symmetry

Article

Regular and Chiral Polyhedra in Euclidean Nets

Daniel Pellicer

Centro de Ciencias Matemáticas, UNAM-Morelia, Antigua Carretera a Patzcuaro 8701, 58087 Morelia, Mexico; pellicer@matmor.unam.mx; Tel.: +52-443-322-2783

Academic Editor: Egon Schulte
Received: 14 September 2016; Accepted: 23 October 2016; Published: 28 October 2016

Abstract: We enumerate the regular and chiral polyhedra (in the sense of Grünbaum's skeletal approach) whose vertex and edge sets are a subset of those of the primitive cubic lattice, the face-centred cubic lattice, or the body-centred cubic lattice.

Keywords: regular polyhedron; chiral polyhedron; net

---

1. Introduction

The primitive cubic lattice, face-centred cubic lattice, and body-centred cubic lattice are well-known geometric objects widely studied in mathematics, and also used in other fields as models of various concepts. There are several interesting topics closely related to these lattices; for example, packings of spheres in Euclidean space [1] and crystal systems [2]. These lattices are a useful tool for the study of the affinely-irreducible discrete groups of isometries of Euclidean space [3], and are the ambient space for other interesting mathematical objects [4–6].

According to [7], a polyhedron is a connected and discrete collection of polygons in Euclidean space where every edge belongs to two cycles (faces). Polyhedra are not required to be convex, are not required to be finite, and their faces are not required to be contained in a plane. Regular and chiral polyhedra admit combinatorial rotations along all their faces and around all their vertices. Regular polyhedra admit combinatorial reflections, while chiral polyhedra do not. Regular polyhedra were classified in [8,9]; chiral polyhedra were classified in [10,11].

In this paper, we prove the following theorem.

Theorem 1. *The only chiral polyhedra in Euclidean space whose vertex and edge sets are subsets of **pcu**, **fcu**, or **bcu** are $P(1,0)$ and $P_1(1,0)$ in **pcu**, and $Q(1,1)$ and $P_2(1,-1)$ in **bcu**.*

The polyhedra $P(1,0)$, $P_1(1,0)$, $Q(1,1)$, and $P_2(1,-1)$ are among the chiral polyhedra described in [10,11]. We recall some of their main aspects and provide some local pictures in Section 4.

In Section 2, we provide background on the primitive cubic lattice **pcu**, face-centred cubic lattice **fcu**, and body-centred cubic lattice **bcu** in the context of nets of Euclidean space. Rotary, chiral, and regular polyhedra are defined and described in Section 3. Finally, in Section 4, we prove Theorem 1 by enumerating the rotary polyhedra whose vertex and edge sets can be taken from those of one of the three lattices mentioned above.

2. Nets

By a net, we mean a connected graph embedded with straight edges in Euclidean space $\mathbb{R}^3$, invariant under translations by three linearly independent vectors. The vertex set of the net must be discrete. Nets arise naturally when modelling periodic structures in chemistry. They are commonly denoted by three letters **abc** that often carry information about chemical compounds whose links can be represented by that net. They are also natural structures for mathematicians to study.

A symmetry of a net is an isometry of $\mathbb{R}^3$ that preserves the vertex and edge sets. A wealth of highly-symmetric nets can be found in the Reticular Chemistry Structure Resource database [12]. In what follows, we describe some of these nets that are relevant for this work. They are all highly related with the cubic tessellation $\mathcal{T}$ of Euclidean space. For convenience, we assume that the vertex set of $\mathcal{T}$ is the set of points with integer coordinates.

The primitive cubic lattice **pcu** consists of all vertices and edges of $\mathcal{T}$ (see Figure 1a). Every vertex is 6-valent and has edges with three distinct direction vectors; namely, those of the coordinate axes.

The face-centred cubic lattice **fcu** can be constructed from $\mathcal{T}$ by taking as vertices all vertices of **pcu** whose sum of coordinates is even (one part of the natural bipartition of **pcu**) and as edges all diagonals of squares of $\mathcal{T}$ with endpoints in the vertex set (see Figure 1b). The vertices are 12-valent, and there are six distinct directions of the edges. There are precisely four vertices of each cube of $\mathcal{T}$ in the vertex set of **fcu**; their convex hull is a regular tetrahedron. Given any vertex of **pcu** that is not a vertex of **fcu**, its six neighbours in **pcu** all belong to **fcu**, and are the vertices of an octahedron. These tetrahedra and octahedra are the cells of tessellation #1 in [13]. Each triangle of that tessellation can be extended to a plane tessellation by equilateral triangles, where all triangles are also triangles of tessellation #1 in [13]. By gluing sets of six triangles together, we can obtain the vertex and edge set of a tessellation by regular hexagons as a subset of the net **fcu**.

The body-centred cubic lattice **bcu** has as vertex set all vertices of **pcu** whose coordinates are either all odd or all even; two vertices are adjacent whenever they are endpoints of a diagonal of a cube of $\mathcal{T}$ (see Figure 1c, where the thin gray lines represent only those lines containing edges of $\mathcal{T}$ where two of the coordinates are even). The vertices are 8-valent, and there are four distinct directions of the edges.

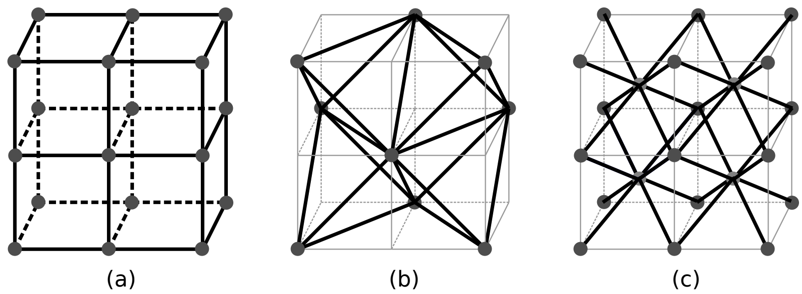

Figure 1. Nets **pcu** (**a**); **fcu** (**b**); and **bcu** (**c**).

The three nets just described are the only nets whose symmetry groups induce only one kind of vertex and whose vertex-stabilizers are isomorphic to the symmetry group $[3,4]$ of the octahedron.

The symmetry group of any net is a crystallographic group, and hence it contains no rotations of order 5 (for example, see [10], Lemma 4.1).

In this paper, we shall think of the nets as rigid objects in Euclidean space, although the same combinatorial structures could be embedded in a less symmetric way. Except in the third column of Tables 2–4, all nets should be understood as geometric objects and not only as combinatorial ones.

We shall need the following straightforward result.

Lemma 1. *The angle between two edges incident to the same vertex is*

- *either π or $\pi/2$ if the edges are in **pcu**;*
- *either $\pi/3$, $\pi/2$, $2\pi/3$, or π if the edges are in **fcu**;*
- *and either $\cos^{-1}(1/3)$, $\cos^{-1}(-1/3)$, or π if the edges are in **bcu**.*

The angles listed in Lemma 1 are highlighted in Figure 2.

