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Symmetries, Lagrangians and Conservation Laws of an Easter Island Population Model

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Abstract: Basener and Ross (2005) proposed a mathematical model that describes the dynamics of growth and sudden decrease in the population of Easter Island. We have applied Lie group analysis to this system and found that it can be integrated by quadrature if the involved parameters satisfy certain relationships. We have also discerned hidden linearity. Moreover, we have determined a Jacobi last multiplier and, consequently, a Lagrangian for the general system and have found other cases independently and dependently on symmetry considerations in order to construct a corresponding variational problem, thus enabling us to find conservation laws by means of Noether's theorem. A comparison with the qualitative analysis given by Basener and Ross is provided.

Keywords: Lie group analysis; Jacobi last multiplier; Lagrangians; Noether's theorem; Easter Island population

1. Introduction

Lie group analysis has been applied in a multitude of physics problems for more than a century, but rarely in biology, maybe because the ordinary differential equations studied in these fields are generally of first order, in contrast with those in physics, which are usually of second order. Yet, when Lie group analysis is successfully applied to biology models, then several instances of integrability, even linearity, are found, which lead to the general solution of the model [1–6].

In [7], a mathematical model that describes the dynamics of growth and sudden decrease in the population of Easter Island was proposed. The model is a nonlinear system of two first-order ordinary differential equations, *i.e.*,

$$\dot{w}_1 = cw_1 \left(1 - \frac{w_1}{K}\right) - hw_2, \quad (1)$$

$$\dot{w}_2 = aw_2 \left(1 - \frac{w_2}{w_1}\right), \quad (2)$$

where w_1 is the amount of resources, R in the original notation, w_2 that of population, P in the original notation, c is the growth rate of the resources, K the carrying capacity, h the harvesting constant and a the growth rate of the population. More details on the construction of this model can be found in [7], where numerical solutions of system (1)–(2) were given and a qualitative analysis of the general behavior of solutions both in finite and infinite time was presented.

In this paper, we apply Lie group analysis to an equivalent second-order equation that can be easily derived from system (1)–(2) and find that it can either be integrated by quadrature or reduced to a linear equation if the involved parameters satisfy certain relationships. We also determine a Jacobi last multiplier for system (1)–(2) in order to construct the corresponding Lagrangians (*i.e.*, variational problems) for both system (1)–(2), and the equivalent second-order equation [8]. We find other Jacobi last multipliers and, consequently, Lagrangians independently and dependently on symmetry considerations. Then, we apply Noether's theorem [9] in order to find conservation laws. Finally, we compare our results to the qualitative analysis in [7].

2. Jacobi Last Multiplier and Its Properties

In this section, we recall the definition and properties of the Jacobi last multiplier, its connection to Lie symmetries and to Lagrangians (namely, calculus of variations).

The method of the Jacobi last multiplier [10,11] (an English translation of [11] is available in [12]) provides a means to determine all of the solutions of the partial differential equation:

$$\mathcal{A}f = \sum_{i=1}^n a_i(x_1, \dots, x_n) \frac{\partial f}{\partial x_i} = 0 \quad (3)$$

or its equivalent associated Lagrange's system:

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \dots = \frac{dx_n}{a_n}. \quad (4)$$

In fact, if one knows the Jacobi last multiplier and all, but one of the solutions, namely $n - 2$ solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier M is given by:

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = M\mathcal{A}f, \quad (5)$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \dots, \omega_{n-1})}{\partial(x_1, x_2, \dots, x_n)} = \det \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \omega_1}{\partial x_1} & & \frac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \omega_{n-1}}{\partial x_1} & \dots & \frac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \quad (6)$$

and $\omega_1, \dots, \omega_{n-1}$ are $n - 1$ solutions of (3) or, equivalently, first integrals of (4) independent of each other. This means that M is a function of the variables (x_1, \dots, x_n) and depends on the chosen $n - 1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier are:

- (a) If one selects a different set of $n - 1$ independent solutions $\eta_1, \dots, \eta_{n-1}$ of Equation (3), then the corresponding last multiplier N is linked to M by the relationship:

$$N = M \frac{\partial(\eta_1, \dots, \eta_{n-1})}{\partial(\omega_1, \dots, \omega_{n-1})}.$$

- (b) Given a non-singular transformation of variables:

$$\tau : (x_1, x_2, \dots, x_n) \longrightarrow (x'_1, x'_2, \dots, x'_n),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(x'_1, x'_2, \dots, x'_n)},$$

where M obviously comes from the $n - 1$ solutions of $\mathcal{A}F = 0$, which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1} .

- (c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^n \frac{\partial(Ma_i)}{\partial x_i} = 0; \quad (7)$$

and *vice versa*, every solution M of this equation is a Jacobi last multiplier.

- (d) If one knows two Jacobi last multipliers M_1 and M_2 of Equation (3), then their ratio is a solution ω of (3) or, equivalently, a first integral of (4). Naturally, the ratio may be quite trivial, namely a constant; *vice versa*, the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1 \omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [13,14]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [15]. If we know $n - 1$ symmetries of (3)/(4), say:

$$\Gamma_i = \sum_{j=1}^n \xi_{ij}(x_1, \dots, x_n) \partial_{x_j}, \quad i = 1, n - 1, \quad (8)$$

a Jacobi last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where:

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \quad (9)$$

There is an obvious corollary to the results of the Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing, the variable associated with that component, *i.e.*, $\partial a_i / \partial x_i = 0$, the last multiplier is a constant, and any other Jacobi last multiplier is a first integral.

Another property of the Jacobi last multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation:

$$\ddot{x} = \phi(t, x, \dot{x}), \quad (10)$$

namely [11,16]:

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}, \quad (11)$$

where $M = M(t, x, \dot{x})$ satisfies the following equation:

$$\frac{d}{dt}(\log M) + \frac{\partial \phi}{\partial \dot{x}} = 0. \quad (12)$$

Then, Equation (10) becomes the Euler–Lagrange equation:

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial L}{\partial x} = 0. \quad (13)$$

The proof is given by taking the derivative of (13) by \dot{x} and showing that this yields (12). If one knows a Jacobi last multiplier, then L can be obtained by a double integration, *i.e.*:

$$L = \int \left(\int M \, d\dot{x} \right) d\dot{x} + \ell_1(t, x)\dot{x} + \ell_2(t, x), \quad (14)$$

where ℓ_1 and ℓ_2 are functions of t and x , which have to satisfy a single partial differential equation related to (10) [17]. As was shown in [17], ℓ_1, ℓ_2 are related to the gauge function $F = F(t, x)$. In fact, we may assume:

$$\begin{aligned} \ell_1 &= \frac{\partial F}{\partial x} \\ \ell_2 &= \frac{\partial F}{\partial t} + \ell_3(t, x) \end{aligned} \quad (15)$$

where ℓ_3 has to satisfy the mentioned partial differential equation and F is obviously arbitrary.

