Symmetry **2015**, *7*, 2150-2180; doi:10.3390/sym7042150

OPEN ACCESS

symmetry

ISSN 2073-8994

www.mdpi.com/journal/symmetry

Article

Quaternifications and Extensions of Current Algebras on S^3

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Academic Editor: Vladimir Shpilrain

Received: 19 August 2015 / Accepted: 16 November 2015 / Published: 27 November 2015

Abstract: Let $\mathbf{H}$ be the quaternion algebra. Let $\mathfrak{g}$ be a complex Lie algebra and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. The quaternification $\mathfrak{g}^{\mathbf{H}} = (\mathbf{H}\otimes U(\mathfrak{g}), [\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}})$ of $\mathfrak{g}$ is defined by the bracket $\left[\mathbf{z}\otimes X\,,\,\mathbf{w}\otimes Y\right]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{z}\cdot\mathbf{w})\otimes(XY) - (\mathbf{w}\cdot\mathbf{z})\otimes(YX)$, for $\mathbf{z}$, $\mathbf{w}\in\mathbf{H}$ and the basis vectors X and Y of $U(\mathfrak{g})$. Let $S^3\mathbf{H}$ be the (non-commutative) algebra of $\mathbf{H}$-valued smooth mappings over S^3 and let $S^3\mathfrak{g}^{\mathbf{H}} = S^3\mathbf{H}\otimes U(\mathfrak{g})$. The Lie algebra structure on $S^3\mathfrak{g}^{\mathbf{H}}$ is induced naturally from that of $\mathfrak{g}^{\mathbf{H}}$. We introduce a 2-cocycle on $S^3\mathfrak{g}^{\mathbf{H}}$ by the aid of a tangential vector field on $S^3\subset\mathbf{C}^2$ and have the corresponding central extension $S^3\mathfrak{g}^{\mathbf{H}}\oplus(\mathbf{C}a)$. As a subalgebra of $S^3\mathbf{H}$ we have the algebra of Laurent polynomial spinors $\mathbf{C}[\phi^{\pm}]$ spanned by a complete orthogonal system of eigen spinors $\{\phi^{\pm(m,l,k)}\}_{m,l,k}$ of the tangential Dirac operator on S^3. Then $\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})$ is a Lie subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}$. We have the central extension $\widehat{\mathfrak{g}}(a) = (\,\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})\,)\oplus(\mathbf{C}a)$ as a Lie-subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}\oplus(\mathbf{C}a)$. Finally we have a Lie algebra $\widehat{\mathfrak{g}}$ which is obtained by adding to $\widehat{\mathfrak{g}}(a)$ a derivation d which acts on $\widehat{\mathfrak{g}}(a)$ by the Euler vector field d_0. That is the $\mathbf{C}$-vector space $\widehat{\mathfrak{g}} = (\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g}))\oplus(\mathbf{C}a)\oplus(\mathbf{C}d)$ endowed with the bracket $\left[\,\phi_1\otimes X_1+\lambda_1 a+\mu_1 d\,,\phi_2\otimes X_2+\lambda_2 a+\mu_2 d\,\right]_{\widehat{\mathfrak{g}}} = (\phi_1\phi_2)\otimes(X_1\,X_2) - (\phi_2\phi_1)\otimes(X_2X_1) + \mu_1 d_0\phi_2\otimes X_2 - \mu_2 d_0\phi_1\otimes X_1 + (X_1|X_2)c(\phi_1,\phi_2)a$. When $\mathfrak{g}$ is a simple Lie algebra with its Cartan subalgebra $\mathfrak{h}$ we shall investigate the weight space decomposition of $\widehat{\mathfrak{g}}$ with respect to the subalgebra $\widehat{\mathfrak{h}} = (\phi^{+(0,0,1)}\otimes\mathfrak{h})\oplus(\mathbf{C}a)\oplus(\mathbf{C}d)$.

Keywords: infinite dimensional lie algebras; current algebra; lie algebra extensions; quaternion analysis

MSC classifications: 81R10; 17B65; 17B67; 22E67

1. Introduction

The set of smooth mappings from a manifold to a Lie algebra has been a subject of investigation both from a purely mathematical standpoint and from quantum field theory. In quantum field theory they appear as a current algebra or an infinitesimal gauge transformation group. Loop algebras are the simplest example. Loop algebras and their representation theory have been fully worked out. A loop algebra valued in a simple Lie algebra or its complexification turned out to behave like a simple Lie algebra and the highly developed theory of finite dimensional Lie algebra was extended to such loop algebras. Loop algebras appear in the simplified model of quantum field theory where the space is one-dimensional and many important facts in the representation theory of loop algebra were first discovered by physicists. As is well known, A. Belavin *et al.* [1] constructed two-dimensional conformal field theory based on the irreducible representations of Virasoro algebra. It turned out that in many applications to field theory one must deal with certain extensions of the associated loop algebra rather than the loop algebra itself. The central extension of a loop algebra is called an affine Lie algebra and the highest weight theory of finite dimensional Lie algebra was extended to this case. [2–5] are good references to study these subjects.

In this paper we shall investigate a generalization of affine Lie algebras to the Lie algebra of mappings from three-sphere S^3 to a Lie algebra. As an affine Lie algebra is a central extension of the Lie algebra of smooth mappings from S^1 to the complexification of a Lie algebra, so our objective is an extension of the Lie algebra of smooth mappings from S^3 to the quaternification of a Lie algebra. As for the higher dimensional generalization of loop groups, J. Mickelsson introduced an abelian extension of current groups $Map(S^3, SU(N))$ for $N \geq 3$ [6]. It is related to the Chern-Simons function on the space of $SU(N)$-connections and the associated current algebra $Map(S^3, su(N))$ has an abelian extension $Map(S^3, su(N)) \oplus \mathcal{A}_3^*$ by the affine dual of the space $\mathcal{A}_3$ of connections over S^3 [7]. In [4] it was shown that, for any smooth manifold M and a simple Lie algebra $\mathfrak{g}$, there is a universal central extension of the Lie algebra $Map(M, \mathfrak{g})$. The kernel of the extension is given by the space of complex valued 1-forms modulo exact 1-forms; $\Omega^1(M)/d\Omega^0(M)$. It implies that any extension is a weighted linear combination of extensions obtained as a pull back of the universal extension of the loop algebra $L\mathfrak{g}$ by a smooth loop $f : S^1 \longrightarrow M$. We are dealing with central extensions of the Lie algebra of smooth mappings from S^3 to the quaternification of a Lie algebra. Now we shall give a brief explanation of each section.

Let $\mathbf{H}$ be the quaternion numbers. In this paper we shall denote a quaternion $a + jb \in \mathbf{H}$ by $\begin{pmatrix} a \\ b \end{pmatrix}$. This comes from the identification of $\mathbf{H}$ with the matrix algebra

$$\mathfrak{mj}(2, \mathbf{C}) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \ : \quad a, b \in \mathbf{C} \right\}.$$

$\mathbf{H}$ becomes an associative algebra and the Lie algebra structure $(\,\mathbf{H}, [\ ,\ ]_{\mathbf{H}}\,)$ is induced on it. The trace of $\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbf{H}$ is defined by $tr\,\mathbf{a} = a + \bar{a}$. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}$ we have $tr\,([\mathbf{u}, \mathbf{v}]_{\mathbf{H}} \cdot \mathbf{w}\,) = tr\,(\mathbf{u} \cdot [\mathbf{v}, \mathbf{w}]_{\mathbf{H}}\,)$.

Let $(\mathfrak{g}, [\quad,\quad]_{\mathfrak{g}})$ be a complex Lie algebra. Let $U(\mathfrak{g})$ be the enveloping algebra. The quaternification of $\mathfrak{g}$ is defined as the vector space $\mathfrak{g}^{\mathbf{H}} = \mathbf{H} \otimes U(\mathfrak{g})$ endowed with the bracket

$$\left[\mathbf{z} \otimes X\,,\, \mathbf{w} \otimes Y\right]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{z} \cdot \mathbf{w}) \otimes (XY) - (\mathbf{w} \cdot \mathbf{z}) \otimes (YX)\,, \tag{1}$$

for $\mathbf{z}$, $\mathbf{w} \in \mathbf{H}$ and the basis vectors X and Y of $U(\mathfrak{g})$. It extends the Lie algebra structure $(\mathfrak{g}, [\quad,\quad]_{\mathfrak{g}})$ to $\left(\mathfrak{g}^{\mathbf{H}}, [\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}}\right)$. The quaternions $\mathbf{H}$ give also a half spinor representation of $Spin(4)$. That is, $\Delta = \mathbf{H} \otimes \mathbf{C} = \mathbf{H} \oplus \mathbf{H}$ gives an irreducible complex representation of the Clifford algebra $\mathrm{Clif}(\mathbf{R}^4)$: $\mathrm{Clif}(\mathbf{R}^4) \otimes \mathbf{C} \simeq \mathrm{End}(\Delta)$, and Δ decomposes into irreducible representations $\Delta^{\pm} = \mathbf{H}$ of $\mathrm{Spin}(4)$. Let $S^{\pm} = \mathbf{C}^2 \times \Delta^{\pm}$ be the trivial even (respectively odd) spinor bundle. A section of spinor bundle is called a spinor. The space of even half spinors $C^{\infty}(S^3, S^+)$ is identified with the space $S^3\mathbf{H} = Map(S^3, \mathbf{H})$. Now the space $S^3\mathfrak{g}^{\mathbf{H}} = S^3\mathbf{H} \otimes U(\mathfrak{g})$ becomes a Lie algebra with respect to the bracket:

$$[\,\phi \otimes X\,,\, \psi \otimes Y\,]_{S^3\mathfrak{g}^{\mathbf{H}}} = (\phi\psi) \otimes (XY) - (\psi\phi) \otimes (YX), \tag{2}$$

for the basis vectors X and Y of $U(\mathfrak{g})$ and $\phi,\, \psi \in S^3\mathbf{H}$. In the sequel we shall abbreviate the Lie bracket $[\,,\,]_{S^3\mathfrak{g}^{\mathbf{H}}}$ simply to $[\,,\,]$. Such an abbreviation will be often adopted for other Lie algebras.

Recall that the central extension of a loop algebra $L\mathfrak{g} = \mathbf{C}[z, z^{-1}] \otimes \mathfrak{g}$ is the Lie algebra $(L\mathfrak{g} \oplus \mathbf{C}a\,, [\,,\,]_c\,)$ given by the bracket

$$[P \otimes X, Q \otimes Y]_c = PQ \otimes [X, Y] + (X|Y)c(P, Q)a\,,$$

with the aid of the 2-cocycle $c(P, Q) = \frac{1}{2\pi}\int_{S^1}(\frac{d}{dz}P)\,Q\,dz$, where $(\cdot|\cdot)$ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$, [2]. We shall give an analogous 2-cocycle on $S^3\mathbf{H}$. Let θ be the vector field on S^3 defined by

$$\theta = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}. \tag{3}$$

For $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in S^3\mathbf{H}$, we put

$$\Theta\,\varphi = \frac{1}{2\sqrt{-1}}\begin{pmatrix} \theta\, u \\ \theta\, v \end{pmatrix}.$$

Let $c:\ S^3\mathbf{H} \times S^3\mathbf{H} \longrightarrow \mathbf{C}$ be the bilinear form given by

$$c(\phi_1, \phi_2) = \frac{1}{2\pi^2}\int_{S^3} tr[\,\Theta\,\phi_1 \cdot \phi_2\,]d\sigma\,, \quad \phi_1,\, \phi_2 \in S^3\mathbf{H}. \tag{4}$$

c defines a 2-cocycle on the algebra $S^3\mathbf{H}$. That is, c satisfies the following equations:

$$c(\phi_1,\, \phi_2) = -\,c(\phi_2,\, \phi_1)$$

and

$$c(\,\phi_1 \cdot \phi_2\,,\,\phi_3) + c(\,\phi_2 \cdot \phi_3\,,\,\phi_1) + c(\,\phi_3 \cdot \phi_1\,,\,\phi_2) = 0\,.$$

We extend c to the 2-cocycle on $S^3\mathfrak{g}^{\mathbf{H}}$ by

$$c(\,\phi_1 \otimes X\,,\,\phi_2 \otimes Y\,) = \,(X|Y)\,c(\phi_1, \phi_2), \tag{5}$$

where $(\,\cdot|\cdot)$ is the non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ extended to $U(\mathfrak{g})$.

Let a be an indefinite element. The Lie algebra extension of $S^3\mathfrak{g}^{\mathbf{H}}$ by the 2-cocycle c is the $\mathbf{C}$-vector space $S^3\mathfrak{g}^{\mathbf{H}} \oplus \mathbf{C}a$ endowed with the following bracket:

$$\begin{aligned} [\,\phi \otimes X\,,\,\psi \otimes Y\,]^{\wedge} &= (\phi \cdot \psi) \otimes (X\,Y) - (\psi \cdot \phi) \otimes (Y\,X) + c(\phi, \psi)(X|Y)\,a\,, \\ [\,a\,, \phi \otimes X\,]^{\wedge} &= 0\,, \end{aligned} \tag{6}$$

for the basis vectors X and Y of $U(\mathfrak{g})$ and $\phi,\,\psi\,\in S^3\mathbf{H}$.