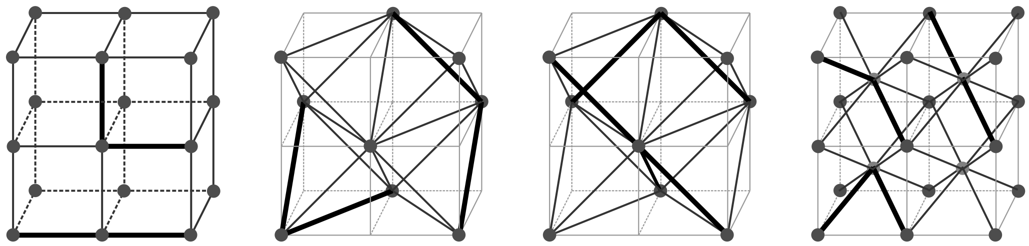

Figure 2. The angles formed by edges of the nets **pcu**, **fcu** and **bcu**.

3. Regular and Chiral Polyhedra

Highly symmetric convex polyhedra have been studied for centuries. It is well-known that there are only five convex regular polyhedra and 13 Archimedean solids, which, together with the infinite families of prisms and antiprisms, are the only convex polyhedra whose faces are regular polygons and whose symmetry groups induce only one kind of vertex.

In order to get a richer theory, we admit other combinatorial structures in $\mathbb{R}^3$ as polyhedra. In particular, we allow infinite polyhedra and even infinite polygons, and we abandon the idea of polygons being spanned by a 2-dimensional membrane.

3.1. Definitions

For us, a polygon is an embedding to $\mathbb{R}^3$ of a connected 2-regular graph; that is, of a cycle or of a two-sided infinite path. Polygons are explicitly allowed to be • skew (non-planar) and infinite, but we require the vertex set to be discrete.

A polyhedron is a (finite or infinite) collection of polygons (also called faces) with the properties that

- the set of vertices is discrete,
- the graph determined by all vertices and edges is connected,
- every edge belongs to exactly two faces,
- the vertex-figure at every vertex is a finite polygon. (The *vertex-figure* at a vertex v is the graph whose vertices are the neighbours of v, two of them joined by an edge whenever they are the neighbours of v in some face of the polyhedron.)

Convex polyhedra clearly satisfy the previous definition, as also do face-to-face tilings of the Euclidean plane (embedded in $\mathbb{R}^3$) and many more interesting structures.

A symmetry of a polyhedron $\mathcal{P}$ is an isometry of $\mathbb{R}^3$ that preserves $\mathcal{P}$. The group of all isometries of $\mathcal{P}$ is denoted $G(\mathcal{P})$.

Whenever there is a symmetry of $\mathcal{P}$ that cyclically permutes the vertices of a face F, we say that $\mathcal{P}$ has *abstract rotations along* F. Similarly, if there is a symmetry of $\mathcal{P}$ that cyclically permutes the neighbours of a given vertex v, we say that $\mathcal{P}$ has *abstract rotations around* v. An *abstract reflection* of $\mathcal{P}$ is a symmetry that, for some triple of mutually incident vertex, edge, and face, it preserves two of the elements while moving the third.

When $\mathcal{P}$ is a convex polyhedron, abstract rotations and abstract reflections are indeed rotations around some axes and reflections with respect to planes. However, if the faces are not planar, or they are not finite, then the abstract rotations about the faces are determined by the nature of the faces, and cannot be rotations about lines.

Some polygons admitting abstract rotations are shown in Figure 3. An abstract rotation of a finite polygon that has all vertices on a plane may be either a geometric rotation or a rotatory reflection (composition of a rotation about a line l and a reflection with respect to a plane perpendicular to l). If the polygon is finite but skew (no plane contains all vertices), then an abstract rotation mapping a vertex to an adjacent vertex is necessarily a rotatory reflection. Some examples of finite polygons can be seen in the left side of Figure 3. The abstract rotation that maps a vertex of a planar zigzag to a neighbouring vertex is either a twist (composition of a translation and a rotation with axis in the direction of the translation) or a glide reflection (composition of a translation and a reflection about the plane containing the direction of the translation). If the polygon is a helix then such an abstract rotation is necessarily a twist. A helix and a zigzag are shown in the right side of Figure 3. In addition to the polygons mentioned above, polygons having all edges in the same line also admit abstract rotations, but they are not relevant for this work.

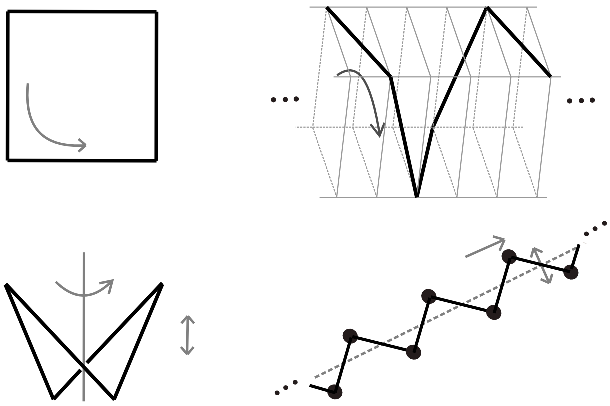

Figure 3. Abstract rotations of polygons.

We say that a polyhedron is *rotary* whenever it admits all possible abstract rotations around all its faces and around all its vertices. A rotary polyhedron is *regular* if in addition it admits an abstract reflection, and *chiral* otherwise. The Platonic solids are the only convex polyhedra that are regular under this definition. Furthermore, there are no convex chiral polyhedra.

The definitions of polyhedron, regular, and chiral in this section are equivalent to those in [10,11,14], and differ mildly only on the condition on the vertex-figure with those in [7]. The above use of the term "chiral" is widely accepted in the community studying abstract polytopes and related topics; to avoid confusion, the reader should bear in mind that in chemistry this word is used to denote a substantially different property.

All faces of a regular or chiral polyhedron have the same number p of edges, and all vertices have the same degree q. The pair $\{p,q\}$ is called the *Schläfli type* (or just *type*) of the polytope. When studying rotary polyhedra, p is allowed to be ∞, but q must be finite to prevent the vertex set from being non-discrete.

A *Petrie polygon* of a polyhedron $\mathcal{P}$ is a closed walk on the vertex and edge sets of $\mathcal{P}$, where any two consecutive edges—but not three—belong to the same face. The structure obtained from the vertex and edge set of $\mathcal{P}$, but considering the Petrie polygons as faces is often a polyhedron. When this occurs, it is called the *Petrial* of $\mathcal{P}$.

The Petrial of the cube is outlined in Figure 4. The four faces are the hexagonal Petrie polygons in thick lines. Note that every edge belongs to precisely two such polygons.

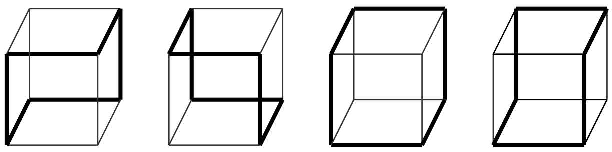

Figure 4. Petrial of the cube.

3.2. Regular Polyhedra

In [7], Grünbaum introduced the idea of polyhedron used in this and many other papers. There, he also described 47 regular polyhedra. The classification of the 48 regular polyhedra was achieved by Dress in [8,9]. A shorter proof of the completeness of the classification can be found in [14]. Throughout, we shall use the names of the polyhedra given in [14].

There are 18 finite regular polyhedra. They are the five Platonic solids, the four Kepler–Poinsot polyhedra, and the Petrials of the previous nine (see [7,15] for further details).

