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A Modification on the Hesitant Fuzzy Set Lexicographical Ranking Method

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Abstract: Recently, a novel hesitant fuzzy set (HFS) ranking technique based on the idea of lexicographical ordering is proposed and an example is presented to demonstrate that the proposed ranking method is invariant with multiple occurrences of any element of a hesitant fuzzy element (HFE). In this paper, we show by examples that the HFS lexicographical ordering method is sometimes invalid, and a modified ranking method is presented. In comparison with the HFS lexicographical ordering method, the modified ranking method is more reasonable in more general cases.

Keywords: hesitant fuzzy set; lexicographical ordering; ranking method

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1. Lexicographical Ordering of HFSs

As a generalization of fuzzy set [1], hesitant fuzzy set [2] is very useful in handling a situation where people have hesitancy to make a decision. It permits the membership degree of an element to a set to be several possible values between 0 and 1[3]. Since its appearance, it has attracted a lot of research attention, and a large amount of literature has been published on hesitant fuzzy set theory and applications [4–19]. Up to now, some researchers have proposed the HFE ranking methods [3,5,7,20–23]. Farhadinia [24] gave a brief study of the existing HFS ranking methods, and then proposed a novel one based on the idea of lexicographical ordering. The purpose of this paper is to point out an error in Farhadinia's method [24] and present a modified ranking method for HFEs. In what follows, we introduce some basic concepts related to hesitant fuzzy sets.

Definition 1 [2,3]. *Let X be a fixed set, then a hesitant fuzzy set on X is defined in terms of a function that when applied to X returns a subset of [0, 1].*

To be understood easily, Xia and Xu [3] utilized the following mathematical symbol to express a hesitant fuzzy set:

$$E = \{ \langle x, h_E(x) \rangle | x \in X \} \tag{1}$$

where $h_E(x)$ is a set of several values in [0, 1], denoting the possible membership degree of $x \in X$ to the set. For convenience, $h = h_E(x)$ is called a hesitant fuzzy element (HFE) [3].

In order to compare the HFEs, Xia and Xu [3] gave the following comparison rule:

Definition 2 [3]. *For a HFE h, the score function of h is defined as*

$$s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma \tag{2}$$

where l_h denotes the number of the elements in h. For any two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

However, in some cases, the comparison rule does not work. In order to address this issue, some researchers have proposed many HFS ranking methods, which have characters of higher discrimination [3,5,7,20–23]. Farhadinia [24] pointed out the shortcomings of the existing HFS ranking techniques by counterexamples and proposed a novel method based on the idea of lexicographical ordering. Before presenting the main results, the following assumption is required.

Assumption 1 [24]. *The arrangement of elements in a HFE h is in an increasing order.*

Definition 3 [24]. *For $X, Y \in R^n$, the lexicographical ordering on the Euclidean space R^n, denoted by $<_{lex}$, is defined as follows:*

$$X = (x_1, x_2, \cdots, x_n) <_{lex} Y = (y_1, y_2, \cdots, y_n)$$

if and only if there is $i \in \{1, 2, \cdots, n\}$ such that

$x_j = y_j$ holds for $j < i$, and $x_i < y_i$.

Furthermore, $\leq_{lex}$ means that $X <_{lex} Y$ or $X = Y$.

Definition 4 [24]. *Let h be a HFE, denoted by $h = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(l)}\}$, and l stands for the number of the elements in h. The ranking vector associated with HFE h can be denoted by*

$$R(h) = (S(h), V_\phi(h)) \tag{3}$$

where $S(h) = \frac{1}{l}\sum_{i=1}^{l} \gamma^{(i)}$ and $V_\phi(h) = \sum\limits_{i=1}^{l-1} \phi(\gamma^{(i+1)} - \gamma^{(i)})$. Here, $V_\phi(h)$ is the successive deviation function of HFE h where ϕ: [0, 1] $\rightarrow$ [0, 1] is an increasing real function with $\phi(0) = 0$.

Then, a comparison rule based on the HFE lexicographical ordering can be derived. For two HFEs $h_1 = \{\gamma_1^{(1)}, \gamma_1^{(2)}, \cdots, \gamma_1^{(l_1)}\}$ and $h_2 = \{\gamma_2^{(1)}, \gamma_2^{(2)}, \cdots, \gamma_2^{(l_2)}\}$, where l_1 and l_2 denote the number of values in h_1 and h_2, respectively,

(i) $h_1 > h_2$ if and only if $R(h_1) >_{lex} R(h_2)$,
(ii) $h_1 \geq h_2$ if and only if $R(h_1) \geq_{lex} R(h_2)$, and
(iii) if $h_1 = h_2$, then $R(h_1) =_{lex} R(h_2)$.