In Section 2 we shall review the theory of spinor analysis after [8,9]. Let $D\,:\,S^+\longrightarrow S^-$ be the (half spinor) Dirac operator. Let $D = \gamma_+(\frac{\partial}{\partial n} - \partial\!\!\!/)$ be the polar decomposition on $S^3 \subset \mathbf{C}^2$ of the Dirac operator, where $\partial\!\!\!/$ is the tangential Dirac operator on S^3 and γ_+ is the Clifford multiplication of the unit normal derivative on S^3. The eigenvalues of $\partial\!\!\!/$ are given by $\{\frac{m}{2},\,-\frac{m+3}{2}\,;\,m = 0, 1, \cdots\}$, with multiplicity $(m+1)(m+2)$. We have an explicitly written formula for eigenspinors $\{\phi^{+(m,l,k)},\,\phi^{-(m,l,k)}\}_{0\le l\le m,\,0\le k\le m+1}$ corresponding to the eigenvalue $\frac{m}{2}$ and $-\frac{m+3}{2}$ respectively and they give rise to a complete orthogonal system in $L^2(S^3, S^+)$. A spinor ϕ on a domain $G \subset \mathbf{C}^2$ is called a *harmonic spinor* on G if $D\phi = 0$. Each $\phi^{+(m,l,k)}$ is extended to a harmonic spinor on $\mathbf{C}^2$, while each $\phi^{-(m,l,k)}$ is extended to a harmonic spinor on $\mathbf{C}^2 \setminus \{0\}$. Every harmonic spinor φ on $\mathbf{C}^2 \setminus \{0\}$ has a Laurent series expansion by the basis $\phi^{\pm(m,l,k)}$:

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z). \tag{7}$$

If only finitely many coefficients are non-zero it is called a *spinor of Laurent polynomial type* . The algebra of spinors of Laurent polynomial type is denoted by $\mathbf{C}[\phi^{\pm}]$. $\mathbf{C}[\phi^{\pm}]$ is a subalgebra of $S^3\mathbf{H}$ that is algebraically generated by $\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\phi^{+(1,0,1)} = \begin{pmatrix} z_2 \\ -\overline{z}_1 \end{pmatrix}$ and $\phi^{-(0,0,0)} = \begin{pmatrix} z_2 \\ \overline{z}_1 \end{pmatrix}$.

As a Lie subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}$, $\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})$ has the central extension by the 2-cocycle c. That is, the $\mathbf{C}$-vector space $\widehat{\mathfrak{g}}(a) = \mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g}) \oplus \mathbf{C}a$ endowed with the Lie bracket Equation (6) becomes an extension of $\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})$ with 1-dimensional center $\mathbf{C}a$. Finally we shall construct the Lie algebra which is obtained by adding to $\widehat{\mathfrak{g}}(a)$ a derivation d which acts on $\widehat{\mathfrak{g}}(a)$ by the Euler vector field d_0 on S^3. The Euler vector field is by definition $d_0 = \frac{1}{2}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + \overline{z}_2\frac{\partial}{\partial \overline{z}_2})$. We have the following fundamental property of the cocycle c .

$$c(\,d_0\phi_1\,, \phi_2\,) + c(\,\phi_1\,, d_0\phi_2\,) = 0.$$

Let $\widehat{\mathfrak{g}} = (\,\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})\,) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d)$. We endow $\widehat{\mathfrak{g}}$ with the bracket defined by

$$\begin{aligned} [\,\phi \otimes X\,,\,\psi \otimes Y\,]_{\widehat{\mathfrak{g}}} &= [\,\phi \otimes X\,,\,\psi \otimes Y\,]^{\wedge}\,, \qquad [\,a\,, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} = 0\,, \\ [\,d,\,a\,]_{\widehat{\mathfrak{g}}} &= 0\,, \qquad [\,d, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} = d_0\phi \otimes X\,. \end{aligned}$$

Then $(\,\widehat{\mathfrak{g}}\,,\,[\ ,\ ]_{\widehat{\mathfrak{g}}}\,)$ is an extension of the Lie algebra $\widehat{\mathfrak{g}}(a)$ on which d acts as d_0. In Section 4, when $\mathfrak{g}$ is a simple Lie algebra with its Cartan subalgebra $\mathfrak{h}$, we shall investigate the weight space decomposition of $\widehat{\mathfrak{g}}$ with respect to the subalgebra $\widehat{\mathfrak{h}} = (\phi^{+(0,0,1)} \otimes \mathfrak{h}) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d)$, the latter is a commutative subalgebra and $ad(\widehat{\mathfrak{h}})$ acts on $\widehat{\mathfrak{g}}$ diagonally. For this purpose we look at the representation of the adjoint action of $\mathfrak{h}$ on the enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{g} = \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$. Let $\Pi = \{\alpha_i;\, i = 1, \cdots, r = \mathrm{rank}\,\mathfrak{g}\} \subset \mathfrak{h}^*$ be the set of simple roots and $\{\alpha_i^\vee\,;\, i = 1, \cdots, r\,\} \subset \mathfrak{h}$ be the set of simple coroots. The Cartan matrix $A = (\,a_{ij}\,)_{i,j=1,\cdots,r}$ is given by $a_{ij} = \langle \alpha_i^\vee,\, \alpha_j\rangle$. Fix a standard set of generators $H_i = \alpha_i^\vee$, $X_i = X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, $Y_i = X_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$, so that $[X_i,\, Y_j] = H_j\delta_{ij}$, $[H_i,\, X_j] = -a_{ji}X_j$ and $[H_i,\, Y_j] = a_{ji}Y_j$. We see that the set of weights of the representation $(\,U(\mathfrak{g}),\, ad(\mathfrak{h}))$ becomes

$$\Sigma = \{\,\sum_{i=1}^{r} k_i\alpha_i \in \mathfrak{h}^*\,; \quad k_i \in \mathbf{Z},\, i = 1, \cdots, r\,\}. \tag{8}$$

The weight space of $\lambda \in \Sigma$ is by definition

$$\mathfrak{g}^U_\lambda \;=\; \{\xi \in U(\mathfrak{g})\,;\, ad(h)\xi = \lambda(h)\xi,\, \forall h \in \mathfrak{h}\}, \tag{9}$$

when $\mathfrak{g}^U_\lambda \neq 0$. Then, given $\lambda = \sum_{i=1}^{r} k_i\alpha_i$, we have

$$\mathfrak{g}^U_\lambda = \mathbf{C}[Y_1^{q_1} \cdots Y_r^{q_r}\, H_1^{l_1} \cdots H_r^{l_r}\, X_1^{p_1} \cdots X_r^{p_r}\,;\, p_i,\, q_i,\, l_i \in \mathbf{N} \cup 0,\;\, k_i = p_i - q_i\,,\, i = 1, \cdots, r\,]\,.$$

The weight space decomposition becomes

$$U(\mathfrak{g}) = \bigoplus_{\lambda\in\Sigma} \mathfrak{g}^U_\lambda\,, \quad \mathfrak{g}^U_0 \supset U(\mathfrak{h}). \tag{10}$$

Now we proceed to the representation $(\,\widehat{\mathfrak{g}},\, ad(\widehat{\mathfrak{h}})\,)$. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ can be regarded naturally as a subspace of $\widehat{\mathfrak{h}}^*$. So $\Sigma \subset \mathfrak{h}^*$ is seen to be a subset of $\widehat{\mathfrak{h}}^*$. We define $\delta \in \widehat{\mathfrak{h}}^*$ by putting $\langle \delta,\, h_i\,\rangle = \langle \delta, a\rangle = 0$, $1 \leqq i \leqq r$, and $\langle \delta, d\rangle = 1$. Then the set of weights $\widehat{\Sigma}$ of the representation $(\,\widehat{\mathfrak{g}},\, ad(\widehat{\mathfrak{h}})\,)$ is

$$\begin{aligned} \widehat{\Sigma} \;&=\; \left\{\frac{m}{2}\delta + \lambda; \quad \lambda \in \Sigma\,,\, m \in \mathbf{Z}\right\} \\ &\quad \bigcup \left\{\frac{m}{2}\delta; \quad m \in \mathbf{Z}\right\}. \end{aligned} \tag{11}$$

The weight space decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} \;=\; \bigoplus_{m\in\mathbf{Z}} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \bigoplus \left(\bigoplus_{\lambda\in\Sigma,\, m\in\mathbf{Z}} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda}\right). \tag{12}$$

Each weight space is given as follows

$$\begin{aligned}
\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} &= \mathbf{C}[\phi^{\pm};\, m\,] \otimes \mathfrak{g}^U_\lambda \qquad \text{for } m \neq 0 \text{ and } \lambda \neq 0\,, \\
\widehat{\mathfrak{g}}_{\frac{m}{2}\delta} &= \mathbf{C}[\phi^{\pm};\, m\,] \otimes \mathfrak{g}^U_0 \qquad \text{for } m \neq 0\,, \\
\widehat{\mathfrak{g}}_{0\delta} &= (\,\mathbf{C}[\phi^{\pm}; 0\,] \otimes \mathfrak{g}^U_0\,) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d) \,\supset\, \widehat{\mathfrak{h}}\,,
\end{aligned}$$

where

$$\mathbf{C}[\phi^{\pm}; m\,] \,=\, \left\{\varphi \in \mathbf{C}[\phi^{\pm}];\; |z|^m \varphi(\frac{z}{|z|}) = \varphi(z)\right\}.$$

2. Quaternification of a Lie Algebra

2.1. Quaternion Algebra

The quaternions $\mathbf{H}$ are formed from the real numbers $\mathbf{R}$ by adjoining three symbols $i,\ j,\ k$ satisfying the identities:

$$\begin{aligned}
i^2 &= j^2 = k^2 = -1\,, \\
ij &= -ji = k, \quad jk = -kj = i, \quad ki = -ik = j\,.
\end{aligned} \tag{13}$$

A general quaternion is of the form $x = x_1 + x_2 i + x_3 j + x_4 k$ with $x_1, x_2, x_3, x_4 \in \mathbf{R}$. By taking $x_3 = x_4 = 0$ the complex numbers $\mathbf{C}$ are contained in $\mathbf{H}$ if we identify i as the usual complex number. Every quaternion x has a unique expression $x = z_1 + jz_2$ with $z_1, z_2 \in \mathbf{C}$. This identifies $\mathbf{H}$ with $\mathbf{C}^2$ as $\mathbf{C}$-vector spaces. The quaternion multiplication will be from the right $x \longrightarrow xy$ where $y = w_1 + jw_2$ with $w_1,\ w_2 \in \mathbf{C}$:

$$xy = (z_1 + jz_2\,)(w_1 + jw_2\,) = (z_1 w_1 - \overline{z}_2 w_2) + j(\overline{z}_1 w_2 + z_2 w_1). \tag{14}$$

The multiplication of a $g = a + jb \in \mathbf{H}$ to $\mathbf{H}$ from the left yields an endomorphism in $\mathbf{H}$: $\{x \longrightarrow gx\} \in End_{\mathbf{H}}(\mathbf{H})$. If we look on it under the identification $\mathbf{H} \simeq \mathbf{C}^2$ mentioned above we have the $\mathbf{C}$-linear map

$$\mathbf{C}^2 \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbf{C}^2\,. \tag{15}$$

This establishes the $\mathbf{R}$-linear isomorphism

$$\mathbf{H} \ni a + jb \xrightarrow{\simeq} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathfrak{mj}(2, \mathbf{C}), \tag{16}$$

where we defined

$$\mathfrak{mj}(2,\mathbf{C}) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : \quad a, b \in \mathbf{C} \right\}. \tag{17}$$

The complex matrices corresponding to $i,\ j,\ k \in \mathbf{H}$ are

$$e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{18}$$

These are the basis of the Lie algebra $\mathfrak{su}(2)$. Thus we have the identification of the following objects

$$\mathbf{H} \simeq \mathfrak{mj}(2,\mathbf{C}) \simeq \mathbf{R} \oplus \mathfrak{su}(2). \tag{19}$$

The correspondence between the elements is given by

$$a + jb \equiv \begin{pmatrix} a \\ b \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \longleftrightarrow s + pe_1 + qe_2 + re_3\,, \tag{20}$$

where $a = s + ir$, $b = q + ip$.

$\mathbf{H}$ becomes an associative algebra with the multiplication law defined by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1 w_1 - \overline{z}_2 w_2 \\ \overline{z}_1 w_2 + z_2 w_1 \end{pmatrix}, \tag{21}$$

which is the rewritten formula of Equation (14) and the right-hand side is the first row of the matrix multiplication

$$\begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} w_1 & -\overline{w}_2 \\ w_2 & \overline{w}_1 \end{pmatrix}.$$

It implies the Lie bracket of two vectors in $\mathbf{H}$, that becomes

$$\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} z_2 \overline{w}_2 - \overline{z}_2 w_2 \\ (w_1 - \overline{w}_1) z_2 - (z_1 - \overline{z}_1) w_2 \end{pmatrix}. \tag{22}$$

These expressions are very convenient to develop the analysis on $\mathbf{H}$, and give an interpretation on the quaternion analysis by the language of spinor analysis.

Proposition 1. *Let* $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbf{H}$. *Then the trace of* $\mathbf{z} \cdot \mathbf{w} \in \mathbf{H} \simeq \mathfrak{mj}(2,\mathbf{C})$ *is given by*

$$tr\,(\mathbf{z} \cdot \mathbf{w}) = 2\mathrm{Re}(z_1 w_1 - \overline{z}_2 w_2), \tag{23}$$

and we have, for $\mathbf{z}_1,\ \mathbf{z}_2,\ \mathbf{z}_3 \in \mathbf{H}$,

$$tr\,(\,[\,\mathbf{z}_1,\ \mathbf{z}_2\,] \cdot \mathbf{z}_3\,) = tr\,(\,\mathbf{z}_1 \cdot [\,\mathbf{z}_2\,,\ \mathbf{z}_3\,]\,)\,. \tag{24}$$

The center of the Lie algebra $\mathbf{H}$ is $\left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \in \mathbf{H};\, t \in \mathbf{R} \right\} \simeq \mathbf{R}$, and Equation (19) says that $\mathbf{H}$ is the trivial central extension of $\mathfrak{su}(2)$.