In [18], it was shown that a system of two first-order ordinary differential equations:

$$\begin{aligned} \dot{u}_1 &= \phi_1(t, u_1, u_2) \\ \dot{u}_2 &= \phi_2(t, u_1, u_2) \end{aligned} \quad (16)$$

always admits a linear Lagrangian of the form:

$$L = U_1(t, u_1, u_2)\dot{u}_1 + U_2(t, u_1, u_2)\dot{u}_2 - V(t, u_1, u_2). \quad (17)$$

The key is a function W , such that:

$$W = -\frac{\partial U_1}{\partial u_2} = \frac{\partial U_2}{\partial u_1} \quad (18)$$

and

$$\frac{d}{dt}(\log W) + \frac{\partial \phi_1}{\partial u_1} + \frac{\partial \phi_2}{\partial u_2} = 0. \quad (19)$$

It is obvious that Equation (19) is Equation (7) of the Jacobi last multiplier for system (16), as it was point out in [8]. Therefore, once a Jacobi last multiplier $M(t, u_1, u_2)$ has been found, then a Lagrangian of system (16) can be obtained by two integrations, *i.e.*,

$$L = \left(\int M du_1 \right) \dot{u}_2 - \left(\int M du_2 \right) \dot{u}_1 + g(t, u_1, u_2) + \frac{d}{dt}G(t, u_1, u_2), \quad (20)$$

where $g(t, u_1, u_2)$ satisfies two linear differential equations of first order that can be always integrated and $G(t, u_1, u_2)$ is the arbitrary gauge function that should be taken into consideration in order to apply correctly Noether's theorem [9]. If a Noether symmetry:

$$\Gamma = \xi(t, u_1, u_2)\partial_t + \eta_1(t, u_1, u_2)\partial_{u_1} + \eta_2(t, u_1, u_2)\partial_{u_2} \quad (21)$$

exists for the Lagrangian L in (20), then a first integral of system (16) is:

$$-\xi L - \frac{\partial L}{\partial \dot{u}_1}(\eta_1 - \xi \dot{u}_1) - \frac{\partial L}{\partial \dot{u}_2}(\eta_2 - \xi \dot{u}_2) + G(t, u_1, u_2). \quad (22)$$

We underline that \dot{u}_1 and \dot{u}_2 always disappear from the expression of the first integral (22) thanks to the linearity of the Lagrangian (20) and formula (20).

3. Lie Symmetries of System (1)–(2)

It is well known to practitioners of Lie group analysis that a first-order system of ordinary differential equations admits an infinite-dimensional Lie symmetry algebra; e.g., see [19]. Lie's theory allows us to integrate system (1)–(2) by quadrature, if we find at least a two-dimensional Lie algebra. In order to find it, we derive w_1 from (2), *i.e.*,

$$w_1 = -\frac{aw_2^2}{\dot{w}_2 - aw_2}. \quad (23)$$

Consequently we obtain a second-order ordinary differential equation (ODE) in $u \equiv w_2$,

$$\ddot{u} = \frac{(2a - h)\dot{u}^2}{au} - (a + c - 2h)\dot{u} - \frac{ac}{K}u^2 + a(c - h)u. \quad (24)$$

and search for its Lie symmetry algebra. An operator Γ :

$$\Gamma = V(t, u)\partial_t + G(t, u)\partial_u \quad (25)$$

is said to generate a Lie point symmetry group of an equation of second-order, e.g.,

$$\ddot{u} = F(t, u, \dot{u}), \quad (26)$$

if its second prolongation:

$$\Gamma_{\frac{2}{2}} = \Gamma + \left(\frac{dG}{dt} - \dot{u} \frac{dV}{dt} \right) \partial_{\dot{u}} + \left[\frac{d}{dt} \left(\frac{dG}{dt} - \dot{u} \frac{dV}{dt} \right) - \ddot{u} \frac{dV}{dt} \right] \partial_{\ddot{u}}$$

applied to (26), on its solutions, is identically equal to zero, *i.e.*:

$$\Gamma_{\frac{2}{2}} \Big|_{(26)} = 0. \quad (27)$$

A trivial Lie point symmetry of system (1)–(2) and also of Equation (24) is translation in time, *i.e.*,

$$\partial_t. \quad (28)$$

Using *ad hoc* REDUCE interactive programs [20], we find that Equation (24) admits at least another symmetry in two cases. Of course, any of those symmetries corresponds to a symmetry of system (1)–(2), as we show below.

Case (A)

If the following relationship among the parameters h , a and c is satisfied (condition $3a - 2c = 0$ does not yield a second Lie symmetry of Equation (24)):

$$h = \frac{a(2a - c)}{3a - 2c}, \quad (29)$$

then Equation (24) admits a two-dimensional Lie symmetry algebra generated by (if $a = c$, then (29) implies that $h = c$; this is a particular example of Case (A), and we discuss it in Remark 3 of Section 4):

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \exp((a - c)t) (\partial_t - 2(a - c)u\partial_u). \quad (30)$$

This Lie symmetry algebra is non-Abelian and transitive, *i.e.*, of Type III in Lie's classification of two-dimensional algebras in the real plane [14] (this classification can also be found in Bianchi's book [15] and in modern textbooks, e.g., [21]). Therefore, to integrate Equation (24), we have to transform it into its canonical form, as given by Lie himself, *i.e.*,

$$\frac{d^2 \tilde{u}}{d\tilde{t}^2} = \frac{1}{\tilde{t}} \mathfrak{F} \left(\frac{d\tilde{u}}{d\tilde{t}} \right), \quad (31)$$

where \tilde{t} , \tilde{u} are the new independent and dependent variables, respectively, \mathfrak{F} is an arbitrary function of $\frac{d\tilde{u}}{d\tilde{t}}$ and the two-dimensional canonical Lie algebra of Type III is generated by the following operators:

$$\partial_{\tilde{t}}, \quad \tilde{t}\partial_{\tilde{t}} + \tilde{u}\partial_{\tilde{u}}. \quad (32)$$

We determine:

$$\tilde{t} = \frac{\exp(-(a - c)t)}{\sqrt{u}}, \quad \tilde{u} = -\frac{\exp(-(a - c)t)}{(a - c)^2} \quad (33)$$

and consequently, (24) becomes:

$$\frac{d^2\tilde{u}}{d\tilde{t}^2} = \frac{a}{2K(3a-2c)\tilde{t}} \left(c(3a^3 - 2c^3 + 7ac^2 - 8a^2c) \left(\frac{d\tilde{u}}{d\tilde{t}} \right)^2 + 2K \right) \frac{d\tilde{u}}{d\tilde{t}} \quad (34)$$

which can be integrated by two quadratures to yield:

$$\tilde{u} = \int \sqrt{\frac{2A_1K^2(3a-2c)}{\tilde{t}^{-2aK^2} + A_1cK(20ac^3 - 9a^4 - 37a^2c^2 + 30a^3c - 4c^4)}} d\tilde{t} + A_2, \quad (35)$$

with A_1, A_2 two arbitrary constants.