Hereafter, we take the increasing real function $\phi = t^2$ into consideration, which is also used in [24].

2. Modified Lexicographical Ordering of HFSs

Farhadinia [24] presented an example to illustrate that multiple occurrences of any element of a HFE should not affect its ranking result.

Example 1 [24]. *A situation is considered, where a group of five decision-makers discuss the membership degree of an element h to a given set. They are hesitant among some possible values, such as 0.1, 0.3, 0.3, 0.3, and 0.5, and they cannot persuade each other. For such cases, a HFE $h_1 = \{0.1, 0.3, 0.3, 0.3, 0.5\}$ can be used to model the hesitance experienced by the five decision-makers. Following from the set theory, the HFE h_1 may be represented as $h_2 = \{0.1, 0.3, 0.3, 0.5\}$ and $h_3 = \{0.1, 0.3, 0.5\}$, where multiple occurrences of any element of a HFE are permitted and should not affect the ranking result. In this situation, all identical HFEs h_1, h_2 and h_3 should have the same ranking value.*

By the HFE lexicographical ranking method and Equation (3), where

$$V_\phi(h) = \sum_{i=1}^{l-1} \left(\gamma^{(i+1)} - \gamma^{(i)}\right)^2$$

we obtain

$$R(h_1) = R(h_2) = R(h_3) = (0.3, 0.08)$$

which implies that

$$h_1 = h_2 = h_3.$$

This is what is expected of the theory of sets. However, we find that the HFE lexicographical ranking method proposed by Farhadinia [24] is not invariant with multiple occurrences of any element of a HFE.

Example 2. *Consider the situation which is discussed in Example 1. If the five decision-makers are hesitant among some possible values, such as 0.1, 0.1, 0.1, 0.3, and 0.5, then the hesitance can be modeled by a HFE* $h_4 = \{0.1, 0.1, 0.1, 0.3, 0.5\}$. *Following from the set theory, the HFE* h_4 *may be represented as* $h_5 = \{0.1, 0.1, 0.3, 0.5\}$ *and* $h_3 = \{0.1, 0.3, 0.5\}$.

According to Farhadinia [24], multiple occurrences of any element of a HFE should not affect the ranking result, and all identical HFEs h_4, h_5, and h_3 should have the same ranking value. By the HFE lexicographical ranking method and Equation (3), where

$$V_\phi(h) = \sum_{i=1}^{l-1} (\gamma^{(i+1)} - \gamma^{(i)})^2$$

We obtain

$$R(h_4) = (0.22, 0.08), R(h_5) = (0.25, 0.08), \text{ and } R(h_3) = (0.3, 0.08).$$

Therefore,

$$h_4 < h_5 < h_3$$

which is contradictory.

Example 3. *Consider the situation which is discussed in Example 1. If the five decision-makers are hesitant among some possible values, such as 0.1, 0.3, 0.5, 0.5, and 0.5, then the hesitance can be modeled by a HFE* $h_7 = \{0.1, 0.3, 0.5, 0.5, 0.5\}$. *Following from the set theory, the HFE* h_7 *may be represented as* $h_8 = \{0.1, 0.3, 0.5, 0.5\}$ *and* $h_3 = \{0.1, 0.3, 0.5\}$.

According to Farhadinia [24], multiple occurrences of any element of a HFE should not affect the ranking result, and all identical HFEs h_7, h_8 and h_3 should have the same ranking value. By the HFE lexicographical ranking method and Equation (3), where

$$V_\phi(h) = \sum_{i=1}^{l-1} (\gamma^{(i+1)} - \gamma^{(i)})^2$$

We obtain

$$R(h_7) = (0.38, 0.08), R(h_8) = (0.35, 0.08), \text{ and } R(h_3) = (0.3, 0.08).$$

Therefore,

$$h_7 > h_8 > h_3$$

which is contradictory.

From the above examples, we can draw a conclusion that the HFE lexicographical ranking method proposed by Farhadinia [24] is not invariant with respect to multiple occurrences of any element of a HFE. In order to overcome the drawbacks of the HFE lexicographical ranking method, we propose a modified one, which is invariant with respect to multiple occurrences of any element of a HFE.

