symmetry

MDPI

Article

On Consistent Nonparametric Statistical Tests of Symmetry Hypotheses

Jean-François Quessy

Département de mathématiques et d'informatique, Université du Québec à Trois-Rivières, Trois-Rivières, QC G9A 5H7, Canada; jean-francois.quessy@uqtr.ca; Tel.: +1-819-376-5011 (ext. 3814)

Academic Editor: Purushottam D. Gujrati
Received: 30 November 2015; Accepted: 19 April 2016; Published: 6 May 2016

Abstract: Being able to formally test for symmetry hypotheses is an important topic in many fields, including environmental and physical sciences. In this paper, one concentrates on a large family of nonparametric tests of symmetry based on Cramér–von Mises statistics computed from empirical distribution and characteristic functions. These tests possess the highly desirable property of being universally consistent in the sense that they detect any kind of departure from symmetry as the sample size becomes large. The asymptotic behaviour of these test statistics under symmetry is deduced from the theory of first-order degenerate V-statistics. The issue of computing valid p-values is tackled using the multiplier bootstrap method suitably adapted to V-statistics, yielding elegant, easy-to-compute and quick procedures for testing symmetry. A special focus is put on tests of univariate symmetry, bivariate exchangeability and reflected symmetry; a simulation study indicates the good sampling properties of these tests. Finally, a framework for testing general symmetry hypotheses is introduced.

Keywords: characteristic function; Cramér–von Mises functional; exchangeability; multiplier bootstrap; reflected symmetry; V-statistics

1. Introduction

In many scientific fields, a natural or experimentally-controlled phenomenon is observed and a dataset is collected. From these observations, one may be interested in testing basic assumptions with respect to some theoretical model. One of these assumptions that often appears in physical models is the so-called symmetry hypothesis; see, for example, [1]. In order to validate a model under investigation, one typically wants to thoroughly test these kinds of hypotheses with the help of a statistical method.

There are various types of symmetry that need to be distinguished first. The most common concerns random variables taking values in the space $\mathbb{R}$ of real numbers. In this context, a random variable $X \in \mathbb{R}$ is said to be symmetric around the origin if $X \stackrel{d}{=} -X$, where here and in the sequel, $\stackrel{d}{=}$ means equality in distribution. More generally, X is symmetric around $a \in \mathbb{R}$ if and only if $X - a \stackrel{d}{=} a - X$. For a pair (X, Y) of random variables taking values in $\mathbb{R}^2$, many types of symmetry have been proposed in the literature. The pair (X, Y) is said to be exchangeable if and only if $(X, Y) \stackrel{d}{=} (Y, X)$. This definition entails that X and Y have the same distribution. Another notion is reflected symmetry: (X, Y) is reflection symmetric around $(a, b) \in \mathbb{R}^2$ if and only if $(X - a, Y - b) \stackrel{d}{=} (a - X, b - Y)$. This definition entails in particular the symmetry of X around a and the symmetry of Y around b. While this paper focuses on the two above-mentioned notions of bivariate symmetry, other definitions have been proposed, e.g., joint symmetry and spherical symmetry.

In the statistics and probability literature, there are two main ways to characterize the stochastic behaviour of random variables and random vectors. The most widely used is the distribution function approach. In that case, one works with the function $\mathrm{P}(X \leq x)$ in the univariate case and with the joint distribution $\mathrm{P}(X \leq x, Y \leq y)$ in the bivariate case. An alternative, yet less popular approach, uses the so-called characteristic functions associated with random variables and random vectors. Since one can recover the distribution function of a random variable (or vector) from its characteristic function, and *vice versa*, the various hypotheses of symmetry described previously can equivalently be stated in terms of distribution functions or using characteristic functions. As will be seen, these two approaches lead to different and competing statistical procedures.

This paper focuses on consistent nonparametric tests of symmetry based on Cramér–von Mises functionals of empirical distribution and characteristic functions. These tests are attractive since they do not require any assumptions on the form of the underlying distribution and provide universally-consistent procedures. In addition, as will be seen, these test statistics for symmetry can be expressed as V-statistics. This representation allows for the derivation of their asymptotic behaviour and, most importantly, suggests a resampling method based on the multiplier bootstrap for the computation of p-values. Compared to permutation methods, which are generally employed when testing symmetry, this strategy is substantially quicker and provides elegant formulas that make the tests easy to implement. The main features of this work are the following:

(i) Describe a general family of Cramér–von Mises test statistics for symmetry hypotheses based on empirical distributions and characteristic functions. In the case of univariate symmetry, exchangeability and reflected symmetry, some of these statistics have already been proposed in the literature.
(ii) Deduce the asymptotic behaviour of these test statistics under the null hypothesis upon noting that they are related to degenerate V-statistics.
(iii) Suggest an efficient alternative to the use of permutations based on the multiplier bootstrap method adapted to V-statistics.
(iv) Present the results of a simulation study that investigates the properties of the tests under the null hypothesis, as well as under violations of symmetry hypotheses.
(v) Develop a general framework for testing a broad class of symmetry hypotheses.