Case (B)

If the following relationship among the parameters a and c is satisfied:

$$a = 2c, \quad (36)$$

then Equation (24) admits a two-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \exp(-ct) (\partial_t + 2cu\partial_u). \quad (37)$$

This Lie symmetry algebra is also non-Abelian and transitive. Therefore, we can derive the corresponding canonical transformation, *i.e.*,

$$\tilde{t} = \frac{\exp(ct)}{\sqrt{u}}, \quad \tilde{u} = -\frac{\exp(ct)}{c}, \quad (38)$$

and consequently, (24) becomes:

$$\frac{d^2\tilde{u}}{d\tilde{t}^2} = \frac{1}{cK} \left(K(c-h) - c^3 \left(\frac{d\tilde{u}}{d\tilde{t}} \right)^2 \right) \frac{d\tilde{u}}{d\tilde{t}}, \quad (39)$$

which can be integrated by two quadratures to yield:

$$\tilde{u} = A_1 \sqrt{K(h-c)} \int \frac{\exp(\tilde{t})}{\sqrt{A_1^{2h/c} \exp(2h\tilde{t}/c) - c^3 A_1^2 \exp(2\tilde{t})}} d\tilde{t} + A_2, \quad (40)$$

with A_1, A_2 two arbitrary constants.

Subcase (B.1)

If in addition to the condition (36), the following relationship among the parameters h and c is satisfied:

$$h = \frac{3}{2}c \quad (41)$$

then Equation (24), *i.e.*,

$$\ddot{u} = \frac{5}{4u} \dot{u}^2 - 2\frac{c^2}{K} u^2 - c^2 u, \quad (42)$$

admits a three-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \exp(-ct) (\partial_t + 2cu\partial_u) \quad \Gamma_3 = \exp(ct) (\partial_t - 2cu\partial_u). \quad (43)$$

This algebra is a representation of $sl(2, \mathbb{R})$. It was shown in [22] that if we treat (42) as a first integral, namely if we solve the equation with respect to $1/K$ (this choice, instead of K , is just for convenience), *i.e.*:

$$\frac{1}{K} = -\frac{1}{2c^2u^2}\ddot{u} - \frac{1}{2u} + \frac{5}{8c^2u^3}\dot{u}^2, \quad (44)$$

then a linearizable third-order equation is obtained, *i.e.*,

$$\ddot{u} = \dot{u} \left(c^2 + \frac{9}{2u}\dot{u} - \frac{15}{4u^2}\dot{u}^2 \right). \quad (45)$$

In fact, this equation admits a seven-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned} X_1 &= \exp(ct) (\partial_t - 2uc\partial_u), \quad X_2 = \exp(-ct) (\partial_t + 2uc\partial_u), \quad X_3 = \partial_t, \\ X_4 &= u\partial_u, \quad X_5 = u^{3/2} \exp(ct)\partial_u, \quad X_6 = u^{3/2} \exp(-ct)\partial_u, \quad X_7 = u^{3/2}\partial_u. \end{aligned} \quad (46)$$

We remark that we could not solve Equation (42) with respect to c since it is not an essential constant: in fact, we could have eliminated it from system (1)–(2) by rescaling. Indeed, the third-order equation that one gets through c , namely:

$$\ddot{u} = \frac{\dot{u}}{2u^2(K+2u)} (18u^2\ddot{u} + 7Ku\dot{u} - 5K\dot{u}^2 - 15u\dot{u}^2),$$

admits a two-dimensional Lie symmetry algebra, generated by ∂_t , $t\partial_t$, and consequently, it is not linearizable.

Following Lie [14], the linearizing transformation of equation (45) is given by finding a two-dimensional Abelian intransitive subalgebra and putting it into the canonical form $\partial_{\tilde{u}}$, $\tilde{t}\partial_{\tilde{t}}$. Since a two-dimensional Abelian intransitive subalgebra is that generated by X_5 and X_6 , then the point transformation:

$$\tilde{t} = \exp(-2ct), \quad \tilde{u} = -2u^{-1/2} \exp(-ct) \quad (47)$$

takes (45) into the following linear equation:

$$\frac{d^3\tilde{u}}{d\tilde{t}^3} = -\frac{3}{2\tilde{t}} \frac{d^2\tilde{u}}{d\tilde{t}^2}. \quad (48)$$

Its general solution simply is:

$$\tilde{u} = A_3 + A_2\tilde{t} + A_1\sqrt{\tilde{t}}, \quad (49)$$

with A_i ($i = 1, 2, 3$) arbitrary constants, that, replaced into (45), yields the general solution:

$$u = \frac{4}{\exp(2ct) (A_3 + A_2 \exp(-2ct) + A_1 \exp(-ct))^2}. \quad (50)$$

Since the original Equation (42) is of second order, then one of the constants of integrations A_i should be superfluous. Indeed, if we replace (50) into (44), then the following condition is obtained:

$$\frac{1}{K} = \frac{1}{2}A_2A_3 - \frac{1}{8}A_1^2 \quad (51)$$

that yields:

$$A_3 = \frac{8 + A_1^2 K}{4A_2 K}. \quad (52)$$

Finally, the general solution of Equation (42) is:

$$u = \frac{64A_2^2 K}{\exp(2ct) (8 + A_1^2 K + 4A_2^2 K \exp(-2ct) + 4A_1 A_2 K \exp(-ct))^2}, \quad (53)$$

and consequently, by means of (23), the general solution of system (1)–(2), *i.e.*,

$$\dot{w}_1 = cw_1 \left(1 - \frac{w_1}{K}\right) - \frac{3c}{2} w_2, \quad (54)$$

$$\dot{w}_2 = 2cw_2 \left(1 - \frac{w_2}{w_1}\right), \quad (55)$$

is given by:

$$w_1 = \frac{32A_2^2 K^2 \exp(-2ct)}{(8 + A_1^2 K + 4A_2^2 K \exp(-2ct) + 4A_1 A_2 K \exp(-ct)) (8 + A_1^2 K + 2A_1 A_2 K \exp(-ct))},$$

$$w_2 = \frac{64A_2^2 K \exp(-2ct)}{(8 + A_1^2 K + 4A_2^2 K \exp(-2ct) + 4A_1 A_2 K \exp(-ct))^2}. \quad (56)$$

Since $\lim_{t \rightarrow +\infty} w_1 = 0$, $\lim_{t \rightarrow +\infty} w_2 = 0$, this solution corresponds to asymptotic death for all [7]. We present two particular instances of the general solution (56) in Figures 1 and 2.