Six of the infinite regular polyhedra are in fact planar. Three of them are the regular tessellations of $\mathbb{R}^2$ by squares, equilateral triangles, and regular hexagons, denoted by $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$, respectively. The remaining three are the Petrials $\{\infty,4\}_4$, $\{\infty,6\}_3$, and $\{\infty,3\}_6$ of these tessellations. Figure 5 shows two Petrie polygons of $\{\infty,4\}_4$, three of $\{\infty,6\}_3$, and three of $\{\infty,3\}_6$; all other Petrie polygons are translates of these.

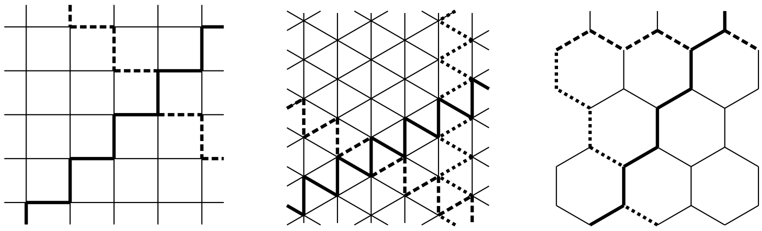

Figure 5. Petrie polygons of the planar polyhedra.

The remaining 24 infinite regular polyhedra live properly in $\mathbb{R}^3$, and can be evenly divided into those that are *blended* (their automorphism groups permute the translates of some plane) and those that are *pure* (not blended).

Each blended polyhedron $\mathcal{P}$ has a regular planar polyhedron $\mathcal{Q}$ as its image under the orthogonal projection to some plane Π, with the property that edges are mapped to edges and faces to faces. The orthogonal projection of $\mathcal{P}$ to the line $\Pi^\perp$ perpendicular to Π is either a line segment $\{\}$ (the only regular polytope of rank 1) or a tessellation $\{\infty\}$ of $\Pi^\perp$ by equal segments (the only regular polygon on the line), and we shall denote it by $\mathcal{R}$ in either case. The polyhedron is then denoted by $\mathcal{Q}\#\mathcal{R}$.

The vertices of the polyhedron $\mathcal{P}\#\{\}$ are contained in two parallel planes, and every edge joins a vertex in one plane to a vertex in the other. The polyhedron $\{4,4\}\#\{\}$ is illustrated in Figure 6a; the faces are skew quadrillaterals that project to the lower (or upper) plane into squares. The faces of $\mathcal{P}\#\{\infty\}$ are helices over the faces of $\mathcal{P}$. If two such faces share an edge, then one is obtained from the other by the reflection about a wall of the helix. Figure 6c shows three faces of $\{4,4\}\#\{\infty\}$, one in solid lines, one in dotted lines and one in dashed lines. One zigzag of $\{\infty,4\}_4\#\{\}$ and one zigzag of $\{\infty,4\}_4\#\{\infty\}$ are shown in Figure 6b,d, respectively.

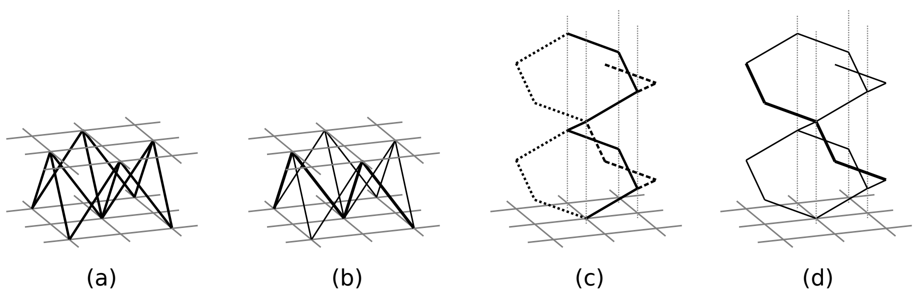

Figure 6. Blended polyhedra $\{4,4\}\#\{\}$ (**a**); $\{\infty,4\}_4\#\{\}$ (**b**); $\{4,4\}\#\{\infty\}$ (**c**) and $\{\infty,4\}_4\#\{\infty\}$ (**d**).

To each of the 12 blended polyhedra, we may associate a real positive parameter β corresponding to the ratio between the lengths of an edge of $\mathcal{Q}$ and a line segment of $\mathcal{R}$. The parameter β determines the angle α between consecutive edges of a face. Assuming that θ is the angle between two consecutive edges of a face of the planar polyhedron $\mathcal{Q}$, the parameter α satisfies that $0 < \alpha < \theta$ when $\mathcal{P}$ is $\mathcal{Q}\#\{\}$, whereas $\theta < \alpha < \pi$ if $\mathcal{P}$ is $\mathcal{Q}\#\{\infty\}$. The parameter α completely determines $\mathcal{P}$ up to similarity (see the polyhedra in Class 6 of [7]).

Three of the pure polyhedra have finite planar faces and skew vertex-figures. Two of them were discovered by Petrie, and the remaining by Coxeter (see [16]). The faces of $\{4,6|4\}$ are squares of the cubic tessellation, while the faces of $\{6,4|4\}$ and of $\{6,6|3\}$ are hexagons in the lattice **fcu**. Partial views of the polyhedra $\{4,3|4\}$ and $\{6,4|4\}$ are shown in Figure 7.

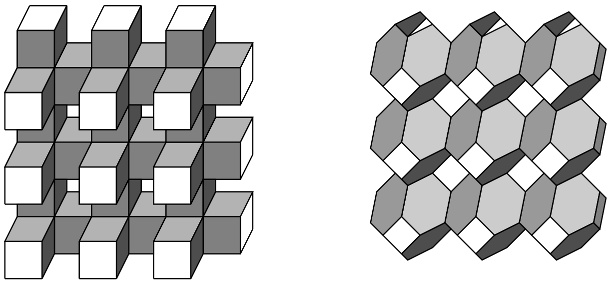

Figure 7. The polyhedra $\{4,3|4\}$ and $\{6,4|4\}$. Squares and hexagons in the same shade of gray represent polygons in parallel planes

Three infinite pure polyhedra have finite skew faces and planar vertex figures. The faces of the polyhedron $\{6,4\}_6$ consist of one Petrie polygon of each cube in the cubic tessellation, suitably chosen; this polyhedron is self-Petrial. The faces of the polyhedron $\{4,6\}_6$ are Petrie polygons of tetrahedra of the tiling of tetrahedra and octahedra; the faces of the polyhedron $\{6,6\}_4$ are Petrie polygons of octahedra of the same tiling. These two polyhedra are Petrials of each other.

The remaining six pure regular polyhedra have helical faces; three have skew vertex-figures and three planar vertex-figures. The polyhedra $\{\infty,6\}_{4,4}$, $\{\infty,4\}_{6,4}$, and $\{\infty,3\}_{6,3}$ are the Petrials of $\{4,6|4\}$, $\{6,4|4\}$, and $\{6,6|3\}$, respectively, and therefore have skew vertex-figures. The faces of $\{\infty,6\}_{4,4}$ and $\{\infty,4\}_{6,4}$ are helices over triangles, whereas those of $\{\infty,3\}_{6,3}$ are helices over squares. The polyhedra $\{\infty,3\}^{(a)}$ and $\{\infty,3\}^{(b)}$ are Petrials of each other; the faces of the former are helices over triangles, and those of the latter are helices over squares. The polyhedron $\{\infty,4\}_{\cdot,*3}$ is self-Petrial, and its facets are helices over triangles.