The paper is organized as follows. Section 2 provides some results on degenerate V-statistics and their multiplier versions that will prove useful throughout the paper. Section 3 focuses on tests of symmetry for random variables, while Section 4 is devoted to tests of bivariate exchangeability and reflected symmetry. The results of an extensive simulation study are presented and discussed in Section 5. A unified framework that contains as special cases the univariate and bivariate tests of symmetry encountered in Sections 3 and 4, but also many other types of symmetry, is developed in Section 6. Technical arguments are relegated to the Appendix.

2. Some Preliminaries on V-statistics

All of the test statistics for symmetry that will be encountered in this work are related to first-order degenerate V-statistics. Therefore, their asymptotic behaviour can be derived using results that one can find, for instance, in the books by [2] and [3]. In what follows, $X_1, \ldots, X_n$ are identically distributed independent observations in $\mathbb{R}^p$. Some of the test statistics that will be described are of the form:

$$V_n = \frac{1}{n} \sum_{j,j'=1}^{n} \psi\left(X_j, X_{j'}\right) \tag{1}$$

where $\psi : \mathbb{R}^{p \times p} \to \mathbb{R}$ is a symmetric kernel of degree two that is first-order degenerate in the sense that $\mathrm{E}\{\psi(x_1, X_2)\} = 0$ for all $x_1 \in \mathbb{R}^p$. In that case,

$$V_n = U_n^{(1)} + \left(\frac{n-1}{n}\right) n\, U_n^{(2)}$$

where $U_n^{(1)}$ and $U_n^{(2)}$ are the U-statistics:

$$U_n^{(1)} = \frac{1}{n}\sum_{j=1}^{n} \psi\left(X_j, X_j\right) \quad \text{and} \quad U_n^{(2)} = \binom{n}{2}^{-1} \sum_{j<j'} \psi\left(X_j, X_{j'}\right)$$

The following result is a straightforward consequence of Theorem 1, p. 79, in [2].

Proposition 1. *If* $\mathrm{E}\{\psi^2(X_1, X_2)\} < \infty$, *the statistic* V_n *converges in distribution to:*

$$\mathrm{E}\left\{\psi\left(X_1, X_1\right)\right\} + \sum_{\kappa=1}^{\infty} \lambda_\kappa \left(Z_\kappa^2 - 1\right)$$

where $(Z_\kappa)_{\kappa=1}^{\infty}$ *are independent* $\mathbb{N}(0,1)$ *random variables and* $(\lambda_\kappa)_{\kappa=1}^{\infty}$ *are the eigenvalues of the mapping* $\eta \mapsto \mathrm{E}\{\psi(\cdot, X_2)\, \eta(X_2)\}$.

Now, consider the statistic:

$$W_n = \frac{1}{n^2} \sum_{j,j',k=1}^{n} \phi\left(X_j, X_{j'}, X_k\right) \tag{2}$$

where $\phi : \mathbb{R}^{p\times p\times p} \to \mathbb{R}$ is a kernel of degree three that satisfies the following assumptions:

$\mathcal{A}_1$. $\phi(x_1, x_2, x_3) = \phi(x_2, x_1, x_3)$ for all $(x_1, x_2, x_3) \in \mathbb{R}^{p\times p\times p}$, *i.e.*, ϕ is symmetric with respect to its first two components;

$\mathcal{A}_2$. $\mathrm{E}\{\phi(x_1, X_2, x_3)\} = 0$ for all $(x_1, x_3) \in \mathbb{R}^{p\times p}$.

The large-sample behaviour of W_n is stated as a proposition whose proof is deferred to the Appendix.