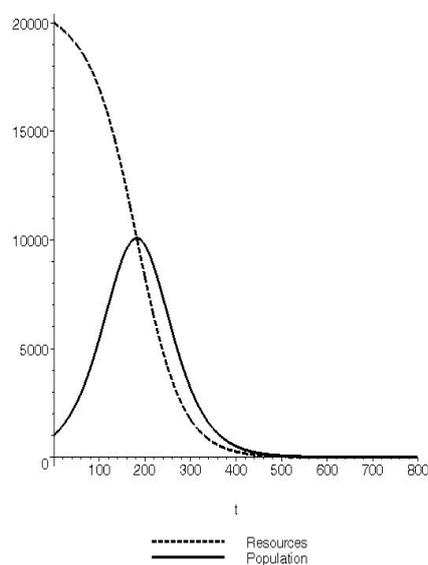


Figure 1. The amount of resources $R \equiv w_1$ and that of the population $P \equiv w_2$ for the values of the parameters $K = 20000$, $c = 0.01$, and for the initial conditions $P(0) = 1000$, $R(0) = 20000$.

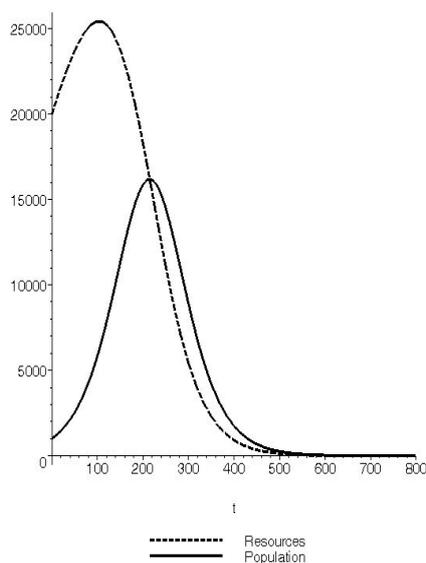


Figure 2. The amount of resources $R \equiv w_1$ and that of the population $P \equiv w_2$ for the values of the parameters $K = 40000$, $c = 0.01$, and for the initial conditions $P(0) = 1000$, $R(0) = 20000$.

It is interesting to remark that if we double the carrying capacity K leaving the other parameters and initial conditions unaltered, the amount of resources $R \equiv w_1$ actually grows along with the population $P \equiv w_2$ at least for a period of time (approximately a century in Figure 2) before decreasing dramatically.

4. Jacobi Last Multipliers, Lagrangians and First Integrals of System (1)–(2)

A Jacobi last multiplier for system (1)–(2) has to satisfy Equation (7), *i.e.*,

$$\frac{d}{dt}(\log M_{[w]}) + \frac{\partial a_1}{\partial w_1} + \frac{\partial a_2}{\partial w_2} = 0. \tag{57}$$

As in [8], we assume that $M_{[w]}$ has the following form:

$$M_{[w]} = w_1^{a_1} w_2^{a_2} \exp(a_0 t), \tag{58}$$

where a_i , ($i = 0, 1, 2$) are constants to be determined. Replacing this $M_{[w]}$ into system (1)–(2) yields:

$$M_{[w]} = w_1^{-2} w_2^{2h/a-2} \exp((a + c - 2h)t). \tag{59}$$

Therefore, the following Lagrangian of system (1) and (2) is obtained by means of (20):

$$L_{[w]} = w_1^{-2} w_2^{2h/a-2} \exp((a + c - 2h)t) \left(\frac{a}{a - 2h} w_2 \dot{w}_1 - w_1 \dot{w}_2 + a w_2 \frac{2c w_1^2 + K(a - c - 2h)w_1 + K(2h - a)w_2}{K(a - 2h)} \right). \tag{60}$$

It is obvious that if $a = 2h$, then the Lagrangian $L_{[w]}$ does not exist. Indeed, if:

$$a = 2h, \tag{61}$$

then the Jacobi Last Multiplier (59) becomes:

$$M_{[w]}^* = w_1^{-2} w_2^{-1} \exp(ct), \quad (62)$$

and consequently, the Lagrangian of system (1)–(2) is:

$$L_{[w]}^* = w_1^{-2} w_2^{-1} \exp(ct) \left(-w_2 \log(w_2) \dot{w}_1 - w_1 \dot{w}_2 - \frac{2c}{K} \log(w_2) w_1^2 w_2 - 2hw_2^2 + c \log(w_2) w_1 w_2 + 4hw_1 w_2 \right). \quad (63)$$

The Jacobi Last Multiplier (59) yields the following Jacobi last multiplier of Equation (24) by means of the application of Property (b):

$$M_{[u]} = M_{[w]} \frac{\partial(w_2, w_1)}{\partial(u, \dot{u})} = \begin{vmatrix} 1 & 0 \\ -au \frac{2\dot{u} - au}{(\dot{u} - au)^2} & \frac{au^2}{(\dot{u} - au)^2} \end{vmatrix} = \frac{u^{2h/a-4}}{a} \exp[(a + c - 2h)t], \quad (64)$$

and consequently, the following Lagrangian is derived by means of (14):

$$L_{[u]} = u^{\frac{2h}{a}-4} \exp[(a + c - 2h)t] \left(\frac{\dot{u}^2}{2a} + \frac{acu^3}{K(a - 2h)} - \frac{a(c - h)u^2}{2(a - h)} \right). \quad (65)$$