Definition 5. *Let h be a HFE, denoted by $h = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(l)}\}$, and l stands for the number of the elements in h. The ranking vector associated with HFE h can be denoted by*

$$R_M(h) = (S_M(h), V_\phi(h)) \tag{4}$$

where $S_M(h) = \frac{\sum_{i=1}^{l} \frac{\gamma^{(i)}}{count(\gamma^{(i)})}}{\sum_{i=1}^{l} \frac{1}{count(\gamma^{(i)})}}$ and $V_\phi(h) = \sum_{i=1}^{l-1} \phi(\gamma^{(i+1)} - \gamma^{(i)})$. Here, $count(\cdot)$ denotes the counting function, which provides us with the number of times a value occurs in a HFE, and $V_\phi(h)$ is the successive deviation function of HFE h where ϕ: [0, 1] $\to$ [0, 1] is an increasing real function with $\phi(0) = 0$.

Then, a comparison rule based on the modified HFE lexicographical ranking method can be derived. For two HFEs, $h_1 = \{\gamma_1^{(1)}, \gamma_1^{(2)}, \cdots, \gamma_1^{(l_1)}\}$ and $h_2 = \{\gamma_2^{(1)}, \gamma_2^{(2)}, \cdots, \gamma_2^{(l_2)}\}$, where l_1 and l_2 denote the number of values in h_1 and h_2, respectively,

(i) $h_1 > h_2$ if and only if $R_M(h_1) >_{lex} R_M(h_2)$,
(ii) $h_1 \geq h_2$ if and only if $R_M(h_1) \geq_{lex} R_M(h_2)$, and
(iii) if $h_1 = h_2$, then $R_M(h_1) =_{lex} R_M(h_2)$.

Example 4. *See Example 1. Let $h_1 = \{0.1, 0.3, 0.3, 0.3, 0.5\}, h_2 = \{0.1, 0.3, 0.3, 0.5\}$ and $h_3 = \{0.1, 0.3, 0.5\}$ be three HFEs, which are discussed in Example 1.*

By the modified HFE lexicographical ranking method and Equation (4), we obtain

$$R_M(h_1) = R_M(h_2) = R_M(h_3) = (0.3, 0.08)$$

which implies that

$$h_1 = h_2 = h_3.$$

Example 5. *See Example 2. Let $h_4 = \{0.1, 0.1, 0.1, 0.3, 0.5\}, h_5 = \{0.1, 0.1, 0.3, 0.5\}$ and $h_3 = \{0.1, 0.3, 0.5\}$ be three HFEs, which are discussed in Example 2.*

By the modified HFE lexicographical ranking method and Equation (4), we obtain

$$R_M(h_4) = R_M(h_5) = R_M(h_3) = (0.3, 0.08)$$

which implies that

$$h_4 = h_5 = h_3.$$

Example 6. *See Example 3. Let $h_7 = \{0.1, 0.3, 0.5, 0.5, 0.5\}, h_8 = \{0.1, 0.3, 0.5, 0.5\}$ and $h_3 = \{0.1, 0.3, 0.5\}$ be three HFEs, which are discussed in Example 3.*

By the modified HFE lexicographical ranking method and Equation (4), we obtain

$$R_M(h_7) = R_M(h_8) = R_M(h_3) = (0.3, 0.08)$$

which implies that

$$h_7 = h_8 = h_3.$$

It is noteworthy that the modified HFE lexicographical ranking method is robust to multiple occurrences of any element of a HFE. As a matter of fact, the HFE lexicographical ranking method proposed by Farhadinia [24] is only invariant with respect to multiple occurrences of the arithmetic-mean $S(h)$ of a HFE h.

Proposition 1. *If a HFE h contains an element $S(h)$, which is the arithmetic-mean of HFE h, then the HFE lexicographical ranking method proposed by Farhadinia [24] is invariant with respect to multiple occurrences of the arithmetic-mean $S(h)$ of HFE h, where $h = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(l)}\}$ and $S(h) = \frac{1}{l}\sum_{i=1}^{l} \gamma^{(i)}$.*

Proof. Assume that the arithmetic-mean $S(h)$ of a HFE h appears m times. Then a novel HFE $\overline{h}$ can be obtained, i.e.,

$$\overline{h} = \{\gamma^{(1)}, \gamma^{(2)}, \cdots \underbrace{S(h), \cdots S(h),}_{m\,times} \cdots \gamma^{(l)}\}.$$