Proposition 2. *The test statistic* W_n *is asymptotically equivalent to the V-statistic with degenerate bivariate kernel* $\Phi(x_1, x_2) = \mathrm{E}\{\phi(x_1, x_2, X_3)\}$, *i.e.,*

$$W_n = \frac{1}{n} \sum_{j,j'=1}^{n} \Phi\left(X_j, X_{j'}\right) + o_{\mathrm{P}}(1) \tag{3}$$

As a consequence, if $\mathrm{E}\{\Phi^2(X_1, X_2)\} < \infty$, *then* W_n *converges in distribution to:*

$$\mathbb{W} = \mathrm{E}\left\{\Phi(X_1, X_1)\right\} + \sum_{\kappa=1}^{\infty} \zeta_\kappa \left(Z_\kappa^2 - 1\right)$$

where $(Z_\kappa)_{\kappa=1}^{\infty}$ *are independent* $\mathbb{N}(0,1)$ *random variables and* $(\zeta_\kappa)_{\kappa=1}^{\infty}$ *are the eigenvalues of the mapping* $\eta \mapsto \mathrm{E}\{\Phi(\cdot, X_2)\, \eta(X_2)\}$.

As mentioned in the Introduction, the proposed methodology for the computation of p-values will be based on the multiplier bootstrap. Specifically, a multiplier sample is obtained by generating, independently of the data, a random sample $\xi_1, \ldots, \xi_n$ of independent and identically distributed random variables, such that $\mathrm{E}(\xi_j) = 0$ and $\mathrm{var}(\xi_j) = 1$. The suggested multiplier versions of V_n and W_n are given, respectively, by:

$$\begin{aligned} \widehat{V}_n &= \frac{1}{n} \sum_{j,j'=1}^{n} \xi_j\, \xi_{j'}\, \psi\left(X_j, X_{j'}\right) \\ \widehat{W}_n &= \frac{1}{n^2} \sum_{j,j',k=1}^{n} \xi_j\, \xi_{j'}\, \phi\left(X_j, X_{j'}, X_k\right) = \frac{1}{n} \sum_{j,j'=1}^{n} \xi_j\, \xi_{j'} \left\{ \frac{1}{n} \sum_{k=1}^{n} \phi\left(X_j, X_{j'}, X_k\right) \right\} \end{aligned} \tag{4}$$

From a slight adaptation of Theorem 3.1 in [4], which applies to first-order degenerate U-statistics, one obtains that $\widehat{V}_n$ is a valid replicate of V_n asymptotically. For W_n, one could show using arguments similar as those in the proof of Proposition 2 that $\widehat{W}_n$ is asymptotically equivalent to:

$$\widehat{W}_n^{\star} = \frac{1}{n} \sum_{j,j'=1}^{n} \xi_j \, \xi_{j'} \, \Phi\left(X_j, X_{j'}\right)$$

so that the validity of $\widehat{W}_n$ to replicate W_n asymptotically can be deduced, as well.

For computational purposes, define the matrices $A, A^{\star} \in \mathbb{R}^{n \times n}$, such that:

$$A_{jj'} = \psi\left(X_j, X_{j'}\right) \quad \text{and} \quad A_{jj'}^{\star} = \frac{1}{n} \sum_{k=1}^{n} \phi\left(X_j, X_{j'}, X_k\right)$$

Letting $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, one can then write:

$$V_n = \frac{1}{n} \mathbf{1} \, A \, \mathbf{1}^{\top}, \quad \widehat{V}_n = \frac{1}{n} \boldsymbol{\xi} \, A \, \boldsymbol{\xi}^{\top}, \quad W_n = \frac{1}{n} \mathbf{1} \, A^{\star} \, \mathbf{1}^{\top} \quad \text{and} \quad \widehat{W}_n = \frac{1}{n} \boldsymbol{\xi} \, A^{\star} \, \boldsymbol{\xi}^{\top}$$

In practice, the multiplier procedure is repeated B times by generating independent vectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(B)}$ of multiplier random variables, *i.e.*, for each $b \in \{1, \ldots, B\}$, $\boldsymbol{\xi}^{(b)} = (\xi_1^{(b)}, \ldots, \xi_n^{(b)})$. Then, one computes $V_n, \widehat{V}_n^{(1)}, \ldots, \widehat{V}_n^{(B)}$ and $W_n, \widehat{W}_n^{(1)}, \ldots, \widehat{W}_n^{(B)}$ using the above formulas. These replicates of V_n and W_n are very quick to compute since the matrices A and $A^{\star}$ need to be evaluated only once from the data.