$\mathbf{R}^3$ being a vector subspace of $\mathbf{H}$:

$$\mathbf{R}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Longleftrightarrow \begin{pmatrix} ir \\ q+ip \end{pmatrix} = ir + j(q+ip) \in \mathbf{H}, \tag{25}$$

we have the action of $\mathbf{H}$ on $\mathbf{R}^3$.

2.2. *Lie Algebra Structure on* $\mathbf{H} \otimes U(\mathfrak{g})$

Let $(\mathfrak{g}, [\quad, \quad]_{\mathfrak{g}})$ be a complex Lie algebra. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbf{H}} = \mathbf{H} \otimes U(\mathfrak{g})$ and define the following bracket on $\mathfrak{g}^{\mathbf{H}}$:

$$[\,\mathbf{z} \otimes X\,,\, \mathbf{w} \otimes Y\,]_{\mathfrak{g}^{\mathbf{H}}} = (\mathbf{z} \cdot \mathbf{w}) \otimes\, (XY) - (\mathbf{w} \cdot \mathbf{z}) \otimes (YX) \tag{26}$$

for the basis vectors X and Y of $U(\mathfrak{g})$ and $\mathbf{z},\, \mathbf{w} \in\, \mathbf{H}$.

By the quaternion number notation every element of $\mathbf{H} \otimes \mathfrak{g}$ may be written as $X + jY$ with $X, Y \in \mathfrak{g}$. Then the above definition is equivalent to

$$\begin{aligned} \left[\, X_1 + jY_1\,,\, X_2 + jY_2\,\right]_{\mathfrak{g}^{\mathbf{H}}} \;&=\; [X_1, X_2]_{\mathfrak{g}} \,-\, (\,\overline{Y}_1 Y_2 \,-\, \overline{Y}_2 Y_1\,) \\ &\quad + j\left(\overline{X}_1 Y_2 - Y_2 X_1 + Y_1 X_2 - \overline{X}_2 Y_1\right)\,, \end{aligned} \tag{27}$$

where $\overline{X}$ is the complex conjugate of X.

Proposition 2. *The bracket* $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}^{\mathbf{H}}}$ *defines a Lie algebra structure on* $\mathbf{H} \otimes U(\mathfrak{g})$.

In fact the bracket defined in Equations (26) or (27) satisfies the antisymmetry equation and the Jacobi identity.

Definition 1. *The Lie algebra* $\left(\mathfrak{g}^{\mathbf{H}} = \mathbf{H} \otimes U(\mathfrak{g})\,,\, [\quad,\quad]_{\mathfrak{g}^{\mathbf{H}}}\right)$ *is called the quaternification of the Lie algebra* $\mathfrak{g}$.

3. Analysis on H

In this section we shall review the analysis of the Dirac operator on $\mathbf{H} \simeq \mathbf{C}^2$. The general references are [10,11], and we follow the calculations developed in [8,9,12].

3.1. Harmonic Polynomials

The Lie group SU(2) acts on $\mathbf{C}^2$ both from the right and from the left. Let *dR*(g) and *dL*(g) denote respectively the right and the left infinitesimal actions of the Lie algebra $\mathfrak{su}(2)$. We define the following vector fields on $\mathbf{C}^2$:

$$\theta_i = dR\left(\frac{1}{2}e_i\right), \quad \tau_i = dL\left(\frac{1}{2}e_i\right), \qquad i = 1,2,3, \tag{28}$$

where $\{e_i;\, i = 1,2,3\}$ is the normal basis of $\mathfrak{su}(2)$, Equation (18) . Each of the triple $\theta_i(z)$, $i = 1,2,3$, and $\tau_i(z)$, $i = 1,2,3$, gives a basis of the vector fields on the three sphere $\{|z| = 1\} \simeq S^3$.

It is more convenient to introduce the following vector fields:

$$e_+ = -z_2\frac{\partial}{\partial \bar{z}_1} + z_1\frac{\partial}{\partial \bar{z}_2} = \theta_1 - \sqrt{-1}\theta_2, \tag{29}$$

$$e_- = -\bar{z}_2\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial z_2} = \theta_1 + \sqrt{-1}\theta_2, \tag{30}$$

$$\theta = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2} = 2\sqrt{-1}\,\theta_3. \tag{31}$$

$$\hat{e}_+ = -\bar{z}_1\frac{\partial}{\partial \bar{z}_2} + z_2\frac{\partial}{\partial z_1} = \tau_1 - \sqrt{-1}\tau_2, \tag{32}$$

$$\hat{e}_- = \bar{z}_2\frac{\partial}{\partial \bar{z}_1} - z_1\frac{\partial}{\partial z_2} = \tau_1 + \sqrt{-1}\tau_2, \tag{33}$$

$$\hat{\theta} = z_2\frac{\partial}{\partial z_2} + \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2} - z_1\frac{\partial}{\partial z_1} = 2\sqrt{-1}\,\tau_3. \tag{34}$$

We have the commutation relations;

$$[\theta, e_+] = 2e_+, \quad [\theta, e_-] = -2e_-, \quad [e_+, e_-] = -\theta. \tag{35}$$

$$[\hat{\theta}, \hat{e}_+] = 2\hat{e}_+, \quad [\hat{\theta}, \hat{e}_-] = -2\hat{e}_-, \quad [\hat{e}_+, \hat{e}_-] = -\hat{\theta}. \tag{36}$$

Both Lie algebras spanned by (e_+, e_-, θ) and $(\hat{e}_+, \hat{e}_-, \hat{\theta})$ are isomorphic to $\mathfrak{sl}(2, \mathbf{C})$.

In the following we denote a function $f(z, \bar{z})$ of variables $z, \bar{z}$ simply by $f(z)$. For $m = 0, 1, 2, \cdots$, and $l, k = 0, 1, \cdots, m$, we define the polynomials:

$$v^k_{(l,m-l)} = (e_-)^k z_1^l z_2^{m-l}, \tag{37}$$

$$w^k_{(l,m-l)} = (\hat{e}_-)^k z_2^l \bar{z}_1^{m-l}. \tag{38}$$

Then $v^k_{(l,m-l)}$ and $w^k_{(l,m-l)}$ are harmonic polynomials on $\mathbf{C}^2$;

$$\Delta v^k_{(l,m-l)} = \Delta w^k_{(l,m-l)} = 0\,,$$

where $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$.

$\left\{ \frac{1}{\sqrt{2\pi}} v^k_{(l,m-l)}\,;\, m = 0, 1, \cdots, 0 \le k, l \le m \right\}$ forms a $L^2(S^3)$-complete orthonormal basis of the space of harmonic polynomials, as well as $\left\{ \frac{1}{\sqrt{2\pi}} w^k_{(l,m-l)}\,;\, m = 0, 1, \cdots, 0 \le k, l \le m \right\}$.

Proposition 3.

$$\begin{aligned} e_+ v^k_{(l,m-l)} &= -k(m-k+1)v^{k-1}_{(l,m-l)}, \\ e_- v^k_{(l,m-l)} &= v^{k+1}_{(l,m-l)}, \\ \theta v^k_{(l,m-l)} &= (m-2k)v^k_{(l,m-l)} \,. \end{aligned} \tag{39}$$

$$\begin{aligned} \hat{e}_+ w^k_{(l,m-l)} &= -k(m-k+1)w^{k-1}_{(l,m-l)}, \\ \hat{e}_- w^k_{(l,m-l)} &= w^{k+1}_{(l,m-l)}, \\ \hat{\theta} w^k_{(l,m-l)} &= (m-2k)w^k_{(l,m-l)}. \end{aligned} \tag{40}$$

Therefore the space of harmonic polynomials on $\mathbf{C}^2$ is decomposed by the right action of SU(2) into $\sum_m \sum_{l=0}^m H_{m,l}$. Each $H_{m,l} = \sum_{k=0}^m \mathbf{C}\, v^k_{(l,m-l)}$ gives an (m + 1) dimensional irreducible representation of $SU(2)$ with the highest weight $\frac{m}{2}$, [13].

We have the following relations.

$$w^k_{(l,m-l)} = (-1)^k \frac{l!}{(m-k)!}\, v^{m-l}_{(k,m-k)} \,, \tag{41}$$

$$\overline{v^k_{(l,m-l)}} = (-1)^{m-l-k} \frac{k!}{(m-k)!} v^{m-k}_{(m-l,l)} \,. \tag{42}$$

3.2. Harmonic Spinors

$\Delta = \mathbf{H} \otimes \mathbf{C} = \mathbf{H} \oplus \mathbf{H}$ gives an irreducible complex representation of the Clifford algebra $\mathrm{Clif}(\mathbf{R}^4)$:

$$\mathrm{Clif}(\mathbf{R}^4) \otimes \mathbf{C} \simeq \mathrm{End}(\Delta) \,.$$

Δ decomposes into irreducible representations $\Delta^{\pm} = \mathbf{H}$ of $\mathrm{Spin}(4)$. Let $S = \mathbf{C}^2 \times \Delta$ be the trivial spinor bundle on $\mathbf{C}^2$. The corresponding bundle $S^+ = \mathbf{C}^2 \times \Delta^+$ (resp. $S^- = \mathbf{C}^2 \times \Delta^-$) is called the even (resp. odd) spinor bundle and the sections are called even (resp. odd) spinors. The set of even spinors or odd spinors on a set $M \subset \mathbf{C}^2$ is nothing but the smooth functions on M valued in $\mathbf{H}$:

$$Map(M, \mathbf{H}) = C^{\infty}(M, S^+) \,. \tag{43}$$

The Dirac operator is defined by

$$\mathcal{D} = c \circ d \tag{44}$$

where $d : S \to S \otimes T^*\mathbf{C}^2 \simeq S \otimes T\mathbf{C}^2$ is the exterior differential and $c : S \otimes T\mathbf{C}^2 \to S$ is the bundle homomorphism coming from the Clifford multiplication. By means of the decomposition $S = S^+ \oplus S^-$ the Dirac operator has the chiral decomposition:

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty(\mathbf{C}^2, S^+ \oplus S^-) \to C^\infty(\mathbf{C}^2, S^+ \oplus S^-). \tag{45}$$

We find that D and $D^\dagger$ have the following coordinate expressions;

$$D = \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial \bar{z}_2} \\ & \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z}_1} \end{pmatrix}, \quad D^\dagger = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} \\ & \\ -\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix}. \tag{46}$$

An even (resp. odd) spinor φ is called a *harmonic spinor* if $D\varphi = 0$ (resp. $D^\dagger\varphi = 0$).

We shall introduce a set of harmonic spinors which, restricted to S^3, forms a complete orthonormal basis of $L^2(S^3, S^+)$.

Let ν and μ be vector fields on $\mathbf{C}^2$ defined by

$$\nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}, \quad \mu = z_2 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}. \tag{47}$$

Then the radial vector field is defined by

$$\frac{\partial}{\partial n} = \frac{1}{2|z|}(\nu + \bar{\nu}) = \frac{1}{2|z|}(\mu + \bar{\mu}). \tag{48}$$

We shall denote by γ the Clifford multiplication of the radial vector $\frac{\partial}{\partial n}$, Equation (48). γ changes the chirality:

$$\gamma : S^+ \oplus S^- \longrightarrow S^- \oplus S^+; \quad \gamma^2 = 1.$$

The matrix expression of γ becomes as follows:

$$\gamma|S^+ = \frac{1}{|z|}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix}, \quad \gamma|S^- = \frac{1}{|z|}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}. \tag{49}$$

In the sequel we shall write γ_+ (resp. γ_-) for $\gamma|S^+$ (resp. $\gamma|S^+$).

Proposition 4. *The Dirac operators D and $D^\dagger$ have the following polar decompositions:*

$$\begin{aligned} D &= \gamma_+ \left(\frac{\partial}{\partial n} - \partial\!\!\!/ \right), \\ D^\dagger &= \left(\frac{\partial}{\partial n} + \partial\!\!\!/ + \frac{3}{2|z|} \right) \gamma_-, \end{aligned}$$

where the tangential (nonchiral) Dirac operator $\partial\!\!\!/$ is given by

$$\partial\!\!\!/ = -\left[\sum_{i=1}^{3}\left(\frac{1}{|z|}\theta_i\right)\cdot\nabla_{\frac{1}{|z|}\theta_i}\right] = \frac{1}{|z|}\begin{pmatrix}-\frac{1}{2}\theta & e_+\\ -e_- & \frac{1}{2}\theta\end{pmatrix}.$$

Proof. In the matrix expression Equation (46) of D and $D^\dagger$, we have $\frac{\partial}{\partial z_1} = \frac{1}{|z|^2}(\bar{z}_1\nu - z_2 e_-)$ *etc.*, and we have the desired formulas. □

The tangential Dirac operator on the sphere $S^3 = \{|z| = 1\}$;

$$\partial\!\!\!/|S^3 : C^\infty(S^3, S^+) \longrightarrow C^\infty(S^3, S^+)$$

is a self adjoint elliptic differential operator.

We put, for $m = 0, 1, 2, \cdots; l = 0, 1, \cdots, m$ and $k = 0, 1, \cdots, m+1$,

$$\phi^{+(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}}\begin{pmatrix}kv^{k-1}_{(l,m-l)}\\ \\ -v^k_{(l,m-l)}\end{pmatrix}, \tag{50}$$

$$\phi^{-(m,l,k)}(z) = \sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}}\left(\frac{1}{|z|^2}\right)^{m+2}\begin{pmatrix}w^k_{(m+1-l,l)}\\ \\ w^k_{(m-l,l+1)}\end{pmatrix}. \tag{51}$$

$\phi^{+(m,l,k)}$ is a harmonic spinor on $\mathbf{C}^2$ and $\phi^{-(m,l,k)}$ is a harmonic spinor on $\mathbf{C}^2\backslash\{0\}$ that is regular at infinity.

From Proposition 3 we have the following.

