}} \\ m_{e\mu}e^{i\gamma_{e\mu}} & m_{\mu\mu}e^{i\gamma_\mu} \end{array} \right) \tag{209}$$

So it is common practice that both the electron neutrino and the muon neutrino are described by left-chiral fields. Since in the present formalism, $\epsilon\chi_R^*$ transforms accordingly, we describe the mass terms in the Lagrangian $\mathcal{L}_m$ in a similar manner as above

$$\mathcal{L} = -\frac{1}{2} \left(\begin{array}{cc} \chi_L^+ & \chi_R^{\mathrm{T}}\epsilon \end{array} \right) M \left(\begin{array}{c} \epsilon\chi_L^* \\ \chi_R \end{array} \right) = -\frac{1}{2} \left(\begin{array}{cc} \chi_L^+ & \chi_R^{\mathrm{T}}\epsilon \end{array} \right) \left(\begin{array}{cc} m_L & \eta_D m_D \\ \eta_D m_D & m_R \end{array} \right) \left(\begin{array}{c} \epsilon\chi_L^* \\ \chi_R \end{array} \right) + h.c. \tag{210}$$

where we made use of the fact that that it is possible to eliminate two of three phases in a two-Majorana particle theory. Note that in Equation (210), one has $\chi_R^{\mathrm{T}}\chi_L^* = -(\chi_R^{\mathrm{T}}\chi_L^*)^{\mathrm{T}} = -\chi_L^+\chi_L$ due to the Fermi statistics, therefore the mass matrix is symmetric, but not Hermitian.

A short calculation shows that the unitary matrix

$$U = \left(\begin{array}{cc} \cos\vartheta & -\sin\vartheta e^{i\alpha} \\ \sin\vartheta e^{-i\alpha} & \cos\vartheta \end{array} \right) \tag{211}$$

diagonalizes M

$$U^{\mathrm{T}}MU = \left(\begin{array}{cc} \hat{m}_1 & 0 \\ 0 & \hat{m}_2 \end{array} \right) \tag{212}$$

if ϑ and α fulfill the constraint

$$2\eta_D m_D \cos(2\vartheta) = \sin(2\vartheta)(m_L e^{i\alpha} - m_R e^{-i\alpha}) \tag{213}$$

The Majorana fields

$$\left(\begin{array}{c} \epsilon\chi_L^{\prime *} \\ \chi_R^{\prime} \end{array} \right) = U^+ \left(\begin{array}{c} \epsilon\chi_L^* \\ \chi_R \end{array} \right) \tag{214}$$

would diagonalize the mass Lagrangian, but since $\hat{m}_1$ and $\hat{m}_2$ are complex, they do not yet correspond to the "physical states" in our model. However, writing $\hat{m}_{1,2} = e^{i\Phi_{1,2}}$, we have

$$U^{\mathrm{T}}MU = \Phi \left(\begin{array}{cc} \hat{m}_1 & 0 \\ 0 & \hat{m}_2 \end{array} \right) \Phi \tag{215}$$

where

$$\Phi = \left(\begin{array}{cc} e^{i\Phi_1/2} & 0 \\ 0 & e^{i\Phi_2/2} \end{array} \right) \tag{216}$$

$U' = U\Phi$ brings the mass matrix into diagonal form with real and positive eigenvalues. These can be interpreted as the physical mass of the neutrino fields.

The present discussion is the starting point of many models which aim at a description of lepton mixing and mass hierarchies, like, e.g., the seesaw mechanism [16,17]. In realistic models with three-generation Majorana neutrinos, even two non-trivial Majorana phases appear, which are related to the important issue of CP non-conservation [18].

We finally comment (once more) on the Majorana fields constructed in this section. It is a simple exercise to demonstrate that the CPT transformed four-component spinor Ψ in Equation (193)

$$CPT[\Psi(x)] = \Psi^{cpt} = i\gamma^5\Psi(-x) = i\gamma^5\Psi(-x^0, -\vec{x}) \tag{217}$$

again fulfills Equation (193), but the CP symmetry is broken when Majorana masses and a CP violating phase are inherent in the theory. CPT transforms spin and momentum of a particle. Whereas the spin changes sign, the momentum is flipped twice by the P and the T operation and remains invariant. However, a massive CPT transformed Majorana particle state can be transformed back into the original state by an appropriate Poincaré transformation. The helicity of the particle is not preserved in this process and changes sign, but the chirality, *i.e.*, the transformation property of the field which describes the particle remains invariant. The situation is completely different for the degenerate case of Dirac particles, where CPT changes the charge of the particle, which is a Lorentz invariant quantity.

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