3. Tests of Univariate Symmetry

Many tests of univariate symmetry have been proposed over the years. An early contribution is that of [5] based on a Cramér–von Mises statistic. Tests of symmetry about an unspecified point have been studied by [6,7]; see also the more recent contribution by [8], where invariant tests based on the empirical characteristic function are proposed. Extensions of these tests are investigated by [9]. Tests based on kernel density estimation have been investigated by [10,11], where the computation of p-values relies on the bootstrap. Data-driven smooth tests of symmetry have been proposed by [12].

Here, one focuses on consistent tests based on distribution and characteristic functions in the case of a known center of symmetry. To this end, let $X_1, \ldots, X_n$ be independent and identically distributed copies of a continuous random variable X. For $x \in \mathbb{R}$, let $\mathrm{P}(X \le x) = F(x)$ be the distribution function of X, and for $t \in \mathbb{R}$, let $c(t) = \mathrm{E}(e^{itX}) = \int_{\mathbb{R}} e^{itx} \mathrm{d}F(x)$ be its characteristic function. Here and in the sequel, $i^2 = -1$, and E is the expectation operator. The goal in this section is to describe test procedures for the null hypothesis $\mathbb{H}_0^{\text{univ}} : X - a \stackrel{d}{=} a - X$. One can focus on the case $a = 0$ only, *i.e.*, $\mathbb{H}_0^{\text{univ}} : X \stackrel{d}{=} -X$. Indeed, the methodology extends easily to the case $a \neq 0$ by observing that $\mathbb{H}_0^{\text{univ}} : X - a \stackrel{d}{=} a - X$ is equivalent to $\mathbb{H}_0^{\text{univ}} : \widetilde{X} \stackrel{d}{=} -\widetilde{X}$, where $\widetilde{X} = X - a$, and by working with the sample of transformed data $\widetilde{X}_1, \ldots, \widetilde{X}_n$, where $\widetilde{X}_j = X_j - a$ for each $j \in \{1, \ldots, n\}$.

The first step is to note that one can write the null hypothesis $\mathbb{H}_0^{\text{univ}} : X \stackrel{d}{=} -X$ from a distribution function or a characteristic function point-of-view. If $\mathbb{H}_0^{\text{univ}}$ is true, then $F(-x) = \mathrm{P}(X \le -x) = \mathrm{P}(-X \le -x) = \mathrm{P}(X \ge x) = 1 - F(x^-)$ for all $x \in \mathbb{R}$ and $c(t) = \mathrm{E}\left(e^{itX}\right) = \mathrm{E}\left(e^{it(-X)}\right) = \mathrm{E}\left(e^{i(-t)X}\right) = c(-t)$ for all $t \in \mathbb{R}$. Hence, the null hypothesis can be written equivalently as:

$$\mathbb{H}_0^{\text{univ}} : F(-x) = 1 - F(x^-) \quad \forall x \in \mathbb{R};$$
$$\mathbb{H}_0^{\text{univ}} : c(t) = c(-t) \quad \forall t \in \mathbb{R}.$$

As a consequence, consistent test statistics can be based either on the empirical version of F or on the empirical version of c given, respectively, by:

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\left(X_j \leq x\right) \quad \text{and} \quad c_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j}$$

Here and in the sequel, $\mathbb{I}(s) = 1$ if the statement s is true and zero otherwise. Natural test statistics for univariate symmetry are therefore given by:

$$\begin{aligned} V_n^{\text{univ}} &= n\int_{\mathbb{R}} \left\{F_n(-x) + F_n(x^-) - 1\right\}^2 \mathrm{d}x \\ W_n^{\text{univ}} &= n\int_{\mathbb{R}} \left\{F_n(-x) + F_n(x^-) - 1\right\}^2 \mathrm{d}F_n(x) \\ V_n^{\text{univ}}(\omega) &= n\int_{\mathbb{R}} |c_n(t) - c_n(-t)|^2 \,\omega(t)\,\mathrm{d}t \end{aligned}$$

where $F_n(x^-) = (1/n)\sum_{j=1}^{n} \mathbb{I}(X_j < x)$ and $|z|$ denotes the modulus of the complex number z. In the definition of the Cramér–von Mises statistic W_n^{univ}, $\mathrm{d}F_n$ puts mass $1/n$ at each element of the sample. This statistic is a special case of the one proposed by [13] when X is continuous. An asymptotically-equivalent version of this test statistic has been investigated by [14]; see also [5]. According to the author's knowledge, V_n^{univ} has not been investigated yet. The test statistic $V_n^{\text{univ}}(\omega)$ uses the characteristic function point-of-view and is based on a nonnegative weight function ω that must be specified by the experimenter. Some examples of weight functions are described in Section 5.2. The following lemma provides formulas for the computation of these test statistics.