Proposition 5. *On $S^3 = \{|z| = 1\}$ we have:*

$$\partial\!\!\!/\phi^{+(m,l,k)} = \frac{m}{2}\phi^{+(m,l,k)}, \tag{52}$$

$$\partial\!\!\!/\phi^{-(m,l,k)} = -\frac{m+3}{2}\phi^{-(m,l,k)}. \tag{53}$$

The eigenvalues of $\partial\!\!\!/$ are

$$\frac{m}{2}, \quad -\frac{m+3}{2}; \quad m = 0, 1, \cdots, \tag{54}$$

and the multiplicity of each eigenvalue is equal to $(m+1)(m+2)$.

The set of eigenspinors

$$\left\{\frac{1}{\sqrt{2}\pi}\phi^{+(m,l,k)}, \quad \frac{1}{\sqrt{2}\pi}\phi^{-(m,l,k)}; \quad m = 0, 1, \cdots, 0 \le l \le m, 0 \le k \le m+1\right\} \tag{55}$$

forms a complete orthonormal system of $L^2(S^3, S^+)$.

The constant for normalization of $\phi^{\pm(m,l,k)}$ is determined by the integral:

$$\int_{S^3} |z_1^a z_2^b|^2 d\sigma = 2\pi^2 \frac{a!b!}{(a+b+1)!}\,, \tag{56}$$

where σ is the surface measure of the unit sphere $S^3 = \{|z| = 1\}$:

$$\int_{S^3} d\sigma_3 = 2\pi^2. \tag{57}$$

3.3. Spinors of Laurent Polynomial Type

If φ is a harmonic spinor on $\mathbf{C}^2 \setminus \{0\}$ then we have the expansion

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z), \tag{58}$$

that is uniformly convergent on any compact subset of $\mathbf{C}^2 \setminus \{0\}$. The coefficients $C_{\pm(m,l,k)}$ are given by the formula:

$$C_{\pm(m,l,k)} = \frac{1}{2\pi^2}\int_{S^3} \langle \varphi,\, \phi^{\pm(m,l,k)}\rangle\, d\sigma, \tag{59}$$

where $\langle\, ,\, \rangle$ is the inner product of S^+.

Lemma 1.

$$\int_{S^3} tr\,\varphi\, d\sigma \;=\; 4\pi^2 Re.C_{+(0,0,1)}, \tag{60}$$

$$\int_{S^3} tr\, J\varphi\, d\sigma \;=\; 4\pi^2 Re.C_{+(0,0,0)}.$$

The formulas follow from Equation (59) if we take $\phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Definition 2.

1. *We call the series Equation (58) a spinor of Laurent polynomial type if only finitely many coefficients $C_{\pm(m,l,k)}$ are non-zero . The space of spinors of Laurent polynomial type is denoted by $\mathbf{C}[\phi^{\pm}]$.*
2. *For a spinor of Laurent polynomial type φ we call the vector $\mathrm{res}\,\varphi = \begin{pmatrix} -C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{pmatrix}$ the residue at 0 of φ.*

We have the residue formula [9].

$$\mathrm{res}\,\varphi = \frac{1}{2\pi^2}\int_{S^3} \gamma_+(z)\varphi(z)\sigma(dz). \tag{61}$$

Remark 1. *To develop the spinor analysis on the 4-sphere S^4 we patch two local coordinates $\mathbf{C}^2_z$ and $\mathbf{C}^2_w$ together by the inversion $w = -\frac{\bar{z}}{|z|^2}$. This is a conformal transformation with the conformal* *weight* $u = -\log|z|^2$. *An even spinor on a subset* $U \subset S^4$ *is a pair of* $\phi \in C^\infty(U \cap \mathbf{C}^2 \times \Delta)$ *and* $\widehat{\phi} \in C^\infty(U \cap \widehat{\mathbf{C}}^2 \times \Delta)$ *such that* $\widehat{\phi}(w) = \overline{|z|^3(\gamma_+\phi)(z)}$ *for* $w = -\frac{\overline{z}}{|z|^2}$. *Let* φ *be a spinor of Laurent polynomial type on* $\mathbf{C}^2 \setminus 0 = \widehat{\mathbf{C}}^2 \setminus \widehat{0}$. *The coefficient* $C_{\pm(m,l,k)}$ *of* φ *and the coefficient* $\widehat{C}_{\pm(m,l,k)}$ *of* $\widehat{\varphi}$ *are related by the formula:*

$$\widehat{C}_{-(m,l,k)} = \overline{C}_{+(m,l,k)}, \qquad \widehat{C}_{+(m,l,k)} = \overline{C}_{-(m,l,k)}. \tag{62}$$

Proposition 6. *The residue of* $\widehat{\varphi}$ *is related to the trace of* φ, *Lemma 1, by*

$$res\,\widehat{\varphi} = \frac{1}{2\pi^2}\int_{S^3} \widehat{\varphi}\, d\sigma = \begin{pmatrix} \overline{C}_{+(0,0,1)} \\ -\overline{C}_{+(0,0,0)} \end{pmatrix}. \tag{63}$$

3.4. Algebraic Generators of $\mathbf{C}[\phi^\pm]$

In the following we show that $\mathbf{C}[\,\phi^\pm\,]$ restricted to S^3 becomes an algebra. The multiplication of two harmonic polynomials on $\mathbf{C}^2$ is not harmonic but its restriction to S^3 is again the restriction to S^3 of some harmonic polynomial. We shall see that this yields the fact that $\mathbf{C}[\phi^\pm\,]$, restricted to S^3, becomes an associative subalgebra of $S^3\mathbf{H}$. Before we give the proof we look at examples that convince us of the necessity of the restriction to S^3 .

Example 1. $\phi^{+(1,0,1)} \cdot \phi^{-(0,0,0)}$ *is decomposed to the sum*

$$\phi^{+(1,0,1)}(z)\cdot\phi^{-(0,0,0)}(z) = \frac{1}{|z|^4}\Big(\,\frac{2}{3}\phi^{+(2,0,1)}(z) + \frac{\sqrt{2}}{3}\phi^{+(2,1,2)}(z)\,\Big) + \frac{1}{|z|^2}\frac{1}{2}\phi^{+(0,0,1)}(z),$$

which is not in $\mathbf{C}[\phi^\pm]$. *But the restriction to* S^3 *is*

$$\frac{2}{3}\phi^{+(2,0,1)} + \frac{\sqrt{2}}{3}\phi^{+(2,1,2)} + \frac{1}{2}\phi^{+(0,0,1)} + \frac{1}{6}\phi^{-(1,1,1)} + \frac{1}{3\sqrt{2}}\phi^{-(1,0,0)} \in \mathbf{C}[\phi^\pm]|_{S^3}.$$

See the table at the end of this subsection.

We start with the following facts:

1. We have the product formula for the harmonic polynomials $v^k_{(a,b)}$.

$$v^{k_1}_{(a_1,b_1)}v^{k_2}_{(a_2,b_2)} = \sum_{j=0}^{a_1+a_2+b_1+b_2} C_j|z|^{2j}\, v^{k_1+k_2-j}_{(a_1+a_2-j,\,b_1+b_2-j)} \tag{64}$$

   for some rational numbers $C_j = C_j(a_1,a_2,b_1,b_2,k_1,k_2)$. See Lemma 4.1 of [12].
2. Let $k = k_1 + k_2$, $a = a_1 + a_2$ and $b = b_1 + b_2$. The above Equation (64) yields that, restricted to S^3, the harmonic polynomial $v^k_{(a,b)}$ is equal to a constant multiple of $v^{k_1}_{(a_1,b_1)}v^{k_2}_{(a_2,b_2)}$ modulo a linear combination of polynomials $v^{k-j}_{(a-j,b-j)}$, $1 \le j \le min(k,a,b)$.
3. $\begin{pmatrix} v^k_{(l,m-l)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ v^{k+1}_{(l,m-l)} \end{pmatrix}$ are written by linear combinations of $\phi^{+(m,l,k+1)}$ and $\phi^{-(m-1,k,l)}$.

4. Therefore the product of two spinors $\phi^{\pm(m_1,l_1,k_1)} \cdot \phi^{\pm(m_2,l_2,k_2)}$ belongs to $\mathbf{C}[\phi^{\pm}]|_{S^3}$. $\mathbf{C}[\phi^{\pm}]|_{S^3}$ becomes an associative algebra.
5. $\phi^{\pm(m,l,k)}$ is written by a linear combination of the products $\phi^{\pm(m_1,l_1,k_1)} \cdot \phi^{\pm(m_2,l_2,k_2)}$ for $0 \le m_1 + m_2 \le m-1$, $0 \le l_1 + l_2 \le l$ and $0 \le k_1 + k_2 \le k$.

Hence we find that the algebra $\mathbf{C}[\phi^{\pm}]|_{S^3}$ is generated by the following $I,\ J,\ \kappa,\ \mu$:

$$I = \phi^{+(0,0,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$
$$\kappa = \phi^{+(1,0,1)} = \begin{pmatrix} z_2 \\ -\overline{z}_1 \end{pmatrix}, \quad \mu = \phi^{-(0,0,0)} = \begin{pmatrix} z_2 \\ \overline{z}_1 \end{pmatrix}. \tag{65}$$

The others are generated by these basis. For example,

$$\lambda = \phi^{+(1,1,1)} = \begin{pmatrix} z_1 \\ \overline{z}_2 \end{pmatrix} = -\kappa J\,, \quad \nu = \phi^{-(0,0,1)} = \begin{pmatrix} -z_1 \\ \overline{z}_2 \end{pmatrix} = -\mu J,$$
$$\phi^{+(1,0,0)} = \sqrt{2}\begin{pmatrix} 0 \\ -z_2 \end{pmatrix} = \frac{1}{\sqrt{2}}J(\kappa+\mu), \quad \phi^{+(1,0,2)} = \sqrt{2}\begin{pmatrix} \overline{z}_1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}J(\mu-\kappa)\,,$$
$$\phi^{+(1,1,2)} = \sqrt{2}\begin{pmatrix} -\overline{z}_2 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}}J(\lambda+\nu), \quad \phi^{+(1,1,0)} = \sqrt{2}\begin{pmatrix} 0 \\ -z_1 \end{pmatrix} = \frac{1}{\sqrt{2}}J(\lambda-\nu),$$
$$\phi^{-(1,0,0)} = \sqrt{2}\begin{pmatrix} z_2^2 \\ z_2\overline{z}_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\nu J(\kappa+\mu), \quad \phi^{-(1,1,0)} = \sqrt{2}\begin{pmatrix} z_2\overline{z}_1 \\ \overline{z}_1^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\mu J(\mu-\kappa),$$
$$\phi^{-(1,1,2)} = \sqrt{2}\begin{pmatrix} -z_1\overline{z}_2 \\ \overline{z}_2^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\nu J(\lambda+\nu), \quad \phi^{-(1,0,2)} = \sqrt{2}\begin{pmatrix} z_1^2 \\ -z_1\overline{z}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\mu J(\lambda-\nu)$$
$$\phi^{-(1,0,1)} = \begin{pmatrix} -2z_1z_2 \\ |z_2|^2-|z_1|^2 \end{pmatrix} = \frac{\nu}{2}(\kappa+\mu+J(\lambda-\nu)),$$
$$\phi^{-(1,1,1)} = \begin{pmatrix} |z_2|^2-|z_1|^2 \\ 2\overline{z}_1\overline{z}_2 \end{pmatrix} = \frac{\mu}{2}(-\kappa+\mu+J(\lambda+\nu))\,.$$

3.5. 2-Cocycle on $S^3\mathbf{H}$

Let $S^3\mathbf{H} = Map(S^3,\mathbf{H}) = C^\infty(S^3,S^+)$ be the set of smooth even spinors on S^3. We define the Lie algebra structure on $S^3\mathbf{H}$ after Equation (22), that is, for even spinors $\phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, we have the Lie bracket

$$[\,\phi_1\,,\,\phi_2\,] = \begin{pmatrix} v_1\bar{v}_2 - \bar{v}_1v_2 \\ (u_2-\bar{u}_2)v_1 - (u_1-\bar{u}_1)v_2 \end{pmatrix}. \tag{66}$$

For a $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in S^3\mathbf{H}$, we put

$$\Theta\,\varphi \;=\; \begin{pmatrix} \frac{1}{2\sqrt{-1}}\theta & 0 \\ 0 & -\frac{1}{2\sqrt{-1}}\overline{\theta} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \;=\; \frac{1}{2\sqrt{-1}} \begin{pmatrix} \theta\, u \\ \theta\, v \end{pmatrix}.$$

Lemma 2. *For any $\phi,\,\psi \in S^3\mathbf{H}$, we have*

$$\Theta\,(\phi\cdot\psi\,) \;=\; (\Theta\,\phi)\cdot\,\psi \,+\, \phi\cdot\,(\Theta\,\psi)\,, \tag{67}$$

$$\int_{S^3} \Theta\,\varphi\, d\sigma \;=\; 0\,. \tag{68}$$

The second assertion follows from the fact

$$\int_{S^3} \theta f\, d\sigma \;=\; 0\,,$$

for any function f on S^3.

Proposition 7.

$$2\sqrt{-1}\,\Theta\,\phi^{+(m,l,k)} \;=\; \frac{m(m+1-k)}{m+1}\,\phi^{+(m,l,k)} \,+\, 2(-1)^l\,\frac{\sqrt{k(m+1-k)}}{m+1}\,\phi^{-(m-1,k-1,l)}\,,$$

$$2\sqrt{-1}\,\Theta\,\phi^{-(m,l,k)} \;=\; (m-2l)\frac{m+3}{m+2}\,\phi^{-(m,l,k)} \,+\, 2(-1)^k\,\frac{\sqrt{(l+1)(m+1-l)}}{m+2}\,\phi^{+(m+1,k,l+1)}\,,$$

on S^3.