If the following condition on the parameters is satisfied, *i.e.*,

$$h = \frac{a + c}{2}, \quad (66)$$

then the trivial Lie symmetry (28) admitted by system (1)–(2) and Equation (24) is also a Noether symmetry (Equation (24) does not admit another Lie symmetry). Consequently, Noether's theorem [9] yields the following first integral:

$$Int = \frac{u^{\frac{c}{a}-3} (K\dot{u}^2 + (2u - K)a^2u^2)}{2aK} \quad (67)$$

of Equation (24), *i.e.*,

$$\ddot{u} = \frac{-a^3Ku^2 + a^2cKu^2 - 2a^2cu^3 + 3aK\dot{u}^2 - cK\dot{u}^2}{2aKu}. \quad (68)$$

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int yields the following first integral of system (1)–(2):

$$I = \frac{w_2^{c/a}}{w_1^2} (2w_1^2 - 2Kw_1 + Kw_2). \quad (69)$$

Remark 1: It was shown in [7], Proposition 6, that if $c > a$, then the first integral (69) yields periodic orbits. In particular, if $a = 1, c = 2$, it is easy to show that the general solution depends on elliptic functions. ♦

It is obvious that if either $a = 2h$ or $a = h$, then the Lagrangian $L_{[u]}$ in (65) does not exist. In fact, if:

$$a = 2h, \quad (70)$$

then from $M_{[u]}$ in (64), *i.e.*,

$$M_{[u]}^* = \frac{u^{-3}}{2h} \exp(ct), \quad (71)$$

the following Lagrangian is obtained by means of (14), *i.e.*,

$$L_{[u]}^* = \frac{u^{-3}}{K} \exp(ct) \left(\frac{K}{4h} \dot{u}^2 - c \log(u) u^3 + K(h-c)u^2 \right). \quad (72)$$

This Lagrangian does not admit any Noether point symmetry unless $c = h$, and consequently, the Lagrangian (72) admits the Lie symmetry $\exp(-ht) (\partial_t + 2hu\partial_u)$ as Noether symmetry and the following first integral is derived:

$$Int_0 = \frac{4h^2 \log(u)u^3 + 4h^2Ku^2 - 8th^3u^3 - 4hKuu\dot{u} + K\dot{u}^2}{4hKu^3}, \quad (73)$$

the gauge function being $F = h(K - 2htu)/(Ku)$.

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int_0 yields the following first integral of system (1)–(2):

$$I_0 = \log(w_2) - 2ht + K \frac{w_2}{w_1^2}. \quad (74)$$

Remark 2: The case $a = 2h, c = h$ is a particular example of Case (B), and consequently, Equation (24), *i.e.*,

$$\ddot{u} = \frac{3K\dot{u}^2 - 2Kuh\dot{u} - 4u^3h^2}{2Ku}, \quad (75)$$

admits the two Lie symmetries (37), *i.e.*,

$$\partial_t, \quad (\partial_t + 2hu\partial_u) \exp(-ht). \quad (76)$$

Another Jacobi last multiplier can be obtained from the reciprocal of (9), *i.e.*,

$$\begin{aligned} \Delta &= \det \begin{bmatrix} 1 & \dot{u} & \frac{3K\dot{u}^2 - 2Kuh\dot{u} - 4u^3h^2}{2Ku} \\ 1 & 0 & 0 \\ \exp(-ht) & 2hu \exp(-ht) & h(3\dot{u} - 2hu) \exp(-ht) \end{bmatrix} \\ &= -\frac{4h^3u^3}{K} \exp(-ht). \end{aligned} \quad (77)$$

Surprisingly, we do not obtain another Jacobi last multiplier. Indeed, $\frac{1}{\Delta} = -\frac{K \exp(ht)}{4h^3u^3}$ is equal to $M_{[u]}^* = \frac{\exp(ht)}{2hu^3}$ in (71), apart from an unessential multiplicative constant. ♦

If

$$a = h, \quad (78)$$

then from $M_{[u]}$ in (64), *i.e.*,

$$M_{[u]}^{**} = \frac{\exp((c-h)t)}{hu^2}, \quad (79)$$

the following Lagrangian is obtained by means of (14), *i.e.*,

$$L_{[u]}^{**} = \frac{\exp((c-h)t)(K\dot{u}^2 - 2chu^3 + 2hK(c-h)\log(u)u^2)}{2hKu^2}. \quad (80)$$

This Lagrangian does not admit any Noether point symmetry unless either of the following further conditions are also satisfied, *i.e.*,

$$h = c \quad (81)$$

or

$$h = 2c. \quad (82)$$

If $h = c$, then the Lagrangian (80) admits the trivial Lie symmetry (28) as Noether symmetry, and consequently, the following first integral is derived:

$$Int_{1a} = \frac{K\dot{u}^2 + 2c^2u^3}{2cKu^2}. \quad (83)$$

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int_{1a} yields the following first integral of system (1)–(2):

$$I_{1a} = \frac{-2Kw_1w_2 + Kw_2^2 + 2w_1^2w_2}{w_1^2}. \quad (84)$$

Remark 3: The case $a = h = c$ is a particular example of Case (A), and consequently, Equation (24), *i.e.*,

$$\ddot{u} = \frac{K\dot{u}^2 - c^2u^3}{Ku}, \quad (85)$$

admits the two Lie symmetries (30), *i.e.*,

$$\partial_t, \quad t\partial_t - 2u\partial_u. \quad (86)$$

Its general solution can be determined through the canonical variables, as explained in the previous section, although in implicit form, *i.e.*,

$$\frac{A_1 - 3(c^2 + 1)u}{3u\sqrt{u}} + \sqrt{\frac{K}{2A_1}} \log\left(\frac{K}{A_1u}\right) + \sqrt{\frac{2K}{A_1}} \log\left(A_1 + \sqrt{A_1(A_1 - c^2u)}\right) = t + A_2. \quad (87)$$

Another Jacobi last multiplier (hence, another Lagrangian) for Equation (85) can be obtained from the reciprocal of (9), *i.e.*,

$$\Delta = \det \begin{bmatrix} 1 & \dot{u} & \frac{K\dot{u}^2 - c^2u^3}{Ku} \\ 1 & 0 & 0 \\ t & -2u & -3\dot{u} \end{bmatrix} = \frac{K\dot{u}^2 + 2c^2u^3}{K}. \quad (88)$$

Because of Property (d), the ratio of the Jacobi last multiplier in (79), *i.e.*, $M_{[u]}^{**} = \frac{1}{cu^2}$, with that found in (88), *i.e.*, $\frac{1}{\Delta} = \frac{K}{K\dot{u}^2 + 2c^2u^3}$, yields a first integral that surprisingly is just Int_{1a} , apart from an unessential multiplicative constant. ♦

If $h = 2c$, then the Lagrangian (80) admits the Lie symmetry $\exp(-ct) (\partial_t + 2cu\partial_u)$ as Noether symmetry, and the following first integral is derived:

$$Int_{1b} = \frac{4c^2Ku^2 + 4c^2u^3 - 4cKui + K\dot{u}^2}{4 \exp(2ct)cKu^2}, \quad (89)$$

the gauge function being $F = c(1 - \log(u)) \exp(-2ct)$.