Lemma 3. *One has:*

$$\begin{aligned} V_n^{\text{univ}} &= \frac{1}{n}\sum_{j,j'=1}^{n} \psi^{\text{univ}}\left(X_j, X_{j'}\right) \\ W_n^{\text{univ}} &= \frac{1}{n^2}\sum_{j,j',k=1}^{n} \phi^{\text{univ}}\left(X_j, X_{j'}, X_k\right) \\ V_n^{\text{univ}}(\omega) &= \frac{1}{n}\sum_{j,j'=1} \psi_\omega^{\text{univ}}\left(X_j, X_{j'}\right) \end{aligned}$$

where $\psi^{\text{univ}}(x_1, x_2) = 2\,\text{sign}(x_1)\,\text{sign}(x_2)\,\min(|x_1|, |x_2|)$,

$$\begin{aligned} \phi^{\text{univ}}(x_1, x_2, x_3) &= \mathbb{I}\left\{x_3 \leq \min(-x_1, -x_2)\right\} - \mathbb{I}\left\{x_3 \leq \min(x_1, -x_2)\right\} \\ &\quad - \mathbb{I}\left\{x_3 \leq \min(-x_1, x_2)\right\} + \mathbb{I}\left\{x_3 \leq \min(x_1, x_2)\right\} \end{aligned}$$

and $\psi_\omega^{\text{univ}}(x_1, x_2) = 4\int_{\mathbb{R}} \sin(tx_1)\,\sin(tx_2)\,\omega(t)\,\mathrm{d}t$.

Since $\psi^{\text{univ}}(x_1, -x_2) = -\psi^{\text{univ}}(x_1, x_2)$ and $\psi_\omega^{\text{univ}}(x_1, -x_2) = -\psi_\omega^{\text{univ}}(x_1, x_2)$, the fact that $X \stackrel{d}{=} -X$ under the null hypothesis entails $\mathrm{E}\{\psi^{\text{univ}}(x_1, X_2)\} = 0$ and $\mathrm{E}\{\psi_\omega^{\text{univ}}(x_1, X_2)\} = 0$. As a consequence, V_n^{univ} and $V_n^{\text{univ}}(\omega)$ are V-statistics of order two with first-order degeneracy, and their large-sample behaviour follows from Proposition 1. Note, however, that an additional requirement on ψ^{univ} is necessary in order that $\mathrm{E}\{\psi^{\text{univ}}(X_1, X_2)^2\} < \infty$. In particular, it will hold true if the moment of order two exists.

Since ϕ^{univ} is symmetric with respect to its first two components and from the fact that $\phi^{\text{univ}}(x_1, -x_2, x_3) = -\phi^{\text{univ}}(x_1, x_2, x_3)$, which entails $\mathrm{E}\{\phi^{\text{univ}}(x_1, X_2, x_3)\} = 0$ for all $(x_1, x_3) \in \mathbb{R}^2$, the asymptotic behaviour of W_n^{univ} is deduced from Proposition 2. Finally, the multiplier versions of V_n^{univ}, W_n^{univ} and $V_n^{\text{univ}}(\omega)$ are derived from the formulas in (4).

4. Tests of Bivariate Symmetry

While less popular than the univariate symmetry hypothesis, many tests of bivariate symmetry have been proposed. The earliest contributions come from [15,16], where nonparametric tests were developed; these tests have been reconsidered by [17]. A test using the empirical distribution function has been suggested by [18]. An investigation comparing some tests of bivariate symmetry was done by [19]. Extensions to tests of multivariate symmetry were considered by [20].

In this section, the focus is put on bivariate exchangeability and reflected symmetry. In the sequel, $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and identically distributed copies of a continuous random pair (X, Y). For $(x, y) \in \mathbb{R}^2$, the joint distribution of (X, Y) is $\mathrm{P}(X \leq x, Y \leq y) = H(x, y)$, and for $(s, t) \in \mathbb{R}^2$, its characteristic function is $C(s, t) = \mathrm{E}(e^{i(sX+tY)}) = \int_{\mathbb{R}^2} e^{i(sx+ty)} \, \mathrm{d}H(x, y)$. The proposed test statistics will be based on the sample versions of H and C, namely:

$$H_n(x, y) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}\left(X_j \leq x, Y_j \leq y\right) \quad \text{and} \quad C_n(s, t) = \frac{1}{n} \sum_{j=1}^{n} e^{i(sX_j + tY_j)} \tag{5}$$