Now we shall introduce a non-trivial 2-cocycle on $S^3\mathbf{H}$.

Definition 3. *For ϕ_1 and $\phi_2 \in S^3\mathbf{H}$, we put*

$$c(\phi_1,\phi_2) \;=\; \frac{1}{2\pi^2}\int_{S^3} tr\,(\,\Theta\phi_1\cdot\phi_2\,)\,d\sigma. \tag{69}$$

Example 2.

$$c(\,\frac{1}{\sqrt{2}}\phi^{+(1,1,2)}\,,\;\frac{\sqrt{-1}}{2}(\phi^{+(1,0,1)} + \phi^{-(0,0,0)})\,) \;=\; \frac{1}{2}. \tag{70}$$

Proposition 8. *c defines a non-trivial 2-cocycle on the algebra $S^3\mathbf{H}$. That is, c satisfies the following equations:*

$$c(\phi_1\,\phi_2) \;=\; -\,c(\phi_2,\,\phi_1)\,, \tag{71}$$

$$c(\phi_1\cdot\phi_2\,,\,\phi_3) + c(\phi_2\cdot\phi_3\,,\,\phi_1\,) + c(\phi_3\cdot\phi_1\,,\,\phi_2\,) = 0. \tag{72}$$

And there is no 1-cochain b such that $c(\phi_1.\phi_2) = b(\,[\phi_1,\phi_2]\,)$.

Proof. By Equation (68) and the Leibnitz rule Equation (67) we have

$$0 \;=\; \int_{S^3} tr\,(\,\Theta\,(\phi_1\cdot\phi_2)\,)\,d\sigma \,=\, \int_{S^3} tr\,\left(\,\Theta\,\phi_1\,\cdot\phi_2\,\right)d\sigma \,+\, \int_{S^3} tr\,\left(\,\phi_1\cdot\,\Theta\,\phi_2\,\right)d\sigma$$

Hence

$$c(\phi_1\,,\,\phi_2\,)\,+\,c(\,\phi_2\,,\,\phi_1\,)\,=0\,.$$

The following calculation proves Equation (72).

$$\begin{aligned}
c(\phi_1\cdot\phi_2\,,\,\phi_3) \;&=\; \int_{S^3} tr\,(\,\Theta(\,\phi_1\cdot\phi_2\,)\cdot\,\phi_3\,)\,d\sigma \\
&=\; \int_{S^3} tr\,(\,\Theta\phi_1\cdot\phi_2\cdot\phi_3\,)d\sigma \,+\, \int_{S^3} tr\,(\,\Theta\phi_2\cdot\,\phi_3\,\cdot\phi_1\,)d\sigma \\
&=\; c(\phi_1\,,\,\phi_2\cdot\phi_3\,)+c(\phi_2\,,\,\phi_3\cdot\phi_1\,)=\,-c(\phi_2\cdot\phi_3\,,\phi_1\,)\,-\,c(\phi_3\cdot\phi_1\,,\,\phi_2).
\end{aligned}$$

Suppose now that c is the coboundary of a 1-cochain $b:\;S^3\mathbf{H}\longrightarrow\mathbf{C}$. Then

$$c(\phi_1,\,\phi_2)=(\delta\,b)(\phi_1,\,\phi_2)\;=\;b(\,[\phi_1,\,\phi_2]\,)$$

for any $\phi_1,\phi_2\in S^3\mathbf{H}$. Take $\phi_1=\frac{1}{\sqrt{2}}\phi^{+(1,1,2)}=\begin{pmatrix}-\overline{z}_2\\0\end{pmatrix}$ and $\phi_2=\frac{\sqrt{-1}}{2}(\phi^{+(1,0,1)}+\phi^{-(0,0,0)})=\begin{pmatrix}\sqrt{-1}z_2\\0\end{pmatrix}$. Then $[\,\phi_1,\,\phi_2\,]=0$, so $(\delta b)(\phi_1,\phi_2)=0$. But $c(\phi_1,\phi_2)=\frac{1}{2}$. Therefore c can not be a coboundary. □

3.6. Calculations of the 2-Cocycle on the Basis

We shall calculate the values of 2-cocycles c for the basis $\{\phi^{\pm(m,l,k)}\}$ of $\mathbf{C}[\phi^{\pm}]$. First we have a lemma that is useful for the following calculations.

Lemma 3.

1.

$$\int_{S^3} v^k_{(a,b)}\,\overline{v}^l_{(c,d)}\,d\sigma \;=\; 2\pi^2\frac{a!b!}{(a+b+1)}\,\frac{k!}{(a+b-k)!}\,\delta_{a,c}\,\delta_{b,d}\,\delta_{k,l}\,. \tag{73}$$

2.

$$\int_{S^3} v^k_{(a,b)}\,v^l_{(c,d)}\,d\sigma \;=(-1)^{b-k}\,2\pi^2\frac{a!b!}{(a+b+1)}\;\delta_{a,d}\,\delta_{b,c}\,\delta_{(a+b-k),l}\,. \tag{74}$$

3.

$$\int_{S^3} w^k_{(a,b)}\,\overline{w}^l_{(c,d)}\,d\sigma \;=\; 2\pi^2\frac{a!b!}{(a+b+1)}\,\frac{k!}{(a+b-k)!}\,\delta_{a,c}\,\delta_{b,d}\,\delta_{k,l}\,. \tag{75}$$

4.

$$\int_{S^3} w^k_{(a,b)}\,w^l_{(c,d)}\,d\sigma \;=(-1)^{b-k}\,2\pi^2\frac{a!b!}{(a+b+1)}\,\delta_{a,d}\,\delta_{b,c}\,\delta_{(a+b-k),l}\,. \tag{76}$$

Lemma 4.

1.
$$c(\phi^{\pm(m,l,k)}, \phi^{\pm(p,q,r)}) = 0\,.$$

2.
$$\begin{aligned} c(\phi^{+(m,l,k)}, \sqrt{-1}\phi^{+(p,q,r)}) &= (-1)^{m-l-k+1}\frac{(m-2k+2)\sqrt{k(m+2-k)}}{m+1}\,\delta_{m,p}\,\delta_{l,p-q}\,\delta_{k,\,p-r+2} \\ &\quad - \frac{(m-2k)(m-k+1)}{m+1}\,\delta_{m,p}\,\delta_{l,q}\,\delta_{k,r}. \end{aligned}$$

3.
$$\begin{aligned} c(\phi^{+(m,l,k)}, \sqrt{-1}\phi^{-(p,q,r)}) &= (-1)^{k-1}\frac{(m-2k+2)\sqrt{(k-1)k}}{m+1}\,\delta_{m,p+1}\,\delta_{l,\,p-r+1}\,\delta_{k,p-q+2} \\ &\quad + (-1)^{l}\,\frac{(m-2k)\sqrt{k(m+1-k)}}{m+1}\,\delta_{m,p+1}\delta_{l,r}\delta_{k,q+1}\,. \end{aligned}$$

4.
$$\begin{aligned} c(\phi^{-(m,l,k)}, \sqrt{-1}\phi^{+(p,q,r)}) &= (-1)^{m+1-l}\frac{(m-2l+1)\sqrt{(m-l+1)(m-l+2)}}{m+2}\,\delta_{m,p-1}\delta_{l,\,p-r+1}\delta_{k,p-q} \\ &\quad +(-1)^{k}\frac{(m-2l+1)\sqrt{(l+1)(m-l+1)}}{m+2}\,\delta_{m,p-1}\delta_{l,r-1}\delta_{k,q}\,. \end{aligned}$$

5.
$$\begin{aligned} c(\phi^{-(m,l,k)}, \sqrt{-1}\phi^{-(p,q,r)}) &= (-1)^{l-k}\frac{(m-2l+1)\sqrt{l(m-l+1)}}{m+2}\,\delta_{m,p}\delta_{l,p-q+1}\delta_{k,p-r+1} \\ &\quad - \frac{(m-2l-1)(l+1)}{m+2}\,\delta_{m,p}\delta_{l,q}\delta_{k,r} \end{aligned}$$

Proof. Since $\theta\, v^{k}_{(a,b)} = (a+b-2k)v^{k}_{(a,b)}$, we have

$$c(\phi^{+(m,l,k)}, \sqrt{-1}\phi^{+(p,q,r)}) = \frac{1}{2\pi^2}\int_{S^3} tr\,[\Theta\phi^{+(m,l,k)}\cdot\sqrt{-1}\phi^{+(p,q,r)}]d\sigma$$

$$= \frac{1}{4\pi^2}\sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}}\sqrt{\frac{(p+1-r)!}{r!q!(p-q)!}}\int_{S^3} tr\,\left[\begin{pmatrix} k(m-2k+2)v^{k-1}_{(l,m-l)} \\ -(m-2k)v^{k}_{(l,m-l)} \end{pmatrix}\cdot\begin{pmatrix} rv^{r-1}_{(q,p-q)} \\ -v^{r}_{(q,p-q)} \end{pmatrix}\right] d\sigma\,.$$

By the above lemma we obtain the value of $c(\phi^{+(m,l,k)}, \sqrt{-1}\phi^{+(p,q,r)})$. The others follow similarly. □

3.7. Radial Derivative on $S^3\mathbf{H}$

We define the following operator d_0 on $C^\infty(S^3)$:

$$d_0\, f(z) = |z|\frac{\partial}{\partial n} f(z) \;=\; \frac{1}{2}(\nu+\bar{\nu})f(z). \tag{77}$$

For an even spinor $\varphi = \begin{pmatrix} u \\ v \end{pmatrix}$ we put

$$d_0\,\varphi = \begin{pmatrix} d_0\, u \\ d_0\, v \end{pmatrix}.$$

Note that if $\varphi \in \mathbf{C}[\phi^\pm]$ then $d_0\varphi \in \mathbf{C}[\phi^\pm]$.

Proposition 9.

1. $$d_0(\phi_1\cdot\phi_2) = (d_0\phi_1)\cdot\phi_2 + \phi_1\cdot(d_0\phi_2)\,. \tag{78}$$

2. $$d_0\phi^{+(m,l,k)} = \frac{m}{2}\,\phi^{+(m,l,k)}, \quad d_0\phi^{-(m,l,k)} \;=\; -\frac{m+3}{2}\,\phi^{-(m,l,k)}. \tag{79}$$

3. *Let $\varphi = \phi_1\cdots\phi_n$ such that $\phi_i = \phi^{+(m_i,l_i,k_i)}$ or $\phi_i = \phi^{-(m_i,l_i,k_i)}$, $i = 1,\cdots,n$. We put*

$$N = \sum_{i:\,\phi_i=\phi^{+(m_i,l_i,k_i)}} m_i \;-\; \sum_{i:\,\phi_i=\phi^{-(m_i,l_i,k_i)}} (m_i+3).$$

   Then

$$d_0(\varphi) = \frac{N}{2}\,\varphi. \tag{80}$$

4. *Let φ be a spinor of Laurent polynomial type:*

$$\varphi(z) = \sum_{m,l,k} C_{+(m,l,k)}\phi^{+(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\phi^{-(m,l,k)}(z). \tag{81}$$

   Then

$$\int_{S^3} tr\,(\,d_0\,\varphi\,)\,d\sigma \;=\; 0\,. \tag{82}$$

Proof. The Formula (79) follows from the Definition Equation (50). The last assertion follows from the fact that the coefficient of $\phi^{+(0,0,1)}$ in the Laurent expansion of $d_0\varphi$ vanishes. □

Definition 4. *Let $\mathbf{C}[\phi^\pm; N]$ be the subspace of $\mathbf{C}[\phi^\pm]$ consisting of those elements that are of homogeneous order N: $\varphi(z) = |z|^N\varphi(\frac{z}{|z|})$.*

$\mathbf{C}[\phi^{\pm};\,N]$ is spanned by the spinors $\varphi=\phi_1\cdots\phi_n$ such that each ϕ_i is equal to $\phi_i=\phi^{+(m_i,l_i,k_i)}$ or $\phi_i=\phi^{-(m_i,l_i,k_i)}$, where $m_i\geq 0$ and $0\leq l_i\leq m_i+1$, $0\leq k_i\leq m_i+2$ as before, and such that

$$N=\sum_{i:\,\phi_i=\phi^{+(m_i,l_i,k_i)}} m_i\;-\sum_{i:\,\phi_i=\phi^{-(m_i,l_i,k_i)}}(m_i+3).$$

$\mathbf{C}[\phi^{\pm}]$ is decomposed into the direct sum of $\mathbf{C}[\phi^{\pm};\,N]$:

$$\mathbf{C}[\phi^{\pm}]\,=\,\bigoplus_{N\in\mathbf{Z}}\mathbf{C}[\phi^{\pm};\,N]\,.$$

Equation (80) implies that the eigenvalues of d_0 on $\mathbf{C}[\phi^{\pm}]$ are $\left\{\frac{N}{2};\,N\in\mathbf{Z}\right\}$ and $\mathbf{C}[\phi^{\pm};\,N]$ is the space of eigenspinors for the eigenvalue $\frac{N}{2}$.