In contrast to the blended polyhedra, the pure polyhedra are unique up to similarity. More details on their geometry can be found in [7].

3.3. Chiral Polyhedra

In 2005, all chiral polyhedra in $\mathbb{R}^3$ were described by Schulte in [10,11]. Here, we briefly summarise that description. They are all infinite and pure, and can be classified into six infinite families.

The chiral polyhedra in the families $P(a,b)$, $Q(c,d)$, and $Q(c,d)^*$ have finite skew faces and skew vertex-figures, whereas those in families $P_1(a,b)$, $P_2(c,d)$, and $P_3(c,d)$ have helical faces and planar vertex-figures. The parameters take real values, not both 0, and a polyhedron with parameters (a,b) or (c,d) is similar to that with parameters (ka,kb) or (kc,kd), respectively, for any $k \neq 0$. This makes it possible to consider the polyhedra in each family to be parametrised by only one real parameter a/b or c/d.

The parameters for polyhedra in the families $P(a,b)$, $Q(c,d)$, and $Q(c,d)^*$ must be rational multiples of each other (or one of them 0), since otherwise the vertex set is not discrete. There is no such restriction for the parameters of polyhedra in the remaining three families. Each of the six families has two distinguished parameters for which the corresponding polyhedra are regular; the polyhedra determined by the remaining parameters are chiral.

The polyhedra in the family $P(a,b)$ have type $\{6,6\}$. Two of them, $P(a,b)$ and $P(a',b')$ (say), are combinatorially isomorphic if and only if $(a,b) \in \{(ka',kb'),(kb',ka')\}$ for some $k \neq 0$, and they are congruent if and only if $(a,b) \in \{\pm(a',b'),\pm(b',a')\}$. The polyhedra $P(1,1)$ and $P(1,-1)$ are the regular polyhedra $\{6,6|3\}$ and $\{6,6\}_4$, respectively.

The polyhedra in the family $Q(c,d)$ have type $\{4,6\}$. Two of them, $Q(c,d)$ and $Q(c',d')$, are combinatorially isomorphic if and only if $(c,d) \in \{(kc',kd'),(-kc',kd')\}$ for some $k \neq 0$, and they are congruent if and only if $(c,d) \in \{\pm(c',d'),\pm(-c',d')\}$. The polyhedra $Q(0,1)$ and $Q(1,0)$ are the regular polyhedra $\{4,6|4\}$ and $\{4,6\}_6$, respectively. When c and d are relatively prime, with c odd and $d \equiv 2$ modulo 4, then the vertex-figure at every vertex is the union of two cycles, and thus in that case, $Q(c,d)$ is not a polyhedron.

The polyhedron $Q(c,d)^*$ is the dual of $Q(c,d)$, meaning that its vertices are at the centres of the faces of $Q(c,d)$, and each of its faces can be constructed around some vertex of $Q(c,d)$. Hence, $Q(c,d)^*$ has type $\{6,4\}$. Furthermore, $Q(c,d)^*$ and $Q(c',d')^*$ are combinatorially isomorphic if $(c,d) \in \{(kc',kd'),(-kc',kd')\}$ for some $k \neq 0$, and they are congruent if and only if $(c,d) \in \{\pm(c',d'),\pm(-c',d')\}$. The polyhedra $Q(0,1)^*$ and $Q(1,0)^*$ are the regular polyhedra $\{6,4|4\}$ and $\{6,4\}_6$, respectively.

If a polyhedron in one of the families $P(a,b)$, $Q(c,d)$, and $Q^*(c,d)$ described above is not combinatorially isomorphic to any of the two regular members of the family, it is geometrically chiral and also chiral as a combinatorial structure.

The polyhedra in the family $P_1(a,b)$ have type $\{\infty,3\}$, and their faces are helices over triangles with the exception of $P_1(1,1)$. The regular instances of this family are $P_1(1,1) = \{3,3\}$ (the tetrahedron) and $P_1(1,-1) = \{\infty,3\}^{(a)}$. Any other member of the family is geometrically chiral but combinatorially isomorphic to $\{\infty,3\}^{(a)}$. Two polyhedra $P_1(a,b)$ and $P_1(a',b')$ are similar when $(a,b) = k(a',b')$ or $(a,b) = k(b',a')$ for some $k \neq 0$; and they are congruent if and only if $(a,b) = \pm(a',b')$ or $(a,b) = \pm(b',a')$.

The polyhedra in the family $P_2(c,d)$ also have type $\{\infty,3\}$, but their faces are helices over squares with the exception of $P_2(0,1)$. The regular instances of this family are $P_2(0,1) = \{4,3\}$ (the cube) and $P_2(1,0) = \{\infty,3\}^{(b)}$. Any other member of the family is geometrically chiral but combinatorially isomorphic to $\{\infty,3\}^{(b)}$. Two polyhedra $P_2(c,d)$ and $P_2(c',d')$ are similar when $(c,d) = k(c',d')$ or $(c,d) = k(-c',d')$ for some $k \neq 0$; and they are congruent if and only if $(c,d) = (\pm c, \pm d)$.

The two regular members of the family $P_3(c,d)$ are $P_3(1,0) = \{3,4\}$ (the octahedron) and $P_3(0,1) = \{\infty,4\}_{\cdot,*3}$. The remaining polyhedra of the family have type $\{\infty,4\}$, and their faces are helices over triangles; they are all geometrically chiral but combinatorially isomorphic to the regular double cover $\{\infty,4\}_{\cdot,*6}$ of $\{\infty,4\}_{\cdot,*3}$. Two polyhedra $P_3(c,d)$ and $P_3(c',d')$ are similar when $(c,d) = k(c',d')$ or $(c,d) = k(-c',d')$ for some $k \neq 0$; and they are congruent if and only if $(c,d) = (\pm c, \pm d)$.

The continuous movement of the parameters of the polyhedra in the last three families can be understood as a continuous movement of the polyhedra $\{\infty,3\}^{(a)}$, $\{\infty,3\}^{(b)}$, and $\{\infty,4\}_{\cdot,*6}$ that preserves at all times the index 2 subgroup of the symmetry group generated by the abstract rotations.

More details about the chiral polyhedra described here can be found in [10,11,17].

4. Polyhedra in Euclidean Lattices

In this section, we prove Theorem 1 by listing the regular and chiral polyhedra whose underlying graph is contained in one of the three lattices defined in Section 2. In other words, we want to find all possible sets $\mathcal{S}$ of polygons (faces) in **pcu**, **fcu**, and **bcu** such that

- the union of the polygons yields a connected graph,
- every edge of the lattice belongs to precisely two polygons, or to none of them,
- every vertex-figure is a finite polygon,
- there are abstract rotations preserving $\mathcal{S}$ along every face,

- there are abstract rotations preserving $\mathcal{S}$ around every vertex.

A large part of this work was done in [18] in a slightly different context, where the nets of the regular infinite polyhedra are studied. Here, we also study polyhedra whose vertex and edge sets are proper subsets of the nets **pcu**, **fcu**, and **bcu**, including the finite polyhedra; we still mention all regular polyhedra for the sake of completeness. The main contribution of this paper, then, is the study of the chiral polyhedra that admit an embedding into the nets **pcu**, **fcu**, and **bcu**.