By the HFE lexicographical ranking method proposed by Farhadinia [24] and Equation (3), where

$$S(h) = \frac{1}{l}\sum_{i=1}^{l} \gamma^{(i)}$$

and

$$V_\phi(h) = \sum_{i=1}^{l-1} (\gamma^{(i+1)} - \gamma^{(i)})^2$$

we obtain

$$\begin{aligned} S(\overline{h}) &= \frac{\sum_{i=1}^{l} \gamma^{(i)} + (m-1) \cdot S(h)}{l+m-1} \\ &= \frac{l \cdot S(h) + (m-1) \cdot S(h)}{l+m-1} \\ &= S(h). \end{aligned}$$

Since

$$V_\phi(h) = V_\phi(\overline{h})$$

we obtain

$$R(h) = R(\overline{h}).$$

Therefore,

$$h = \overline{h}$$

which completes the proof. □

Proposition 2. *The modified HFE lexicographical ranking method is robust to multiple occurrences of any element of a HFE.*

Proof. Let $h = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(l)}\}$ be a HFE. Assume that the first element $\gamma^{(1)}$ appears m_1 times, the second $\gamma^{(2)}$ appears m_2 times,$\cdots\cdots$, and the last $\gamma^{(l)}$ appears m_l times. Then a novel HFE $\widetilde{h}$ can be obtained, i.e.,

$$\widetilde{h} = \{\underbrace{\gamma^{(1)} \cdots \gamma^{(1)}}_{m_1\,times}, \underbrace{\gamma^{(2)} \cdots \gamma^{(2)}}_{m_2\,times}, \cdots, \underbrace{\gamma^{(l)} \cdots \gamma^{(l)}}_{m_l\,times}\}.$$

By the modified HFE lexicographical ranking method and Equation (4), where

$$S_M(h) = \frac{\sum_{i=1}^{l} \frac{\gamma^{(i)}}{count(\gamma^{(i)})}}{\sum_{i=1}^{l} \frac{1}{count(\gamma^{(i)})}}$$

and

$$V_\phi(h) = \sum_{i=1}^{l-1} \phi(\gamma^{(i+1)} - \gamma^{(i)})$$

we obtain

$$S_M(\widetilde{h}) = \frac{(\frac{\gamma^{(1)}}{m_1} + \frac{\gamma^{(1)}}{m_1} + \cdots + \frac{\gamma^{(1)}}{m_1}) + (\frac{\gamma^{(2)}}{m_2} + \frac{\gamma^{(2)}}{m_2} + \cdots + \frac{\gamma^{(2)}}{m_2}) + \cdots + (\frac{\gamma^{(l)}}{m_l} + \frac{\gamma^{(l)}}{m_l} + \cdots + \frac{\gamma^{(l)}}{m_l})}{(\frac{1}{m_1} + \frac{1}{m_1} + \cdots + \frac{1}{m_1}) + (\frac{1}{m_2} + \frac{1}{m_2} + \cdots + \frac{1}{m_2}) + \cdots + (\frac{1}{m_l} + \frac{1}{m_l} + \cdots + \frac{1}{m_l})}$$

$$= \frac{\gamma^{(1)}+\gamma^{(2)}+\cdots+\gamma^{(l)}}{l}$$

$$= S(h).$$

Since

$$V_\phi(h) = V_\phi(\widetilde{h})$$

we obtain

$$R_M(h) = R_M(\widetilde{h}).$$

It implies that

$$h = \widetilde{h}$$

which completes the proof. □

In fact, if the values appear only once in a HFE, then Equation (4) is reduced to Equation (3). In other words, this paper provides an extended form of lexicographical ordering of HFS's proposed by Farhadinia [24]. Furthermore, with this modified approach, the shortcomings in the lexicographical ordering of HFS's are overcome. People can adopt the proposed method to rank HFEs, especially when the values appear more than once in a HFE. Of course, the lexicographical ordering of HFS's proposed by Farhadinia [24] can also be used to avoid unnecessary calculations when the values appear only once in a HFE.

3. Conclusions

Farhadinia [24] proposed a novel HFS ranking technique based on the idea of lexicographical ordering method and pointed out that it is invariant with respect to multiple occurrences of any element of a HFE. In this paper, we presented several counterexamples to explain the error in his method. Moreover, a modified HFE lexicographical ranking method has been put forward to correct the error.

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