4.1. Exchangeability

The goal here is to test for the null hypothesis $\mathbb{H}_0^{\text{exch}} : (X, Y) \stackrel{d}{=} (Y, X)$. When $\mathbb{H}_0^{\text{exch}}$ is true, $H(x, y) = \mathrm{P}(X \leq x, Y \leq y) = \mathrm{P}(Y \leq x, X \leq y) = \mathrm{P}(X \leq y, Y \leq x) = H(y, x)$ and $C(s, t) = \mathrm{E}\left(e^{i(sX+tY)}\right) = \mathrm{E}\left(e^{i(sY+tX)}\right) = \mathrm{E}\left(e^{i(tX+sY)}\right) = C(t, s)$. Hence, the null hypothesis can be written equivalently as:

$$\begin{aligned}
&\mathbb{H}_0^{\text{exch}} : H(x, y) = H(y, x) \quad \forall (x, y) \in \mathbb{R}^2; \\
&\mathbb{H}_0^{\text{exch}} : C(s, t) = C(t, s) \quad \forall (s, t) \in \mathbb{R}^2.
\end{aligned}$$

In view of these two characterizations of the null hypothesis, consider:

$$\begin{aligned}
W_n^{\text{exch}} &= n \int_{\mathbb{R}^2} \left\{H_n(x, y) - H_n(y, x)\right\}^2 \mathrm{d}H_n(x, y) \\
V_n^{\text{exch}}(\Omega) &= n \int_{\mathbb{R}^2} \left|C_n(s, t) - C_n(t, s)\right|^2 \Omega(s, t) \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}$$

where Ω is a nonnegative and integrable weight function. The test statistic W_n^{exch} was introduced by [16], where a test of symmetry is performed using an approximation of the distribution under $\mathbb{H}_0^{\text{exch}}$. Because the latter is inaccurate under high levels of dependence, an alternative procedure was proposed by [21]. Explicit formulas for W_n^{exch} and $V_n^{\text{exch}}(\Omega)$ are provided in the next lemma.

Lemma 4. *One has:*

$$W_n^{\text{exch}} = \frac{1}{n^2} \sum_{j,j',k=1}^{n} \phi^{\text{exch}} \left\{(X_j, Y_j), (X_{j'}, Y_{j'}), (X_k, Y_k)\right\}$$

where:

$$\begin{aligned}
\phi^{\text{exch}} \left\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\right\} = \; & \mathbb{I}\left\{x_3 \geq \max(x_1, x_2), y_3 \geq \max(y_1, y_2)\right\} \\
& - \mathbb{I}\left\{x_3 \geq \max(x_1, y_2), y_3 \geq \max(y_1, x_2)\right\} \\
& - \mathbb{I}\left\{x_3 \geq \max(y_1, x_2), y_3 \geq \max(x_1, y_2)\right\} \\
& + \mathbb{I}\left\{x_3 \geq \max(y_1, y_2), y_3 \geq \max(x_1, x_2)\right\}
\end{aligned}$$

and:

$$V_n^{\text{exch}}(\Omega) = \frac{1}{n} \sum_{j,j'=1}^{n} \psi_\Omega^{\text{exch}} \left\{(X_j, Y_j), (X_{j'}, Y_{j'})\right\}$$

where for $\widetilde{\psi}_\Omega^{\text{exch}}(x,y) = \int_{\mathbb{R}^2} \cos(s\,x + t\,y)\,\Omega(s,t)\,\mathrm{d}s\,\mathrm{d}t$,

$$\begin{aligned}\psi_\Omega^{\text{exch}}\{(x_1,y_1),(x_2,y_2)\} &= \widetilde{\psi}_\Omega^{\text{exch}}(x_1-x_2,y_1-y_2) - \widetilde{\psi}_\Omega^{\text{exch}}(x_1-y_2,y_1-x_2) \\ &\quad - \widetilde{\psi}_\Omega^{\text{exch}}(y_1-x_2,x_1-y_2) + \widetilde{\psi}_\Omega^{\text{exch}}(y_1-y_2,x_1-x_2)\end{aligned}$$