The vertex and edge sets of the cube and of its Petrial can be easily seen as subsets of **pcu**. As explained in Section 2, **fcu** contains subsets of vertices and edges isometric to those of tetrahedra, octahedra, and hence also of their Petrials.

Among the finite regular polyhedra, only the six mentioned above have no 5-fold rotation in their symmetry groups. The underlying graphs of the remaining twelve cannot be embedded in any of the three nets while preserving their symmetries.

The vertex and edge sets of the polyhedra $\{4,4\}$ and $\{\infty,4\}_4$ can be found as subsets of **pcu** in the obvious way. They can also be found in **fcu**, for example, by considering only the vertices and edges of the net whose third coordinates equal to 0. As mentioned in Section 2, the vertices and edges of each of the remaining four planar polyhedra—$\{3,6\}$, $\{6,3\}$, $\{\infty,3\}_6$, and $\{\infty,6\}_3$—can be seen as subsets of those of **fcu**.

In Table 1, we summarise the lattices containing finite and planar regular polyhedra.

Table 1. Finite and planar polyhedra and the nets where they can be embedded.

Polyhedra	Net	Remarks	Polyhedra	Net	Remarks
$\{3,3\}$, $\{4,3\}_3$	**fcu**	finite	$\{3,6\}$, $\{\infty,6\}_3$	**fcu**	planar
$\{3,4\}$, $\{6,4\}_3$	**fcu**	finite	$\{6,3\}$, $\{\infty,3\}_6$	**fcu**	planar
$\{4,3\}$, $\{6,3\}_4$	**pcu**	finite	$\{4,4\}$, $\{\infty,4\}_4$	**pcu**, **fcu**	planar

A blended polyhedron may be embedded in different nets for different values of its parameter α. Several possibilities can be discarded by noting the angles between two edges incident to the same vertex in **pcu**, **fcu**, and **bcu** (see Lemma 1).

The angle between two consecutive edges in a face of $\{4,4\}\#\{\}$ or $\{\infty,4\}_4\#\{\}$ is strictly less than $\pi/2$, and so these polyhedra cannot be found as a subset of **pcu**. A sample square of one embedding of $\{4,4\}\#\{\}$ in **fcu** has vertices

$$(0,0,0),(1,0,1),(1,1,0),(0,1,1),$$

and one in **bcu** has vertices

$$(0,0,0),(1,1,1),(2,0,0),(1,-1,1).$$

In both cases, the polyhedron can be embedded in such a way that the vertices all have third coordinates equal to 0 or to 1.

Consecutive edges on a face of $\{3,6\}\#\{\}$ make an angle smaller than $\pi/3$, and hence the vertex and edge sets of this polyhedron (and of its Petrial) are not subsets of any of the lattices **pcu**, **fcu**, and **bcu**.

The vertex and edge sets of $\{6,3\}\#\{\}$ and of $\{\infty,3\}_6\#\{\}$ can be found as subsets of any of **pcu**, **fcu**, and **bcu**. A sample hexagon of $\{6,3\}\#\{\}$ in each of these lattices has vertex set

$$(0,0,0),(1,0,0),(1,-1,0),(1,-1,1),(0,-1,1),(0,0,1),$$
$$(0,0,0),(1,1,0),(1,0,-1),(2,0,0),(1,-1,0),(1,0,1),$$
$$(0,0,0),(1,1,-1),(2,0,-2),(3,-1,-1),(2,-2,0),(1,-1,1),$$

respectively. When extending these hexagons to the entire polyhedron, half of the vertices are in the plane $x + y + z = 0$, and half in the plane $x + y + z = a$, where $a = 2$ for **fcu** and $a = 1$ for the remaining two nets.

The angle between two consecutive edges in a face of $\{4, 4\}\#\{\infty\}$ or $\{\infty, 4\}_4\#\{\infty\}$ is strictly greater than $\pi/2$ but less than π, and so these polyhedra cannot be found as a subset of **pcu**. A sample helix of one embedding of $\{4, 4\}\#\{\infty\}$ in **fcu** has vertices

$$\ldots, (1, 1, -2), (0, 1, -1), (0, 0, 0), (1, 0, 1), (1, 1, 2), (0, 1, 3), \ldots$$

and one in **bcu** has vertices

$$\ldots, (2, 0, -2), (1, -1, -1), (0, 0, 0), (1, 1, 1), (2, 0, 2), (1, -1, 3), \ldots$$

In both nets, the axes of the helices are parallel to a coordinate axis (the z-axis in the case of the embeddings containing the two helices above).

The vertex and edge sets of the polyhedron $\{3, 6\}\#\{\infty\}$ and of $\{\infty, 6\}_3\#\{\infty\}$ can be found as subsets of any of **pcu**, **fcu**, and **bcu**. A sample hexagonal helix of $\{3, 6\}\#\{\infty\}$ in each of these lattices has vertex set

$$\ldots, (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 1, 1), (2, 2, 1), \ldots$$
$$\ldots, (0, 0, 0), (1, 1, 0), (1, 2, 1), (2, 2, 2), (3, 3, 2), (3, 4, 3), \ldots$$
$$\ldots, (0, 0, 0), (1, 1, -1), (2, 0, 0), (1, 1, 1), (2, 2, 0), (3, 1, 1), \ldots$$

respectively. In all these helices, the axis has direction vector $(1, 1, 1)$. In general, the direction axes of all helices are parallel to exactly one diagonal of a cube of the cubic tessellation.

Consecutive edges on a face of $\{6, 3\}\#\{\infty\}$ make an angle greater than $2\pi/3$, and hence the vertex and edge sets of this polyhedron (and of its Petrial) are not subsets of any of the lattices **pcu**, **fcu**, or **bcu**.

The nets containing blended regular polyhedra are summarised in Table 2. The polyhedra blended with $\{\}$ are combinatorially isomorphic to the planar polyhedra, and their nets are intrinsically planar; the name of these planar nets according to the Reticular Chemistry Structure Resource database appear in the column "Net". The nets of the polyhedra blended with $\{\infty\}$ admit several embeddings in Euclidean space. In the column "Net", we indicate the name of the most symmetric such embedding according to Reticular Chemistry Structure Resource database. The nets **pcu** and **dia** have more symmetries than the blended poyhedra they carry.

Table 2. Blended polyhedra and the nets where they can be embedded.

Polyhedra	Ambient Net	Net
$\{3, 6\}\#\{\}$, $\{\infty, 6\}_3\#\{\}$	none	**hxl**
$\{6, 3\}\#\{\}$, $\{\infty, 3\}_6\#\{\}$	**pcu**, **fcu**, **bcu**	**hcb**
$\{4, 4\}\#\{\}$, $\{\infty, 4\}_4\#\{\}$	**fcu**, **bcu**	**sql**
$\{3, 6\}\#\{\infty\}$, $\{\infty, 6\}_3\#\{\infty\}$	**pcu**, **fcu**, **bcu**	**pcu**
$\{6, 3\}\#\{\infty\}$, $\{\infty, 3\}_6\#\{\infty\}$	none	**acs**
$\{4, 4\}\#\{\infty\}$, $\{\infty, 4\}_4\#\{\infty\}$	**fcu**, **bcu**	**dia**

In Table 3, we list the nets where the pure polyhedra can be embedded. In the column "Net", we indicate the name of the net consisting of the vertex and edge sets of each polyhedron. This table has a large intersection with Table 1 in [18].