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int_{1b} yields the following first integral of system (1)–(2):

$$I_{1b} = \exp(-2ct)w_2 \frac{Kw_2 + w_1^2}{w_1^2}. \quad (90)$$

Remark 4: The case $a = h = 2c$ is another instance of Case (B), and consequently, Equation (24), *i.e.*,

$$\ddot{u} = \frac{K\dot{u}^2 + Kcui - 2Kc^2u^2 - 2c^2u^3}{Ku}, \quad (91)$$

admits the two Lie symmetries (37), *i.e.*,

$$\partial_t, \quad (\partial_t + 2cu\partial_u) \exp(-ct), \quad (92)$$

and its general solution can be determined through the canonical variables, as explained in the previous section. Thus, the general solution of system (1)–(2) is that both w_1 and w_2 can go to infinity at the same finite time, *i.e.*, if:

$$t = \frac{1}{c} \log \left(cA_2 - \frac{c\sqrt{A_1K}}{4} \log(c^4A_1^4K^2) \right)$$

$$w_1 = \frac{4c(A_1K)^{3/2} \exp \left(ct + \frac{2(cA_2 - \exp(ct))}{c\sqrt{A_1K}} \right)}{c^4A_1^4K^2 - \exp \left(4 \frac{cA_2 - \exp(ct)}{c\sqrt{A_1K}} \right)}, \quad (93)$$

$$w_2 = \frac{4A_1K \exp \left(2ct + 2 \frac{cA_2 - \exp(ct)}{c\sqrt{A_1K}} \right)}{\left(c^2A_1^2K + \exp \left(2 \frac{cA_2 - \exp(ct)}{c\sqrt{A_1K}} \right) \right)^2}. \quad (94)$$

Another Jacobi last multiplier for Equation (91) can be obtained from the reciprocal of (9), *i.e.*,

$$\begin{aligned} \Delta &= \det \begin{bmatrix} 1 & i & \frac{K\dot{u}^2 + Kcui - 2Kc^2u^2 - 2c^2u^3}{Ku} \\ 1 & 0 & 0 \\ \exp(-ct) & 2cu \exp(-ct) & c(3\dot{u} - 2cu) \exp(-ct) \end{bmatrix} \\ &= -\frac{c}{K} (K(\dot{u} - 2cu)^2 + 4c^2u^3) \exp(-ct). \end{aligned} \quad (95)$$

Because of Property (d), the ratio of the Jacobi last multiplier in (79), i.e., $\frac{\exp(-ct)}{2cu^2}$, with that found in (95), i.e., $\frac{1}{\Delta} = \frac{K \exp(ct)}{c(K(\dot{u} - 2cu)^2 + 4c^2u^3)}$, yields a first integral that is just Int_{1a} , apart from an unessential multiplicative constant. ♦

Case (A)

If condition (29) is satisfied, then the Lagrangian (65) admits the Lie symmetry Γ_2 in (30) as Noether symmetry, with gauge function $F = u^{\frac{2(c-a)}{3a-2c}} \exp[2a(a-c)t/(3a-2c)](3a^2 - 5ac + 2c^2)/(2a)$, and consequently, Noether's theorem [9] yields the following first integral:

$$Int_A = u^{\frac{2(c-a)}{3a-2c}} \exp\left(\frac{2a(a-c)t}{3a-2c}\right) \left(K \left(\frac{\dot{u}}{u} - 2c \right)^2 + 4aK \frac{\dot{u}}{u} + 2c(3a-2c)u + 4aK(a-2c) \right) \quad (96)$$

of Equation (24), i.e.,

$$\ddot{u} = \frac{(4a-3c)\dot{u}^2}{(3a-2c)u} + \frac{(a-c)(a-2c)\dot{u}}{3a-2c} - \frac{au}{K(3a-2c)} (2K(a-c)^2 + c(3a-2c)u). \quad (97)$$

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int_A yields the following first integral of system (1)–(2):

$$I_A = w_2^{\frac{2(c-a)}{3a-2c}} \exp\left(\frac{2a(a-c)t}{3a-2c}\right) \left(K \left(a \frac{w_2}{w_1} - (3a-2c) \right)^2 + 2c(3a-2c)w_2 \right). \quad (98)$$

Another Jacobi last multiplier, and consequently Lagrangian, can be obtained by means of the two Lie symmetries (30) through (9), i.e.,

$$M_{A[1,2]} = \exp((c-a)t) \frac{3a-2c}{a(a-c)} [k\dot{u}^2 + 4Ku(a-c)\dot{u} + 2u^2(2K(a-c)^2 + c(3a-2c)u)]^{-1}. \quad (99)$$

The corresponding Lagrangian admits one Lie symmetry as Noether symmetry, with gauge function $F = t/K$, i.e., Γ_2 in (30), and consequently, the following first integral:

$$Int_{A[1,2]} = \frac{1}{2(a-c)aK} \left((3a-2p) \log(4K(a-c)^2u^2 + 2(3a-2c)cu^3 + 4K(a-c)u\dot{u} + K\dot{u}^2) + 2(a-c)at - 2(4a-3c) \log(u) \right). \quad (100)$$

We remark that the two first integrals Int_A and $Int_{A[1,2]}$ are not functionally independent.