Example

$$\phi^{+(1,0,1)}\cdot\phi^{-(0,0,0)}\,\in\,\mathbf{C}[\,\phi^{\pm};\,-2\,],$$
$$d_0(\phi^{+(1,0,1)}\cdot\phi^{-(0,0,0)})=-\frac{2}{2}(\phi^{+(1,0,1)}\cdot\phi^{-(0,0,0)}).$$

Proposition 10.

$$c(\,d_0\phi_1\,,\phi_2\,)\,+\,c(\,\phi_1\,,d_0\phi_2\,)\,=\,0\,. \tag{83}$$

In fact, since $\theta\,d_0\,=\,(\nu-\bar\nu)(\nu+\bar\nu)=\nu^2-\bar\nu^2\,=\,d_0\,\theta$, we have

$$\begin{aligned}0\;&=\;\int_{S^3}tr\,(\,d_0(\Theta\phi_1\cdot\phi_2)\,)\,d\sigma=\int_{S^3}tr\,(\,(d_0\Theta\phi_1)\cdot\phi_2+\Theta\phi_1\cdot d_0\phi_2\,)\,d\sigma\\ &=\;\int_{S^3}tr\,((\Theta\,d_0\phi_1)\cdot\phi_2\,)\,d\sigma+\int_{S^3}tr\,(\Theta\phi_1\cdot d_0\phi_2\,)\,d\sigma.\\ &=\;c(d_0\phi_1,\phi_2)+c(\phi_1,d_0\phi_2)\end{aligned}$$

4. Extensions of the Lie Algebra $\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})$

In this section we shall construct a central extension for the 3-dimensional loop algebra $Map(S^3,\mathfrak{g}^{\mathbf{H}})\;=\;S^3\mathbf{H}\otimes U(\mathfrak{g})$ associated to the above 2-cocycle c, and the central extension of $\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})$ induced from it. Then we shall give the second central extension by adding a derivative to the first extension that acts as the radial derivation.

4.1. Extension of $S^3\mathfrak{g}^{\mathbf{H}}\;=S^3\mathbf{H}\otimes U(\mathfrak{g})$

From Proposition 2 we see that $S^3\mathfrak{g}^{\mathbf{H}}\;=S^3\mathbf{H}\otimes U(\mathfrak{g})$ endowed with the following bracket $[\,,\,]_{S^3\mathfrak{g}^{\mathbf{H}}}$ becomes a Lie algebra.

$$[\,\phi\otimes X\,,\,\psi\otimes Y\,]_{S^3\mathfrak{g}^{\mathbf{H}}}=(\phi\cdot\psi)\otimes(XY)\;-(\psi\cdot\phi)\otimes(YX), \tag{84}$$

for the basis vectors X and Y of $U(\mathfrak{g})$ and $\phi,\,\psi\,\in S^3\mathbf{H}$.

We take the non-degenerate invariant symmetric bilinear $\mathbf{C}$-valued form $(\,\cdot\,|\,\cdot\,)$ on $\mathfrak{g}$ and extend it to $U(\mathfrak{g})$. For $X = X_1^{l_1}\cdots X_m^{l_m}$ and $Y = Y_1^{k_1}\cdots Y_m^{k_m}$ written by the basis $X_1,\cdots,X_m,\,Y_1,\cdots,Y_m$ of $\mathfrak{g}$, $(X|Y)$ is defined by

$$(X|Y) = tr(ad(X_1^{l_1})\cdots ad(X_m^{l_m})ad(Y_1^{k_1})\cdots ad(Y_m^{k_m})).$$

Then we define a $\mathbf{C}$-valued 2-cocycle on the Lie algebra $S^3\mathfrak{g}^{\mathbf{H}}$ by

$$c(\,\phi_1\otimes X\,,\,\phi_2\otimes Y\,) = \,(X|Y)\,c(\phi_1,\phi_2). \tag{85}$$

The 2-cocycle property follows from the fact $(XY|Z) = (YZ|X)$ and Proposition 8.

Let a be an indefinite number. There is an extension of the Lie algebra $S^3\mathfrak{g}^{\mathbf{H}}$ by the 1-dimensional center $\mathbf{C}a$ associated to the cocycle c. Explicitly, we have the following theorem.

Theorem 5. *The* $\mathbf{C}$*-vector space*

$$S^3\mathfrak{g}^{\mathbf{H}}(a) \;=\; (\,S^3\mathbf{H}\otimes U(\mathfrak{g})\,)\oplus(\mathbf{C}a), \tag{86}$$

endowed with the following bracket becomes a Lie algebra.

$$\begin{aligned} [\,\phi\otimes X\,,\,\psi\otimes Y\,]^{\widehat{}} \;&=\; (\phi\cdot\psi)\otimes\,(XY)-(\psi\cdot\phi)\otimes(YX)+(X|Y)\,c(\phi,\psi)\,a\,, && (87)\\ [\,a\,,\,\phi\otimes X\,]^{\widehat{}} \;&=\; 0\,, && (88)\end{aligned}$$

for the basis vectors X and Y of $U(\mathfrak{g})$ and $\phi,\,\psi\,\in S^3\mathbf{H}$.

As a Lie subalgebra of $S^3\mathfrak{g}^{\mathbf{H}}$ we have $\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})$.

Definition 6. *We denote by $\widehat{\mathfrak{g}}(a)$ the extension of the Lie algebra $\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})$ by the 1-dimensional center $\mathbf{C}a$ associated to the cocycle c:*

$$\widehat{\mathfrak{g}}(a) = \mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})\,\oplus(\mathbf{C}a). \tag{89}$$

The Lie bracket is given by

$$\begin{aligned} [\,\phi\otimes X\,,\,\psi\otimes Y\,]^{\widehat{}} \;&=\; (\phi\cdot\psi)\otimes\,(XY)-(\psi\cdot\phi)\otimes(YX)+(X|Y)\,c(\phi,\psi)\,a\,, && (90)\\ [\,a\,,\,\phi\otimes X\,]^{\widehat{}} \;&=\; 0\,, && (91)\end{aligned}$$

for $X,Y\in U(\mathfrak{g})$ and $\phi,\,\psi\,\in\mathbf{C}[\phi^{\pm}]$.

4.2. Extension of $\widehat{\mathfrak{g}}(a)$ by the Derivation

We introduced the radial derivative d_0 acting on $S^3\mathbf{H}$. d_0 preserves the space of spinors of Laurent polynomial type $\mathbf{C}[\phi^{\pm}]$. The derivation d_0 on $\mathbf{C}[\phi^{\pm}]$ is extended to a derivation of the Lie algebra $\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})$ by

$$d_0\,(\phi \otimes X\,) \,=\, (d_0\phi\,) \otimes X\,. \tag{92}$$

In fact we have from Equation (78)

$$\begin{aligned} & d_0\left(\,[\,\phi_1 \otimes X_1\,,\,\phi_2 \otimes X_2\,]\widehat{}\;\right) = d_0\,(\,(\phi_1\phi_2) \otimes (X_1X_2) \,-\, (\phi_2\phi_1) \otimes (X_2X_1)\,) \\ & = \,(d_0\phi_1 \cdot \phi_2) \otimes (X_1X_2) \,-\, (\phi_2 \cdot d_0\phi_1) \otimes (X_2X_1) + \,(\phi_1 \cdot d_0\phi_2) \otimes (X_1X_2) \\ & \qquad - (d_0\phi_2 \cdot \phi_1) \otimes (X_2X_1)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} & [\,d_0(\phi_1 \otimes X_1)\,,\,\phi_2 \otimes X_2\,]\widehat{}\, + \,[\,\phi_1 \otimes X_1\,,\,d_0(\phi_2 \otimes X_2)\,]\widehat{} \\ & = \,(d_0\phi_1 \cdot \phi_2) \otimes (X_1X_2) \,-\, (\phi_2 \cdot d_0\phi_1) \otimes (X_2X_1) \,+\, (\phi_1 \cdot d_0\phi_2) \otimes (X_1X_2) \\ & \qquad - (d_0\phi_2 \cdot \phi_1) \otimes (X_2X_1) \,+\, (X_1|X_2)\,(\,c(d_0\phi_1,\,\phi_2) \,+\, c(\phi_1,\,d_0\phi_2))\,a\,. \end{aligned}$$

Since $c(d_0\phi_1,\,\phi_2) \,+\, c(\phi_1,\,d_0\phi_2) = 0$ from Proposition 10 we have

$$\begin{aligned} & d_0\left(\,[\,\phi_1 \otimes X_1\,,\,\phi_2 \otimes X_2\,]\widehat{}\;\right) \\ & = \,[\,d_0(\phi_1 \otimes X_1)\,,\,\phi_2 \otimes X_2\,]\widehat{}\, + \,[\,\phi_1 \otimes X_1\,,\,d_0(\phi_2 \otimes X_2)\,]\widehat{}\;. \end{aligned}$$

Thus d_0 is a derivation that acts on the Lie algebra $\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})$.

We denote by $\widehat{\mathfrak{g}}$ the Lie algebra that is obtained by adjoining a derivation d to $\widehat{\mathfrak{g}}(a)$ which acts on $\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})$ as d_0 and which kills a. More explicitly we have the following

Theorem 7. *Let a and d be indefinite elements. We consider the $\mathbf{C}$ vector space:*

$$\widehat{\mathfrak{g}} \,=\, \left(\mathbf{C}[\phi^{\pm}] \otimes U(\mathfrak{g})\right) \oplus (\mathbf{C}\,a) \oplus (\mathbf{C}d)\,, \tag{93}$$

and define the following bracket on $\widehat{\mathfrak{g}}$. For $\phi,\,\psi \,\in\, \mathbf{C}[\phi^{\pm}]$ and the basis vectors X and Y of $U(\mathfrak{g})$, we put

$$\begin{aligned} [\,\phi \otimes X\,,\,\psi \otimes Y\,]_{\widehat{\mathfrak{g}}} \;&=\; [\,\phi \otimes X\,,\,\psi \otimes Y\,]\widehat{} \qquad\qquad (94) \\ &=\; (\phi \cdot \psi) \otimes\,(XY) - (\psi \cdot \phi) \otimes (YX) + (X|Y)\,c(\phi, \psi)\,a\,, \\ [\,a\,,\,\phi \otimes X\,]_{\widehat{\mathfrak{g}}} \;&=\; 0\,, \qquad [\,d, \phi \otimes X\,]_{\widehat{\mathfrak{g}}} = d_0\phi \otimes X\,, \qquad\qquad (95) \\ [\,d\,,\,a\,]_{\widehat{\mathfrak{g}}} \;&=\; 0\,. \qquad\qquad (96) \end{aligned}$$

Then $(\,\widehat{\mathfrak{g}}\,,\,[\,\cdot,\cdot\,]_{\widehat{\mathfrak{g}}}\,)$ becomes a Lie algebra.

Proof. It is enough to prove the following Jacobi identity:

$$[\,[\,d\,,\,\phi_1\otimes X_1\,]_{\widehat{\mathfrak{g}}}\,,\,\phi_2\otimes X_2\,]_{\widehat{\mathfrak{g}}}+[\,[\phi_1\otimes X_1,\phi_2\otimes X_2\,]_{\widehat{\mathfrak{g}}}\,,\,d\,]_{\widehat{\mathfrak{g}}}\,+\,[\,[\phi_2\otimes X_2,\,d\,]_{\widehat{\mathfrak{g}}},\,\phi_1\otimes X_1\,]_{\widehat{\mathfrak{g}}}=0.$$

In the following we shall abbreviate the bracket $[\,,\,]_{\widehat{\mathfrak{g}}}$ simply to $[\;,\;]$. We have

$$\begin{aligned}[\,[\,d\,,\,\phi_1\otimes X_1\,]\,,\phi_2\otimes X_2\,]=&[\,d_0\phi_1\otimes X_1,\,\phi_2\otimes X_2\,]\\ =&(\,d_0\phi_1\cdot\phi_2)\otimes\,(X_1X_2)-(\,\phi_2\cdot d_0\phi_1\,)\otimes(X_2X_1)\\ &+(X_1|X_2)c(\,d_0\phi_1\,,\,\phi_2)\,a\,.\end{aligned}$$

Similarly

$$\begin{aligned}[\,[\,\phi_2\otimes X_2,\,d\,],\phi_1\otimes X_1\,]=&(\phi_1\cdot d_0\phi_2)\otimes(X_1X_2)-(\,d_0\phi_2\cdot\phi_1)\otimes(X_2X_1)\\ &+(X_1|X_2)\,c(\phi_1,\,d_0\phi_2\,)\,a\,.\\ [\,[\phi_1\otimes X_1,\phi_2\otimes X_2\,]\,,\,d\,]=&-\Big[\,d\,,\,(\phi_1\cdot\phi_2)\otimes(X_1X_2)-(\phi_2\cdot\phi_1)\otimes(X_2X_1)+(X_1|X_2)c(\phi_1\,,\phi_2)\,a\,\Big]\\ =&-\,d_0(\phi_1\cdot\phi_2)\otimes\,(X_1X_2)+d_0(\phi_2\cdot\phi_1)\otimes(X_2X_1)\,.\end{aligned}$$

The sum of three equations vanishes by virtue of Equation (78) and Proposition 10. □

Remember from Definition 4 that $\mathbf{C}[\phi^{\pm};\,N]$ denotes the subspace in $\mathbf{C}[\phi^{\pm}]$ generated by the products $\phi_1\cdots\phi_n$ with each ϕ_i being $\phi_i=\phi^{+(m_i,l_i,k_i)}$ or $\phi_i=\phi^{-(m_i,l_i,k_i)}$, $i=1,\cdots,n$, such that

$$\sum_{i;\,\phi_i=\phi^{+(m_i,l_i,k_i)}}m_i\;-\sum_{i;\,\phi_i=\phi^{-(m_i,l_i,k_i)}}(m_i+3)\;=\;N\,.$$

Proposition 11. *The centralizer of d in $\widehat{\mathfrak{g}}$ is given by*

$$(\,\mathbf{C}[\phi^{\pm};\,0]\,\otimes U(\mathfrak{g})\,)\,\oplus\,\mathbf{C}a\,\oplus\mathbf{C}d\,. \tag{97}$$

The proposition follows from Equation (80) .