We now turn our attention to the chiral polyhedra. The procedure we will follow consists of first determining two consecutive edges at a face of the polyhedra in each family. For simplicity, we choose the common vertex to be the origin, except for the polyhedra $Q(c,d)^*$, where we consider them as the duals of the polyhedra $Q(c,d)$. To determine whether these two edges at the origin can be embedded in **pcu**, **fcu**, or **bcu**, we use the standard inner product to take the cosine of the angle between them and compare with the cosine of the angles described in Lemma 1. That is, the cosine must equal 0 or -1 if the edges are in **pcu**; $1/2$, 0, $-1/2$, or -1 if the edges are in **fcu**; and $1/3$, $-1/3$, or -1 if the edges are in **bcu**. It will then remain to determine if the parameters yield a polyhedron; in particular, if the polyhedron in question has finite faces, we still have to verify if the obtained parameters are rational multiples of each other, or if one of them is 0.

Table 3. Pure polyhedra and the nets where they can be embedded.

Polyhedra	Ambient Net	Net
$\{4,6\|4\}$, $\{\infty,6\}_{4,4}$	**pcu**	**pcu**
$\{6,4\|4\}$, $\{\infty,4\}_{6,4}$	**fcu**	**sod**
$\{6,6\|3\}$, $\{\infty,6\}_{6,3}$	**fcu**	**crs**
$\{4,6\}_6$, $\{6,6\}_4$	**fcu**	**hxg**
$\{6,4\}_6$, $\{\infty,4\}_{\cdot,*3}$	**pcu**	**nbo**
$\{\infty,3\}^{(a)}$, $\{\infty,3\}^{(b)}$	**fcu**	**srs**

4.1. Polyhedra $P(a,b)$

According to Section 5 of [10], the neighbours of the origin in the base face of $P(a,b)$ are $(a,0,b)$ and $(0,-b,-a)$; that is, the image of the origin under S_1 and S_1^{-1}, where S_1 is given by

$$(x,y,z) \mapsto (-y, z-b, x-a).$$

They form an angle α with the origin, given by

$$\cos(\alpha) = \frac{(a,0,b)\cdot(0,-b,-a)}{|(a,0,b)||(0,-b,-a)|} = \frac{-ab}{a^2+b^2}.$$

We use the fact that the polyhedra $P(ka,kb)$ and $P(kb,ka)$ are congruent for any $k \neq 0$ to assume without loss of generality that $a=1$. Then, $\cos(\alpha) = -b/(1+b^2)$. This equals 0 if and only if $b=0$ and the polyhedron is $P(1,0)$. This polyhedron has the same vertex and edge sets as **pcu**; the faces are some Petrie polygons of the cubes in the cubic tiling. The six faces around the origin are illustrated in the left of Figure 8. Two of the six faces are in solid lines, two in dotted lines and two in dashed lines.

On the other hand, $\cos(\alpha) = 1/z$ if and only if $b^2+bz+1=0$. When $z \in \{2,-2\}$, then the polyhedra are $P(1,1)$ and $P(1,-1)$, which are regular. If $z \in \{3,-3\}$, then b is not rational, and so $P(1,b)$ is not a (discrete) polyhedron. The equation has no solution when $z=-1$, and hence $P(1,0)$ is the only chiral polyhedron in the family whose vertex and edge sets are subsets of **pcu**, **fcu**, or **bcu**.

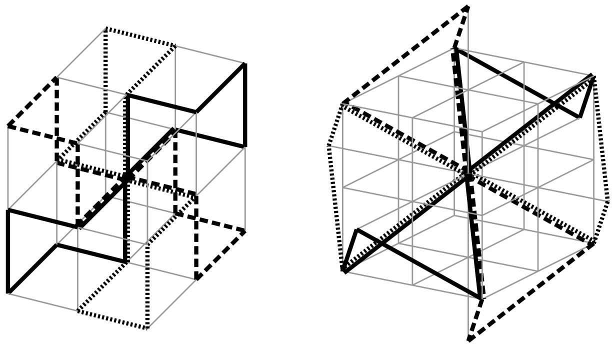

Figure 8. The polyhedra $P(1,0)$ and $Q(1,1)$.

4.2. Polyhedra $Q(c,d)$

The neighbours of the origin in the base face of $Q(c,d)$ are its images of S_1 and S_1^{-1}, where S_1 maps (x,y,z) to $(-x+c, z-d, -y-c)$, as in Section 6 of [10]. Thus, these neighbours are $(c,-c,d)$ and $(c,-d,-c)$, while the fourth vertex of the base face is $(0,-c-d,d-c)$. The neighbours form an angle α with the origin whose cosine is equal to $c^2/(2c^2+d^2)$.

Recall that $Q(0,1)$ is regular, and by similarity of the polyhedra $Q(c,d)$ and $Q(kc,kd)$, we may assume that $c=1$. Then, $\cos(\alpha) = 1/(2+d^2)$. This number is always strictly greater than 0, and it equals $1/z$ if and only if $d^2 - z + 2 = 0$. If $z=2$, then the polyhedron is $Q(1,0)$, which is also regular. Finally, if $z=3$, then we may assume that the polyhedron is $Q(1,1)$, since $Q(1,-1)$ is congruent to $Q(1,1)$.

To describe the polyhedron $Q(1,1)$, we first observe that the vertices of **bcu** are the union of 8 disjoint copies of the vertices of 2**bcu**, the net similar to **bcu** whose edges are twice as long. The following list contains a representative in each of these copies:

$$\{(0,0,0),(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1),(2,0,0),(0,2,0),(0,0,2)\}.$$

The vertices of $Q(1,1)$ are those of **bcu**; the edges are those of **bcu** after removing:

- all edges with direction vector $(1,1,1)$ at vertices in 2**bcu**;
- all edges with direction vector $(1,-1,1)$ at vertices in $(-1,1,1)+2$**bcu**;
- all edges with direction vector $(1,1,-1)$ at vertices in $(1,-1,1)+2$**bcu**;
- all edges with direction vector $(-1,1,1)$ at vertices in $(1,1,-1)+2$**bcu**.

This removes two edges from every vertex of **bcu**, and hence the vertices of $Q(1,1)$ are 6-valent. The faces are skew quadrilaterals congruent to the base quadrilateral with vertices $(0,0,0)$, $(1,-1,-1)$, $(0,-2,0)$, and $(1,-1,1)$. The corresponding net then has only one kind of vertex and one kind of edge under $G(Q(1,1))$, it is bipartite (as a subnet of **bcu**), and its smallest rings have 4-edges. The author does not know if this net already has a name. The six faces of this polyhedron at the origin are shown in Figure 8. Two of the six faces are in solid lines, two in dotted lines and two in dashed lines.