Case (B)

If condition (36) is satisfied, then the Lagrangian (65) admits the Lie symmetry Γ_2 in (37) as Noether symmetry, with gauge function $F = c^2u^{h/c} \exp[2(c-h)t]/((2c-h)u^2)$, and consequently, Noether's theorem [9] yields the following first integral ($h \neq c$):

$$Int_B = \frac{u^{h/c} \exp[2(c-h)t]}{4cK(c-h)u^4} \left(4c^3Ku^2 - 4c^2hKu^2 - 4c^3u^3 - 4c^2Ku\dot{u} + 4chKu\dot{u} + cK\dot{u}^2 - hK\dot{u}^2 \right) \quad (101)$$

of Equation (24), i.e.,

$$\ddot{u} = \frac{4c^3Ku^2 - 4c^2hKu^2 - 4c^3u^3 - 6c^2Kui + 4Khcu + 4cK\dot{u}^2 - Kh\dot{u}^2}{2cKu}. \quad (102)$$

Replacing u with w_2 and \dot{u} with $\dot{w}_2 = aw_2(1 - w_2/w_1)$, namely the RHS of Equation (2), into Int_B yields the following first integral of system (1)–(2):

$$I_B = w_2^{\frac{h-c}{c}} \exp[2(c-h)t] \left(K(c-h)\frac{w_2}{w_1^2} - c \right). \quad (103)$$

Another Jacobi last multiplier, and consequently Lagrangian, can be obtained by means of the two Lie symmetries (37) through (9), i.e.,

$$M_{B[1,2]} = \frac{K \exp(ct)}{K(c-h)\dot{u}^2 - 4cK(c-h)u\dot{u} + 4c^2u^2(Kc - Kh - cu)}. \quad (104)$$

The corresponding Lagrangian admits only the Lie symmetry Γ_2 in (37) as Noether symmetry, with gauge function $F = t$, and consequently, the following first integral:

$$Int_{B[1,2]} = \frac{1}{2c(c-h)} \left(2c(c-h)t + (h-4c)\log(u) + \log(4Kc^2(c-h)^2u^2 - 4c^3u^3 - 4cK(c-h)u\dot{u} + K(c-h)\dot{u}^2) \right). \quad (105)$$

We remark that the two first integrals Int_B and $Int_{B[1,2]}$ are not functionally independent.

Subcase (B.1)

If condition (41) is satisfied, then the Lagrangian (65) admits all three Lie symmetries in (43) as Noether symmetries, with gauge functions $F_2 = \frac{2c}{\sqrt{u} \exp(ct)}$, $F_3 = \frac{2c \exp(ct)}{\sqrt{u}}$, and consequently, Noether's theorem [9] yields the following three first integrals:

$$\begin{aligned} Int1_{B_1} &= \frac{8c^2u^3 - 4c^2Ku^2 + K\dot{u}^2}{4cKu^2\sqrt{u}}, \\ Int2_{B_1} &= \frac{4c^2Ku^2 + 8c^2u^3 - 4cKui + K\dot{u}^2}{4cKu^2\sqrt{u} \exp(ct)}, \\ Int3_{B_1} &= \exp(ct) \frac{4c^2Ku^2 + 8c^2u^3 + 4cKui + K\dot{u}^2}{4cKu^2\sqrt{u}}, \end{aligned} \quad (106)$$

of Equation (24), i.e.,

$$\ddot{u} = \frac{5K\dot{u}^2 - 4c^2Ku^2 - 8c^2u^3}{4Ku}. \quad (107)$$

Three more Jacobi last multipliers, and consequently, three Lagrangians, can be obtained by means of the three Lie symmetries (43) through (9), i.e.,

$$\begin{aligned} M_{B_1[1,2]} &= -\frac{2K \exp(ct)}{c(4c^2Ku^2 + 8c^2u^3 - 4cKui + K\dot{u}^2)}, \\ M_{B_1[1,3]} &= \frac{2K}{c \exp(ct)(4c^2Ku^2 + 8c^2u^3 + 4cKui + K\dot{u}^2)}, \\ M_{B_1[2,3]} &= -\frac{K}{c(4c^2Ku^2 - 8c^2u^3 - K\dot{u}^2)}. \end{aligned} \quad (108)$$

Since the ratio of two Jacobi last multipliers is a first integral, then we can obtain three first integrals of Equation (107), *i.e.*,

$$\begin{aligned}\frac{M_{B_1[1,2]}}{M_{B_1[1,3]}} &= -\exp(2ct) \frac{4c^2Ku^2 + 8c^2u^3 + 4cKu\dot{u} + K\dot{u}^2}{4c^2Ku^2 + 8c^2u^3 - 4cKu\dot{u} + K\dot{u}^2}, \\ \frac{M_{B_1[1,2]}}{M_{B_1[2,3]}} &= 2\exp(ct) \frac{4c^2Ku^2 - 8c^2u^3 - K\dot{u}^2}{4c^2Ku^2 + 8c^2u^3 - 4cKu\dot{u} + K\dot{u}^2}, \\ \frac{M_{B_1[1,3]}}{M_{B_1[2,3]}} &= -2\exp(-ct) \frac{4c^2Ku^2 - 8c^2u^3 - K\dot{u}^2}{4c^2Ku^2 + 8c^2u^3 + 4cKu\dot{u} + K\dot{u}^2}.\end{aligned}\quad (109)$$

The Lagrangian corresponding to $M_{B_1[1,2]}$ admits only the Lie symmetry Γ_2 in (43) as Noether symmetry, with gauge function $F = t$, and consequently, Noether's theorem [9] yields the following first integral:

$$Int_{B_1[1,2]} = \frac{1}{2c} \left(5 \log(u) + 2ct - 2 \log(4c^2Ku^2 + 8c^2u^3 - 4cKu\dot{u} + K\dot{u}^2) \right). \quad (110)$$

The Lagrangian corresponding to $M_{B_1[1,3]}$ admits only the Lie symmetry Γ_3 in (43) as Noether symmetry, with gauge function $F = t$, and consequently, Noether's theorem [9] yields the following first integral:

$$Int_{B_1[1,3]} = \frac{1}{2c} \left(-5 \log(u) + 2ct + 2 \log(4c^2Ku^2 + 8c^2u^3 + 4cKu\dot{u} + K\dot{u}^2) \right). \quad (111)$$

The Lagrangian corresponding to $M_{B_1[2,3]}$ admits only the Lie symmetry Γ_1 in (43) as Noether symmetry, and consequently, Noether's theorem [9] yields the following first integral:

$$Int_{B_1[2,3]} = \frac{1}{4c} \left(-5 \log(u) + 2 \log(4c^2Ku^2 - 8c^2u^3 - K\dot{u}^2) \right). \quad (112)$$

Of course, of all these nine first integrals, only two are functionally independent.

5. Discussion and Final Remarks

In this paper, we have determined different cases that fit the qualitative prediction (in [7], it was assumed $a = 1$) given in [7]. Assuming $a = 1$, then the solution (56) with $c = 1/2, h = 3/4$ corresponds (since $h > c$ and $h < 1$) to Proposition 2, namely the case of asymptotic death for all, with both the population and the resources going to zero at infinity. Furthermore, the ratio of population over resources goes asymptotically to 2, *i.e.*, $\frac{c-1}{h-1}$, as predicted in [7]. Indeed, solution (56) shows that resources decrease, while the population grows exponentially *for an extended period of time* followed by a *catastrophic elimination* [7], as can also be seen in Figure 1. However, for at least a period of time, resources may increase along with the population, as shown in Figure 2, if the carrying capacity K is large enough.