5. Structure of $\widehat{\mathfrak{g}}$

5.1. The Weight Space Decomposition of $U(\mathfrak{g})$

Let $(\,\mathfrak{g}\,,\,[\,,\,]_{\mathfrak{g}}\,)$ be a simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the root space decomposition with the root space $\mathfrak{g}_\alpha=\{X\in\mathfrak{g};\,ad(h)X\;=<\alpha,h>X,\quad\forall h\in\mathfrak{h}\}$. Here $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ is the set of roots and $\dim\,\mathfrak{g}_\alpha=1$. Let $\Pi=\{\alpha_i;\,i=1,\cdots,r=\mathrm{rank}\,\mathfrak{g}\}\subset\mathfrak{h}^*$ be the set of simple roots and $\{\alpha_i^\vee\,;\,i\,=\,1,\cdots,r\,\}\subset\mathfrak{h}$ be the set of simple coroots. The Cartan matrix $A=(\,a_{ij}\,)_{i,j=1,\cdots,r}$ is given by $a_{ij}=\langle\alpha_i^\vee,\,\alpha_j\rangle$. Fix a standard set of generators $H_i=\alpha_i^\vee$, $X_i\in\mathfrak{g}_{\alpha_i}$,

$Y_i \in \mathfrak{g}_{-\alpha_i}$, so that $[\,X_i,\,Y_j] = H_j\delta_{ij}$, $[H_i,\,X_j] = a_{ji}X_j$ and $[H_i,\,Y_j] = -a_{ji}Y_j$. Let $\Delta_\pm$ be the set of positive (respectively negative) roots of $\mathfrak{g}$ and put

$$\mathfrak{n}_\pm = \sum_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha\,.$$

Then $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. The enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has the direct sum decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{n}_-)\cdot U(\mathfrak{h})\cdot U(\mathfrak{n}_+)\,. \tag{98}$$

In the following we summarize the known results on the representation $(ad(\mathfrak{h}),\,U(\mathfrak{g})\,)$ [14,15]. The set

$$\{\,Y_1^{m_1}\cdots Y_r^{m_r}H_1^{l_1}\cdots H_r^{l_r}X_1^{n_1}\cdots X_r^{n_r}\,;\quad m_i,\,n_i,\,l_i\in \mathbf{N}\cup 0\,\}.$$

forms a basis of the enveloping algebra $U(\mathfrak{g})$. The adjoint action of $\mathfrak{h}$ is extended to that on $U(\mathfrak{g})$:

$$ad(h)(x\cdot y) = \,(\,ad(h)x\,)\cdot y\,+\,x\cdot(\,ad(h)y\,)\,.$$

$\lambda\in\mathfrak{h}^*$ is called a weight of the representation $(U(\mathfrak{g}),\,ad(\mathfrak{h})\,)$ if there exists a non-zero $x\in U(\mathfrak{g})$ such that $ad(h)x = hx - xh = \lambda(h)x$ for all $h\in\mathfrak{h}$. Let Σ be the set of weights of the representation $(U(\mathfrak{g}),\,ad(\mathfrak{h})\,)$. The weight space for the weight λ is by definition

$$\mathfrak{g}^U_\lambda \;=\; \{x\in U(\mathfrak{g})\,;\quad ad(h)x = \lambda(h)x,\quad \forall h\in\mathfrak{h}\}.$$

Let $\lambda = \sum_{i=1}^r n_i\alpha_i - \sum_{i=1}^r m_i\alpha_i$, $n_i,\,m_i\geq 0$. For any $l_1,\,l_2,\cdots,l_r\geq 0$,

$$\overline{X}_\lambda = Y_1^{m_1}\cdots Y_r^{m_r}H_1^{l_1}\cdots H_r^{l_r}X_1^{n_1}\cdots X_r^{n_r}$$

gives a weight vector with the weight λ ; $\overline{X}_\lambda\in\mathfrak{g}^U_\lambda$. Conversely any weight λ may be written in the form $\lambda = \sum_{i=1}^r n_i\alpha_i - \sum_{i=1}^r m_i\alpha_i$, though the coefficients n_i , m_i are not uniquely determined.

Lemma 5.

1. *The set of weights of the adjoint representation* $(U(\mathfrak{g}),\,ad(\mathfrak{h}))$ *is*

$$\Sigma = \{\sum k_i\alpha_i\,;\quad \alpha_i\in\Pi\,,\,k_i\in\mathbf{Z}\,\}. \tag{99}$$

If we denote

$$\Sigma_\pm = \{\,\pm\sum n_i\alpha_i\,\in\Sigma\,;\quad n_i>0\,\} \tag{100}$$

then $\Sigma_\pm\cap\Delta = \Delta_\pm$*.*

2. *If* $\lambda\in\Sigma$ *then* $-\lambda\in\Sigma$*.*
3. *For each* $\lambda = \sum_{i=1}^r k_i\alpha_i\,\in\Sigma$*,* $\mathfrak{g}^U_\lambda$ *is generated by the basis*

$$\overline{X}_\lambda(l_1,\cdots,l_r,m_1,\cdots,m_r,n_1,\cdots,n_r) = Y_1^{m_1}\cdots Y_r^{m_r}H_1^{l_1}\cdots H_r^{l_r}X_1^{n_1}\cdots X_r^{n_r}$$

with $n_i,\, m_i,\, l_i \in \mathbf{N}\cup 0$ *such that* $k_i = n_i - m_i\,,\ i=1,\cdots,r\,.$
In particular $\mathfrak{g}_0^U$ *is generated by the basis*

$$\overline{X}_0(l_1,\cdots,l_r,n_1,\cdots,n_r,n_1,\cdots,n_r) = Y_1^{n_1}\cdots Y_r^{n_r}\,H_1^{l_1}\cdots H_r^{l_r}\,X_1^{n_1}\cdots X_r^{n_r}$$

with $n_i,\, l_i \in \mathbf{N}\cup 0\,,\ i=1,\cdots,r\,.$ *In particular*

$$U(\mathfrak{h}) \subset \mathfrak{g}_0^U\,.$$

4.

$$[\,\mathfrak{g}_\lambda^U\,,\,\mathfrak{g}_\mu^U\,] \,\subset\, \mathfrak{g}_{\lambda+\mu}^U\,. \tag{101}$$

5.2. Weight Space Decomposition of $\widehat{\mathfrak{g}}$

In the following we shall investigate the Lie algebra structure of

$$\widehat{\mathfrak{g}} \,=\, \big(\,\mathbf{C}[\phi^{\pm}]\otimes U(\mathfrak{g})\,\big)\oplus(\mathbf{C}\,a)\oplus(\mathbf{C}d)\,. \tag{102}$$

Remember that the Lie bracket was defined by

$$\begin{aligned}
[\,\phi\otimes X\,,\,\psi\otimes Y\,]_{\widehat{\mathfrak{g}}} &= (\phi\,\psi)\otimes(XY)-(\psi\phi)\otimes(YX)+\,(X|Y)\,c(\phi,\psi)\,a\,,\\
[\,a\,,\phi\otimes X\,]_{\widehat{\mathfrak{g}}}=0\,, &\qquad [\,a,\,d\,]_{\widehat{\mathfrak{g}}}=0\,,\\
[\,d,\phi\otimes X\,]_{\widehat{\mathfrak{g}}} &= d_0\phi\otimes X\,,
\end{aligned}$$

for the basis vectors X and Y of $U(\mathfrak{g})$. Since $\phi^{+(0,0,1)} = \begin{pmatrix}1\\0\end{pmatrix}$ we identify $X\in U(\mathfrak{g})$ with $\phi^{+(0,0,1)}\otimes X$. Thus we look $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$:

$$\big[\phi^{+(0,0,1)}\otimes X,\,\phi^{+(0,0,1)}\otimes Y\big]_{\widehat{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}}\,, \tag{103}$$

and we shall write $\phi^{+(0,0,1)}\otimes X$ simply as X.

Let

$$\widehat{\mathfrak{h}} \,=\, (\,(\,\mathbf{C}\,\phi^{+(0,0,1)}\,)\otimes\mathfrak{h})\,\oplus(\mathbf{C}\,a)\oplus(\mathbf{C}\,d)\,\,=\mathfrak{h}\oplus(\mathbf{C}\,a)\oplus(\mathbf{C}d)\,. \tag{104}$$

We write $\hat{h} = h + sa + td \in \widehat{\mathfrak{h}}$ with $h\in\mathfrak{h}$ and $s,\,t\in\mathbf{C}$. For any $h\in\mathfrak{h}$, $\phi\in\mathbf{C}[\phi^{\pm}]$ and $X\in U(\mathfrak{g})$, it holds that

$$\begin{aligned}
[\,\phi^{+(0,0,1)}\otimes h,\,\phi\otimes X\,]_{\widehat{\mathfrak{g}}} &= \phi\otimes(\,hX-Xh\,)\,,\\
[\,d,\,\phi\otimes X\,]_{\widehat{\mathfrak{g}}} &= (d_0\phi\,)\otimes X\,,\\
[\,\phi^{+(0,0,1)}\otimes h,\,a\,]_{\widehat{\mathfrak{g}}} &= 0\,,\quad [\,\phi^{+(0,0,1)}\otimes h,\,d\,]_{\widehat{\mathfrak{g}}}=0,\quad [\,d,a\,]_{\widehat{\mathfrak{g}}}=0\,.
\end{aligned}$$

Then the adjoint action of $\hat{h} = h + sa + td \in \widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{g}}$ is written as follows

$$ad(\hat{h})\,(\phi \otimes X + \mu a + \nu d) = \phi \otimes (hX - Xh) + td_0\phi \otimes X\,, \tag{105}$$

for $\xi = \phi \otimes X + \mu a + \nu d \in \widehat{\mathfrak{g}}$.

An element λ of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ can be regarded as an element of $\widehat{\mathfrak{h}}^*$ by putting

$$\langle \lambda, a \rangle = \langle \lambda, d \rangle = 0. \tag{106}$$

So $\Delta \subset \mathfrak{h}^*$ is seen to be a subset of $\widehat{\mathfrak{h}}^*$. We define the elements $\delta\,,\, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ by

$$\langle \delta, \alpha_i^\vee \rangle = \langle\, \Lambda_0, \alpha_i^\vee \,\rangle = 0, \qquad (1 \leqq i \leqq r), \tag{107}$$

$$\langle\, \delta, a \rangle = 0\,, \qquad \langle\, \delta, d\,\rangle = 1, \tag{108}$$

$$\langle\, \Lambda_0, a \rangle = 1\,, \qquad \langle\, \Lambda_0, d\,\rangle = 0. \tag{109}$$

Then the set $\{\, \alpha_1, \cdots, \alpha_r,\, \Lambda_0,\, \delta\,\}$ forms a basis of $\widehat{\mathfrak{h}}^*$. Similarly Σ is a subset of $\widehat{\mathfrak{h}}^*$.

Since $\widehat{\mathfrak{h}}$ is a commutative subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{g}}$ is decomposed into a direct sum of the simultaneous eigenspaces of $ad\,(\hat{h}),\ \hat{h} \in \widehat{\mathfrak{h}}$.

For $\lambda = \gamma + k_0\delta \in \widehat{\mathfrak{h}}^*$, $\gamma = \sum_{i=1}^r k_i\alpha_i \in \Sigma$, $k_i \in \mathbf{Z}$, $i = 0, 1, \cdots, r$, we put,

$$\widehat{\mathfrak{g}}_\lambda = \left\{ \xi \in \widehat{\mathfrak{g}}\,; \quad [\,\hat{h},\, \xi\,] \,=\, \langle \lambda, \hat{h} \rangle\, \xi \quad \text{for } \forall \hat{h} \in \widehat{\mathfrak{h}}\, \right\}. \tag{110}$$

λ is called a weight of $\widehat{\mathfrak{g}}$ if $\widehat{\mathfrak{g}}_\lambda \neq 0$. $\widehat{\mathfrak{g}}_\lambda$ is called the weight space of λ.

Let $\widehat{\Sigma}$ denote the set of weights of the representation $\left(\widehat{\mathfrak{g}}\,, ad(\widehat{\mathfrak{h}})\right)$.

Theorem 8.

1.

$$\begin{aligned} \widehat{\Sigma} \;&=\; \left\{ \frac{m}{2}\delta + \lambda; \quad \lambda \in \Sigma\,,\, m \in \mathbf{Z} \right\} \\ &\quad \bigcup \left\{ \frac{m}{2}\delta; \quad m \in \mathbf{Z} \right\}. \end{aligned}$$

2. *For $\lambda \in \Sigma$, $\lambda \neq 0$ and $m \in \mathbf{Z}$, we have*

$$\widehat{\mathfrak{g}}_{\frac{m}{2}\delta + \lambda} = \mathbf{C}[\phi^{\pm};\, m\,] \otimes \mathfrak{g}_\lambda^U\,. \tag{111}$$

3.

$$\begin{aligned} \widehat{\mathfrak{g}}_{0\delta} \;&=\; (\,\mathbf{C}[\phi^{\pm}; 0\,] \otimes \mathfrak{g}_0^U\,) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d) \;\supset\; \widehat{\mathfrak{h}}\,, \\ \widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \;&=\; \mathbf{C}[\phi^{\pm};\, m\,] \otimes \mathfrak{g}_0^U\,, \quad \textit{for } 0 \neq m \in \mathbf{Z}\,. \end{aligned}$$

4. $\widehat{\mathfrak{g}}$ *has the following decomposition:*

$$\widehat{\mathfrak{g}} = \bigoplus_{m \in \mathbf{Z}} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta} \bigoplus \bigoplus_{\lambda \in \Sigma,\, m \in \mathbf{Z}} \widehat{\mathfrak{g}}_{\frac{m}{2}\delta + \lambda} \,. \tag{112}$$

Proof. First we prove the second assertion. Let $X \in \mathfrak{g}^U_\lambda$ for a $\lambda \in \Sigma$, $\lambda \neq 0$, and let $\varphi \in \mathbf{C}[\phi^\pm;\, m]$ for a $m \in \mathbf{Z}$. We have, for any $h \in \mathfrak{h}$,

$$\begin{aligned} [\,\phi^{+(0,0,1)} \otimes h,\, \varphi \otimes X\,]_{\widehat{\mathfrak{g}}} &= \varphi \otimes (hX - Xh) \,=\, \langle \lambda, h \rangle\, \varphi \otimes X, \\ [\,d,\, \varphi \otimes X\,]_{\widehat{\mathfrak{g}}} &= \frac{m}{2} \varphi \otimes X, \end{aligned}$$

that is, for every $\hat{h} \in \widehat{\mathfrak{h}}$, we have

$$[\,\hat{h}\,, \varphi \otimes X]_{\widehat{\mathfrak{g}}} = \left\langle \frac{m}{2}\delta + \lambda\,,\, \hat{h} \right\rangle (\varphi \otimes X)\,. \tag{113}$$

Therefore we have $\varphi \otimes X \in \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda}$.