4.3. Polyhedra $Q(c,d)^*$

The polyhedron $Q(c,d)^*$ is the geometric dual of $Q(c,d)$, and so the vertices are in the geometric centres of the faces of $Q(c,d)$. The centre of the base face of $Q(c,d)$, and hence the base vertex of $Q(c,d)^*$ is $v_0 = 1/2(c,-d-c,d-c)$. The base face of $Q(c,d)$ shares consecutive edges with its images under the isometries S_2 and S_2^{-1}, where S_2 maps (x,y,z) to $(-z,-x,-y)$. Therefore, the two neighbours of v_0 in the base face of $Q(c,d)^*$ are $v_0S_2 = 1/2(c-d,-c,c+d)$ and $v_0S_2^{-1} = 1/2(c+d,c-d,-c)$. By translating by $-v_0$, we get that the cosine of the angle α formed by the two neighbours of v_0 with v_0 is $-d^2/(2d^2+4c^2)$.

The polyhedron $Q(0,1)^*$ is regular, and therefore we may assume that $c = 1$. Then, $\cos(\alpha) = -d^2/(4+2d^2)$. This number is in the interval $(-1/2, 0]$, and it is 0 only when $d = 0$. Since $Q(1,0)^*$ is regular, we only need to explore the possibility of $\cos(\alpha) = -1/3$.

If $\cos(\alpha) = -1/3$, then $d = \pm 2$. As for the polyhedra $Q(c,d)$ with c odd and $d \equiv 2$ modulo 4, the structure $Q(1,2)^*$ is not a polyhedron. Here, every edge belongs to more than one hexagon. Hence, none of the polyhedra $Q(c,d)^*$ live in **pcu**, **fcu**, or **bcu**.

4.4. Polyhedra $P_1(a,b)$

According to Section 4 of [11], the neighbours of $(0,0,0)$ in the base helix of the polyhedron $P_1(a,b)$ are $(b,a,0)$ and $(a,0,b)$; that is, the images of $(0,0,0)$ under S_1 and S_1^{-1}, where S_1 maps (x,y,z) to $(-z+b, -x+a, y)$. The cosine of the angle α formed by these two neighbours with $(0,0,0)$ equals $ab/(a^2+b^2)$.

Since $P_1(a,b)$ is similar to $P_1(b,a)$ and to $P(ka,kb)$ for every $k \neq 0$, we may assume that $a = 1$. Hence, $\cos(\alpha) = b/(1+b^2)$. If $b \neq 0$ and $\cos(\alpha) = 1/z$, then $b^2 - bz + 1 = 0$. If $z = -1$, then the equation has no solution. If $z \in \{2, -2\}$, then we obtain one of the regular polyhedra $P(1,1)$ or $P(1,-1)$. On the other hand, if $z \in \{3, -3\}$, then $b \notin \{0, 1, -1\}$. We claim that these choices of b do not yield polyhedra having their vertex and edge sets on **bcu**, although the angles between consecutive edges of a face suggest that they could. To see this, we recall that the three neighbours of $(0,0,0)$ in $P_1(1,b)$ are $(b,1,0)$, $(1,0,b)$, and $(0,b,1)$, and note that the neighbours of $(b,1,0)$ are $(0,0,0)$, $(b-1,1,b)$, and $(b,1-b,1)$ (see ([11], Page 198)). This implies that the directions of the edges at $(0,0,0)$ are $(b,1,0)$, $(1,0,b)$, and $(0,b,1)$; and that at $(b,1,0)$, there are edges in the directions of $(-1,0,b)$ and $(0,-b,1)$. Therefore $P_1(1,b)$ has edges with at least five different directions. Since **bcu** has edges in only four different directions (the main diagonals of a cube of the cubic tiling), there is no chiral polyhedron $P_1(a,b)$ with $a, b \neq 0$ whose vertex and edge sets are subsets of **pcu**, **fcu**, or **bcu**.

The polyhedron $P_1(1,0)$ is described in detail in [19]. Its faces are helices over triangles embedded in **pcu**. The three edges at every vertex in $P_1(1,0)$ are in the directions of the canonical axes. The axes of the helices are in the directions of the diagonals of a cube of the cubic tiling. The 1-skeleton of $P_1(1,0)$ is illustrated in the left of Figure 9. The three helical faces at some point are shown in the right part of the same figure.

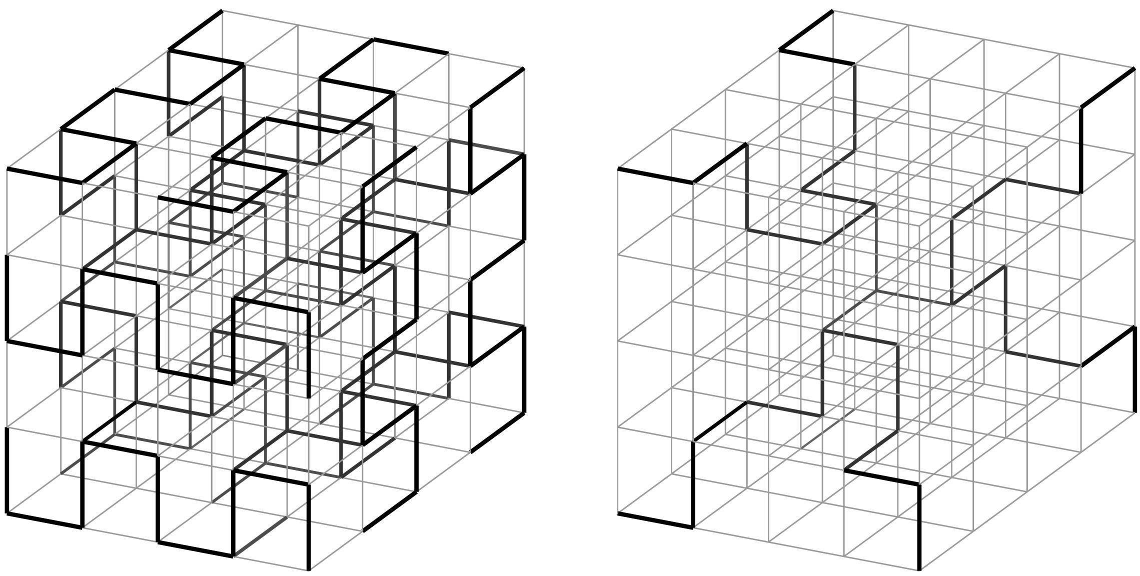

Figure 9. The polyhedron $P_1(1,0)$.

4.5. Polyhedra $P_2(c,d)$

The neighbours of $(0,0,0)$ in the base helix of the polyhedron $P_2(c,d)$ are $(d,c,-c)$ and $(c,-c,d)$; that is, the images of $(0,0,0)$ under S_1 and S_1^{-1}, where S_1 maps (x,y,z) to $(-z+d, y+c, x-c)$ as in Section 5 of [11]. The cosine of the angle α formed by these two neighbours with $(0,0,0)$ equals $-c^2/(d^2+2c^2)$.