The kernel ϕ^{exch} is symmetric with respect to its first two components. In addition, $\mathrm{E}\left[\phi^{\text{exch}}\{(x_1,y_1),(X_2,Y_2),(x_3,y_3)\}\right] = 0$ under the null hypothesis, because $\phi^{\text{exch}}\{(x_1,y_1),(y_2,x_2),(x_3,y_3)\} = -\phi^{\text{exch}}\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$. The asymptotic behaviour of W_n^{exch} can then be deduced from Proposition 2. Similarly, $\psi_\Omega^{\text{exch}}\{(x_1,y_1),(y_2,x_2)\} = -\psi_\Omega^{\text{exch}}\{(x_1,y_1),(x_2,y_2)\}$, so that $\mathrm{E}\left[\psi_\Omega^{\text{exch}}\{(x_1,y_1),(X_2,Y_2)\}\right] = 0$ and $V_n^{\text{exch}}(\Omega)$ is a V-statistic with first-order degeneracy. Its large-sample behaviour then follows from Proposition 1. Multiplier versions of W_n^{exch} and $V_n^{\text{exch}}(\Omega)$ derive from formulas in Equation (4).

4.2. Reflected Symmetry

As mentioned in the Introduction, the null hypothesis of reflected symmetry around $(a,b) \in \mathbb{R}^2$ is $\mathbb{H}_0^{\text{refl}} : (X-a, Y-b) \overset{d}{=} (a-X, b-Y)$. For simplicity, one assumes that $a = b = 0$, so that the focus is put on $\mathbb{H}_0^{\text{refl}} : (X,Y) \overset{d}{=} (-X,-Y)$. The extension to arbitrary $(a,b) \in \mathbb{R}^2$ is straightforward upon noting that the null hypothesis $\mathbb{H}_0^{\text{refl}} : (X-a, Y-b) \overset{d}{=} (a-X, b-Y)$ is equivalent to $\mathbb{H}_0^{\text{refl}} : (\widetilde{X},\widetilde{Y}) \overset{d}{=} (-\widetilde{X},-\widetilde{Y})$, where $\widetilde{X} = X - a$ and $\widetilde{Y} = Y - b$. Hence, one would only have to consider the sample of transformed data $(\widetilde{X}_1,\widetilde{Y}_1),\ldots,(\widetilde{X}_n,\widetilde{Y}_n)$, where $\widetilde{X}_j = X_j - a$ and $\widetilde{Y}_j = Y_j - b$ for each $j \in \{1,\ldots,n\}$.

When $\mathbb{H}_0^{\text{refl}} : (X,Y) \overset{d}{=} (-X,-Y)$ is true, $H(x,y) = \mathrm{P}(-X \le x, -Y \le y) = \mathrm{P}(X \ge -x, Y \ge -y)$ and $C(s,t) = \mathrm{E}(e^{i(s(-X)+t(-Y))}) = \mathrm{E}(e^{i((-s)X+(-t)Y)}) = C(-s,-t)$. Letting $\bar{H}(x,y) = \mathrm{P}(X \ge x, Y \ge y)$, the distribution function and characteristic function versions of $\mathbb{H}_0^{\text{refl}}$ are then respectively:

$$\begin{aligned}&\mathbb{H}_0^{\text{refl}} : H(x,y) = \bar{H}(-x,-y) \quad \forall (x,y) \in \mathbb{R}^2; \\ &\mathbb{H}_0^{\text{refl}} : C(s,t) = C(-s,-t) \quad \forall (s,t) \in \mathbb{R}^2.\end{aligned}$$

Letting $\bar{H}_n(x,y) = (1/n)\sum_{j=1}^n \mathbb{I}(X_j \ge x, Y_j \ge y)$, consider the test statistics:

$$\begin{aligned}W_n^{\text{refl}} &= n \int_{\mathbb{R}^2} \{H_n(x,y) - \bar{H}_n(-x,-y)\}^2 \,\mathrm{d}H_n(x,y) \\ V_n^{\text{refl}}(\Omega) &= n \int_{\mathbb{R}^2} |C_n(s,t) - C_n(-s,-t)|^2\, \Omega(s,t)\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

Explicit formulas are given next.