In [7], one first integral was derived under the condition $h = (c+1)/2$ (Proposition 6). It corresponds to our first integral I in (69) under the condition $h = (c+a)/2$. We have derived many other first integrals and also the general solution of system (1)–(2) in closed form, as well as the general solution of Equation (24) in implicit form:

the first integral I_0 in (74) if $a = 2h, c = h$;

the first integral I_{1a} in (84) and also the general solution (87) if $a = h = c$;

the first integral I_{1b} in (90) and also the general solution (94) if $a = h, h = 2c$;

the first integral I_A in (98) if $h = \frac{a(2a - c)}{3a - 2c}$, that corresponds to Proposition 3 ($h = 2, c = 4/3$);

the first integral I_B in (103) if $a = 2c$, that corresponds to Proposition 1 ($h = 3/4, c = 1/2$).

Last, but not least, we were able to find a Lagrangian (60) for system (1)–(2) and any value of the involved parameters (if $a = 2h$ the Lagrangian is (63)).

Recently, archeological findings have pointed out that Polynesian rats may have greatly contributed to the tree destruction on Easter Island [23]. Consequently, in [24], rats have been included in the mathematical model, leading to three nonlinear first-order differential equations. The chaotic nature of that system has subsequently been determined in [25]. Can symmetries help since they have always been associated with non-chaotic systems? We recall that in [26], Lie symmetries yielded an exact transformation that turned a butterfly into a tornado (and *vice versa*), and the chaotic features of the Lorenz system [27] were thus tamed. Could the chaotic features of the rats-trees-islanders model be equally tamed? We hope to address this question in the near future.

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Author Contributions

The authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Edwards, M.; Nucci, M.C. Application of Lie group analysis to a core group model for sexually transmitted diseases. *J. Nonlinear Math. Phys.* **2006**, *13*, 211–230.
2. Gradassi, A.; Nucci, M.C. Hidden linearity in systems for competition with evolution in ecology and finance. *J. Math. Anal. Appl.* **2007**, *333*, 274–294.
3. Nucci, M.C. Using lie symmetries in epidemiology. *Electron. J. Diff. Eqns. Conf.* **2005**, *12*, 87–101.
4. Nucci, M.C.; Leach, P.G.L. An integrable S-I-S model. *J. Math. Anal. Appl.* **2004**, *290*, 506–518.
5. Nucci, M.C.; Leach, P.G.L. Lie integrable cases of the simplified multistrain/two-stream model for tuberculosis and Dengue fever. *J. Math. Anal. Appl.* **2007**, *333*, 430–449.
6. Torrisi, V.; Nucci, M.C. Application of Lie group analysis to a mathematical model which describes HIV transmission. In *The Geometrical Study of Differential Equations*; Leslie, J.A., Hobart, T.P., Eds.; American Mathematical Society: Providence, RI, USA, 2011; pp. 11–20.

7. Basener, B.; Ross, D.S. Booming and crashing populations and Easter Island. *SIAM J. Appl. Math.* **2005**, *65*, 684–701.
8. Nucci, M.C.; Tamizhmani, K.M. Lagrangians for biological models. *J. Nonlinear Math. Phys.* **2012**, *19*, 1250021:1–1250021:23.
9. Noether, E. Invariante Variationsprobleme. *Nachr. Ges. Wiss. Gött. Math. Phys. Kl.* **1918**, *1918*, 235–257.
10. Jacobi, C.G.J. Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi. *J. Reine Angew. Math.* **1844**, *27*, 199–268, **1845**, *29*, 213–279, 333–376.
11. Jacobi, C.G.J. *Vorlesungen über Dynamik. Nebst Fünf Hinterlassenen Abhandlungen Desselben Herausgegeben von A. Clebsch*; Druck und Verlag von Georg Reimer: Berlin, Germany, 1884.
12. Jacobi, C.G.J. *Jacobi's Lectures on Dynamics*; Clebsch, A., Ed.; Industan Book Agency: New Delhi, India, 2009.
13. Lie, S. Verallgemeinerung und neue Verwerthung der Jacobischen Multiplikator-Theorie. *Christ. Forh.* **1874**, 255–274.
14. Lie, S. *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*; Teubner: Leipzig, Germany, 1912.
15. Bianchi, L. *Lezioni Sulla Teoria Dei Gruppi Continui Finiti di Trasformazioni*; Enrico Spoerri: Pisa, Italy, 1918.
16. Whittaker, E.T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*; Cambridge University Press: Cambridge, UK, 1999.
17. Nucci, M.C.; Leach, P.G.L. Lagrangians galore. *J. Math. Phys.* **2007**, *48*, 123510:1–123510:16.
18. Trubatch, S.L.; Franco, A. Canonical procedures for population dynamics. *J. Theor. Biol.* **1974**, *48*, 299–324.
19. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer-Verlag: Berlin, Germany, 1986.
20. Nucci, M.C. Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: Manual and floppy disk. In *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends in Theoretical Developments and Computational Methods*; Ibragimov, N.H., Ed.; CRC Press: Boca Raton, FL, USA, 1996; pp. 415–481.
21. Ibragimov, N.H. *Elementary Lie Group Analysis and Ordinary Differential Equations*; Wiley: Chichester, UK, 1999.
22. Leach, P.G.L. Equivalence classes of second-order ordinary differential equations with only a three-dimensional Lie algebra of point symmetries and linearisation. *J. Math. Anal. Appl.* **2003**, *284*, 31–48.
23. Hunt, T. Rethinking the fall of Easter Island: New evidence points to an alternative explanation for a civilization's collapse. *Am. Sci.* **2006**, *94*, 412–419.
24. Basener, W.; Brooks, B.P.; Radin, M.A.; Wiandt, T. Dynamics of a discrete population model for extinction and sustainability in ancient civilizations. *Nonlinear Dyn. Psychol. Life Sci.* **2008**, *12*, 29–53.
25. Sprott, J.C. Chaos in Easter Island ecology. *Nonlinear Dyn. Psychol. Life Sci.* **2011**, *15*, 445–454.

26. Nucci, M.C. Lorenz integrable system moves à la Poincaré. *J. Math. Phys.* **2003**, *44*, 4107–4118.
27. Lorenz, E.N. Deterministic nonperiodic flow. *J. Atmos. Sci.* **1963**, *20*, 130–141.

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