Taking on account that $P_2(0,1)$ and $P_2(1,0)$ are regular, and that $\cos(\alpha) \in (-1/2, 0)$ if $c,d \neq 0$, we only need to consider the possibility of $\cos(\alpha) = -1/3$. This gives the parameters $c = d = 1$ (recall here that $P_2(c,d)$ is isomeric to $P_2(c,-d)$). The vertex and edge sets of this polyhedron are indeed subsets of **bcu**. This can be seen by noting that the direction of the three edges at $(0,0,0)$ of this polyhedron are $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$; and that the isometries S_1 and S_2 (the latter mapping (x,y,z) to (y,z,x)) that generate the symmetry group of the polyhedron preserve the set of directions $\{(1,-1,-1),(-1,1,-1),(-1,-1,1),(1,1,1)\}$, all directions of edges of **bcu**. In fact, this polyhedron has its vertices and edges in the diamond net **dia**, which is contained in **bcu**. In the left of Figure 10, we show a portion of the 1-skeleton of $P_2(1,1)$; the three helices at a point are illustrated in the right of the same figure.

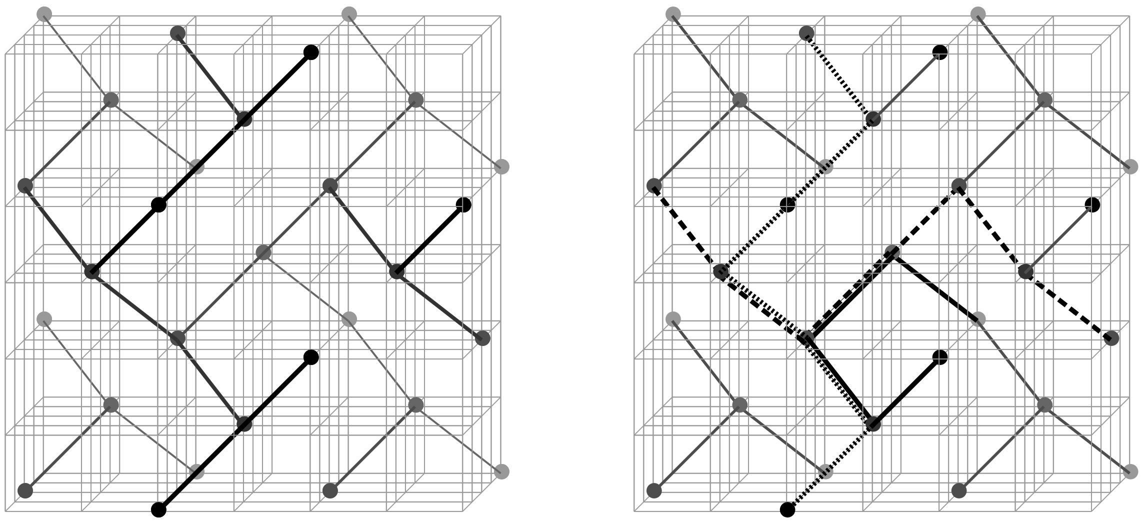

Figure 10. The polyhedron $P_2(1,1)$.

4.6. Polyhedra $P_3(c,d)$

According to Section 6 of [11], the neighbours of $(0,0,0)$ in the base helix of the polyhedron $P_3(c,d)$ are $(-d,-c,c)$ and $(c,-c,d)$; that is, the images of $(0,0,0)$ under S_1 and S_1^{-1}, where S_1 maps (x,y,z) to $(z-d, x-c, y+c)$. The cosine of the angle α formed by these two neighbours with $(0,0,0)$ equals $c^2/(d^2+2c^2)$.

If $c=0$ or $d=0$, then $P_3(c,d)$ is regular, so we may assume that $c=1$ and $d \neq 0$. In this situation, $\cos(\alpha) \in (0, 1/2)$, and it equals $1/3$ whenever $d \in \{1,-1\}$. However, Lemma 6.3 of [11] states that if c/d is a non-zero integer, then $P(c,d)$ is not a geometric polyhedron. In fact, every edge of $P_3(1,\pm 1)$ belongs to three helical faces.

With this, we conclude the proof of Theorem 1. The previous discussion in summarised in the following table.

Table 4. Pure polyhedra and the nets where they can be embedded.

Polyhedra	Ambient Net	Net
$P(1,0)$	**pcu**	**pcu**
$Q(1,1)$	**bcu**	unknown
$P_1(1,0)$	**pcu**	**srs**
$P_2(1,-1)$	**bcu**	**srs**

Acknowledgments: The author thanks the annonymous referees for helpful suggestions that substantially improved this paper. The author was supported by PAPIIT-UNAM under project grant IN101615. This work was elaborated while the author was on sabbatical at the Faculty of Education of the University of Ljubljana. The author thanks that institution and also the program PASPA-DGAPA and UNAM for their support to this sabbatical stay.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Rogers, C.A. The packing of equal spheres. *Proc. Lond. Math. Soc.* **1958**, *8*, 609–620.
2. *International Tables for Crystallography*; Hahn, T., Ed.; Published for International Union of Crystallography, Chester; D. Reidel Publishing Co.: Dordrecht, The Netherlands, 1983; Volume A, p. 869.
3. Conway, J.H.; Burgiel, H.; Goodman-Strauss, C. *The Symmetries of Things*; A K Peters, Ltd.: Wellesley, MA, USA, 2008; p. 444.
4. Pellicer, D.; Schulte, E. Regular polygonal complexes in space, I. *Trans. Am. Math. Soc.* **2010**, *362*, 6679–6714.
5. Pellicer, D.; Schulte, E. Regular polygonal complexes in space, II. *Trans. Am. Math. Soc.* **2013**, *365*, 2031–2061.
6. Schulte, E. Polyhedra, complexes, nets and symmetry. *Acta Crystallogr. Sect. A* **2014**, *7*, 203–216.
7. Grünbaum, B. Regular polyhedra—Old and new. *Aequ. Math.* **1977**, *16*, 1–20.
8. Dress, A.W.M. A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group. *Aequ. Math.* **1981**, *23*, 252–265.
9. Dress, A.W.M. A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration. *Aequ. Math.* **1985**, *29*, 222–243.
10. Schulte, E. Chiral polyhedra in ordinary space. I. *Discrete Comput. Geom.* **2004**, *32*, 55–99.
11. Schulte, E. Chiral polyhedra in ordinary space. II. *Discrete Comput. Geom.* **2005**, *34*, 181–229.
12. O'Keeffe, M.; Peskov, M.A.; Ramsden, S.J.; Yaghi, O.M. Reticular Chemistry Structure Resource. *Accts. Chem. Res.* **2008**, *41*, 1782–1789.
13. Grünbaum, B. Uniform tilings of 3-space. *Geombinatorics* **1994**, *4*, 49–56.
14. McMullen, P.; Schulte, E. Regular polytopes in ordinary space. *Discrete Comput. Geom.* **1997**, *17*, 449–478.
15. Coxeter, H.S.M. *Regular Polytopes*, 3rd ed.; Dover Publications, Inc.: Mineola, NY, USA, 1973; p. 335.
16. Coxeter, H.S.M. Regular skew polyhedra in three and four dimensions, and their topological analogues. *Proc. Lond. Math. Soc.* **1937**, *43*, 33–62.
17. Pellicer, D.; Ivić Weiss, A. Combinatorial structure of Schulte's chiral polyhedra. *Discrete Comput. Geom.* **2010**, *44*, 167–194.
18. O'Keeffe, M. Three-periodic nets and tilings: Regular and related infinite polyhedra. *Acta Crystallogr. Sect. A* **2008**, *64*, 425–429.
19. Pellicer, D. A chiral 4-polytope in $\mathbb{R}^3$. *Ars Math. Contemp.* **2016**, in press.