Conversely, for a given $m \in \mathbf{Z}$ and a $\xi \in \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda}$, we shall show that ξ has the form $\phi \otimes X$ with $\phi \in \mathbf{C}[\phi^\pm; m]$ and $X \in \mathfrak{g}^U_\lambda$. Let $\xi = \phi \otimes X + \mu a + \nu d$ for $\phi \in \mathbf{C}[\phi^\pm],\ X \in U(\mathfrak{g})$ and $\mu,\, \nu \in \mathbf{C}$. ϕ is decomposed to the sum

$$\phi = \sum_{n \in \mathbf{Z}} \phi_n$$

by the homogeneous degree; $\phi_n \in \mathbf{C}[\phi^\pm; n]$. We have

$$\begin{aligned} [\hat{h}, \xi] = [\,\phi^{+(0,0,1)} \otimes h + sa + td\,,\, \phi \otimes X \,+ \mu a + \nu d\,] = \phi \otimes [\,h\,,\, X\,] \\ + t(\sum_{n \in \mathbf{Z}} \frac{n}{2} \phi_n \otimes X\,) \end{aligned}$$

for any $\hat{h} = \phi^{+(0,0,1)} \otimes h + sa + td \in \widehat{\mathfrak{h}}$. From the assumption we have

$$\begin{aligned} [\,\hat{h}, \xi\,] &= \langle \frac{m}{2}\delta + \lambda\,,\, \hat{h} \,\rangle\, \xi \\ &= \,<\lambda, h>\, \phi \otimes X \,+ (\frac{m}{2}t + <\lambda, h>)(\mu a + \nu d) \\ &\quad + \frac{m}{2} t\,(\sum_n \phi_n) \otimes X. \end{aligned}$$

Comparing the above two equations we have $\mu = \nu = 0$, and $\phi_n = 0$ for all n except for $n = m$. Therefore $\phi \in \mathbf{C}[\phi^\pm; m]$. We also have $[\hat{h}, \xi] = \phi \otimes [h, X] = \langle \lambda,\, h \rangle\, \phi \otimes X$ for all $\hat{h} = \phi^{+(0,0,1)} \otimes h + sa + td \in \widehat{\mathfrak{h}}$. Hence $X \in \mathfrak{g}^U_\lambda$ and $\xi = \phi_m \otimes X \in \widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda}$. We have proved

$$\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\lambda} = \mathbf{C}[\phi^\pm; m] \otimes \mathfrak{g}^U_\lambda\,.$$

The proof of the third assertion is also carried out by the same argument as above if we revise it for the case $\lambda = 0$. The above discussion yields the first and the fourth assertions. $\square$

Proposition 12. *We have the following relations:*

1.
$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}}\subset\,\widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta+\alpha+\beta}\ ,\tag{114}$$
for $\alpha,\ \beta\in\widehat{\Sigma}$ *and for* $m,n\in\mathbf{Z}$.

2.
$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}}\subset\,\widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta}\,,\tag{115}$$
for $m,n\in\mathbf{Z}$.

Proof. Let $\phi\otimes X\in\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}$ and $\psi\otimes Y\in\widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}$. Then we have, for $h\in\mathfrak{h}$,

$$\begin{aligned}
[\,h,[\,\phi\otimes X,\psi\otimes Y]\,] &= -[\,\phi\otimes X,[\,\psi\otimes Y,h\,]\,]-[\,\psi\otimes Y,[\,h,\phi\otimes X]\,]\\
&=<\beta,h>[\,\phi\otimes X,\psi\otimes Y\,]+<\alpha,h>[\,\phi\otimes X,\psi\otimes Y\,]\\
&=<\alpha+\beta,h>[\,\phi\otimes X,\psi\otimes Y\,].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\,d,[\,\phi\otimes X,\psi\otimes Y\,]\,] &= -[\,\phi\otimes X,[\psi\otimes Y,d]\,]-[\,\psi\otimes Y,[\,d,\phi\otimes X]\,]\\
&=\frac{m+n}{2}[\,\phi\otimes X,\psi\otimes Y\,]\,.
\end{aligned}$$

Hence
$$[\,\widehat{h},[\,\phi\otimes X,\psi\otimes Y\,]\,]=\left\langle\frac{m+n}{2}\delta+\alpha+\beta\,,\,\widehat{h}\right\rangle[\,\phi\otimes X,\psi\otimes Y\,]\tag{116}$$
for any $\widehat{h}\in\widehat{\mathfrak{h}}$. Therefore
$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta+\alpha}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta+\beta}\,\right]_{\widehat{\mathfrak{g}}}\subset\,\widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta+\alpha+\beta}\ ,\tag{117}$$

The same calculation for $\phi\otimes H\in\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}$ and $\psi\otimes H'\in\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}$ yields

$$\left[\,\widehat{\mathfrak{g}}_{\frac{m}{2}\delta}\,,\,\widehat{\mathfrak{g}}_{\frac{n}{2}\delta}\,\right]_{\widehat{\mathfrak{g}}}\subset\,\widehat{\mathfrak{g}}_{\frac{m+n}{2}\delta}\,.\tag{118}$$

□

5.3. *Generators of* $\widehat{\mathfrak{g}}$

Let $\{\alpha_i\}_{i=1,\cdots,r}\subset\mathfrak{h}^*$ be the set of simple roots and $\{h_i\}_{i=1,\cdots,r}\subset\mathfrak{h}$ be the set of simple coroots. e_i, f_i, $i=1,\cdots,r$, denote the Chevalley generators;

$$\begin{aligned}
[\,e_i,\,f_j\,] &= \delta_{ij}h_i,\\
[\,h,\,e_i\,] &= \alpha_i(h)\,,\quad [\,h,\,f_i\,]=-\alpha_i(h),\quad \text{for }\forall h\in\mathfrak{h}.
\end{aligned}$$

Let $A=(\,a_{ij}\,)_{i,j=1,\cdots,r}$ be the Cartan matrix of $\mathfrak{g}$; $a_{ij}=\alpha_i(h_j)$.

By the natural embedding of $\mathfrak{g}$ in $\widehat{\mathfrak{g}}$ we have the vectors

$$\widehat{h}_i = \phi^{+(0,0,1)} \otimes h_i \in \widehat{\mathfrak{h}}, \tag{119}$$

$$\widehat{e}_i = \phi^{+(0,0,1)} \otimes e_i \in \widehat{\mathfrak{g}}_{0\delta+\alpha_i}, \quad \widehat{f}_i = \phi^{+(0,0,1)} \otimes f_i \in \widehat{\mathfrak{g}}_{0\delta-\alpha_i}, \qquad i = 1, \cdots, r. \tag{120}$$

It is easy to verify the relations:

$$\left[\widehat{e}_i\,, \widehat{f}_j\right]_{\widehat{\mathfrak{g}}} = \delta_{ij}\,\widehat{h}_i\,, \tag{121}$$

$$\left[\widehat{h}_i\,, \widehat{e}_j\right]_{\widehat{\mathfrak{g}}} = a_{ij}\,\widehat{e}_j, \quad \left[\widehat{h}_i\,, \widehat{f}_j\right]_{\widehat{\mathfrak{g}}} = -a_{ij}\,\widehat{f}_j\,, \quad 1 \le i, j \le r. \tag{122}$$

We have obtained a part of generators of $\widehat{\mathfrak{g}}$ that come naturally from $\mathfrak{g}$.

We recall that for an affine Lie algebra $(\mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus (\mathbf{C}a) \oplus (\mathbf{C}d)$ there is a special Chevalley generator coming from the irreducible representation spaces $t^{\pm 1} \otimes \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$. Let θ be the highest root of $\mathfrak{g}$ and suppose that $e_\theta \in \mathfrak{g}_\theta$ and $e_{-\theta} \in \mathfrak{g}_{-\theta}$ satisfy the relations $(e_\theta | e_{-\theta}) = 1$ and $[e_\theta, e_{-\theta}] = h_\theta$, then we have a Chevalley generator $\{\, t \otimes e_{-\theta},\ t^{-1} \otimes e_\theta,\ -t^0 \otimes h_\theta + a\}$ for the subalgebra $(\mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus (\mathbf{C}a)$ and adding d we have the Chevalley generators of the affine Lie algebra [2,5,16]. In the sequel we shall do a similar observation for our Lie algebra $\widehat{\mathfrak{g}}$. We put

$$\kappa = \phi^{+(1,0,1)}\,, \quad \kappa_* = \sqrt{-1}\begin{pmatrix} \overline{z}_2 \\ \overline{z}_1 \end{pmatrix} = -\frac{\sqrt{-1}}{\sqrt{2}}\phi^{+(1,1,2)} + \frac{\sqrt{-1}}{2}(\phi^{-(0,0,0)} - \phi^{+(1,0,1)})\,.$$

$$\mu = \phi^{-(0,0,0)}\,, \quad \mu_* = \sqrt{-1}\begin{pmatrix} \overline{z}_2 \\ -\overline{z}_1 \end{pmatrix} = -\frac{\sqrt{-1}}{\sqrt{2}}\phi^{+(1,1,2)} - \frac{\sqrt{-1}}{2}(\phi^{-(0,0,0)} - \phi^{+(1,0,1)})\,.$$

We recall that $J = \phi^{+(0,0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Lemma 6.

1.

$$\kappa\,\kappa_* = \kappa_*\,\kappa = \mu\,\mu_* = \mu_*\,\mu = \sqrt{-1}\phi^{+(0,0,1)}. \tag{123}$$

2.

$$c(\kappa, \kappa_*) = c(\mu, \mu_*) = 1. \tag{124}$$

We consider the following vectors of $\widehat{\mathfrak{g}}$:

$$\widehat{f}_J = J \otimes e_{-\theta} \in \widehat{\mathfrak{g}}_{0\delta-\theta}\,, \qquad \widehat{e}_J = (-J) \otimes e_\theta \in \widehat{\mathfrak{g}}_{0\delta+\theta}\,, \tag{125}$$

$$\widehat{f}_\kappa = \kappa \otimes e_{-\theta} \in \widehat{\mathfrak{g}}_{\frac{1}{2}\delta-\theta}\,, \qquad \widehat{e}_\kappa = \kappa_* \otimes e_\theta \in \widehat{\mathfrak{g}}_{\frac{1}{2}\delta+\theta} \oplus \widehat{\mathfrak{g}}_{-\frac{3}{2}\delta+\theta}\,, \tag{126}$$

$$\widehat{f}_\mu = \mu \otimes e_{-\theta} \in \widehat{\mathfrak{g}}_{-\frac{3}{2}\delta-\theta}\,, \qquad \widehat{e}_\mu = \mu_* \otimes e_\theta \in \widehat{\mathfrak{g}}_{\frac{1}{2}\delta+\theta} \oplus \widehat{\mathfrak{g}}_{-\frac{3}{2}\delta+\theta}\,. \tag{127}$$

Then we have the generators of $\widehat{\mathfrak{g}}(a)$ that are given by the following three tuples:

$$\left(\widehat{e}_i, \widehat{f}_i, \widehat{h}_i\right) \quad i = 1, 2, \cdots, r,$$
$$\left(\widehat{e}_\mu, \widehat{f}_\mu, \widehat{h}_\theta\right), \quad \left(\widehat{e}_\kappa, \widehat{f}_\kappa, \widehat{h}_\theta\right), \quad \left(\widehat{e}_J, \widehat{f}_J, \widehat{h}_\theta\right) \; .$$

These three tuples satisfy the following relations.

Proposition 13.

1.
$$\left[\widehat{e}_\pi \, , \, \widehat{f}_i\right]_{\widehat{\mathfrak{g}}} = \left[\widehat{f}_\pi \, , \, \widehat{e}_i\right]_{\widehat{\mathfrak{g}}} = 0 \, , \quad \textit{for } 1 \leq i \leq r, \; \textit{and } \pi = J, \, \kappa, \, \mu \, . \tag{128}$$

2.
$$\left[\widehat{e}_J \, , \, \widehat{f}_J\right]_{\widehat{\mathfrak{g}}} = \widehat{h}_\theta \, , \tag{129}$$

3.
$$\left[\widehat{e}_\mu \, , \, \widehat{f}_\mu\right]_{\widehat{\mathfrak{g}}} = \sqrt{-1}\,\widehat{h}_\theta + a, \quad \left[\widehat{e}_\kappa \, , \, \widehat{f}_\kappa\right]_{\widehat{\mathfrak{g}}} = \sqrt{-1}\,\widehat{h}_\theta + a \, . \tag{130}$$

Acknowledgments

A résumé of these results has appeared in [17]. The present article is devoted to the explanation of these results with detailed proof. The authors would like to express their thanks to Yasushi Homma of Waseda University for his valuable objections to the early version of this paper.

Author Contributions

Tosiaki Kori and Yuto Imai wrote the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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