Lemma 5. *One has:*

$$W_n^{\text{refl}} = \frac{1}{n^2} \sum_{j,j',k=1}^{n} \phi^{\text{refl}}\left\{(X_j,Y_j),(X_{j'},Y_{j'}),(X_k,Y_k)\right\}$$

where:

$$\begin{aligned}\phi^{\text{refl}}\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\} &= \mathbb{I}\{x_3 \ge \max(x_1,x_2), y_3 \ge \max(y_1,y_2)\} \\ &\quad - \mathbb{I}\{x_3 \ge \max(x_1,-x_2), y_3 \ge \max(y_1,-y_2)\} \\ &\quad - \mathbb{I}\{x_3 \ge \max(-x_1,x_2), y_3 \ge \max(-y_1,y_2)\} \\ &\quad + \mathbb{I}\{x_3 \ge \max(-x_1,-x_2), y_3 \ge \max(-y_1,-y_2)\}\end{aligned}$$

and:

$$V_n^{\text{refl}}(\Omega) = \frac{1}{n} \sum_{j,j'=1}^{n} \psi_\Omega^{\text{refl}}\left\{(X_j,Y_j),(X_{j'},Y_{j'})\right\}$$

where $\psi_\Omega^{\mathrm{refl}}\{(x_1,y_1),(x_2,y_2)\} = 4\int_{\mathbb{R}^2} \sin(sx_1+ty_1)\sin(sx_2+ty_2)\,\Omega(s,t)\,\mathrm{d}s\,\mathrm{d}t$.

Proceeding similarly as with ϕ^{exch}, one can show that $\mathrm{E}\left[\phi^{\mathrm{refl}}\{(x_1,y_1),(X_2,Y_2),(x_3,y_3)\}\right] = 0$. Thus, since ϕ^{refl} is symmetric with respect to its first two components, the asymptotic behaviour of W_n^{refl} follows from Proposition 2. Furthermore, since $\psi_\Omega^{\mathrm{refl}}\{(x_1,y_1),(-x_2,-y_2)\} = -\psi_\Omega^{\mathrm{refl}}\{(x_1,y_1),(x_2,y_2)\}$, one deduces $\mathrm{E}\{\psi_\Omega^{\mathrm{refl}}\{(x_1,y_1),(X_2,Y_2)\}\} = 0$, and $V_n^{\mathrm{refl}}(\Omega)$ is a first-order degenerate V-statistic whose large-sample behaviour follows from Proposition 1. Multiplier versions of W_n^{refl} and $V_n^{\mathrm{refl}}(\Omega)$ are derived from Equation (4).

4.3. A Note on Copula Symmetry

A class of bivariate symmetries, yet less known than exchangeability and reflected symmetry, is based on copulas. The latter allows one to shed new light on the understanding of bivariate symmetry. The starting point is a theorem by [22] that states that there exists a function $D : [0,1]^2 \to [0,1]$ called a copula, such that $\mathrm{P}(X \le x, Y \le y) = D\{\mathrm{P}(X \le x), \mathrm{P}(Y \le y)\}$ for all $(x,y) \in \mathbb{R}^2$. If the marginal distributions $F_X(x) = \mathrm{P}(X \le x)$ and $F_Y(y) = \mathrm{P}(Y \le y)$ are continuous, then D is unique. As a consequence, D completely characterizes the dependence between X and Y when (X,Y) is continuous.

Because Sklar's representation entails that the random pair $(U,V) = (F_X(X), F_Y(Y))$ is distributed as D, exchangeability and reflected symmetry can be reformulated as follows:

(i) The pair (X,Y) is exchangeable if and only if $X \overset{d}{=} Y$ and $(U,V) \overset{d}{=} (V,U)$;

(ii) The pair (X,Y) is reflection symmetric around $(a,b) \in \mathbb{R}^2$ if and only if $X - a \overset{d}{=} a - X$, $Y - b \overset{d}{=} b - Y$ and $(\widetilde{U},\widetilde{V}) \overset{d}{=} (-\widetilde{U},-\widetilde{V})$, where $\widetilde{U} = U - 1/2$ and $\widetilde{V} = V - 1/2$.

The reader is referred to [23] for more details on the general theory of copulas.

Assuming the availability of independent random copies $(U_1,V_1),\ldots,(U_n,V_n)$ of (U,V), one can test for the exchangeability and reflected symmetry of the copula only. This setup is equivalent in assuming that the marginal distributions F_X and F_Y are known, so that a random sample $(X_1,Y_1),\ldots,(X_n,Y_n)$ can be transformed to the copula scale by letting $(U_j,V_j) = (F_X(X_j), F_Y(Y_j))$ for each $j \in \{1,\ldots,n\}$. For copula exchangeability, the method described in Subsection 3.2 can be applied directly; for copula reflected symmetry, this corresponds to the case $a = b = 1/2$, and then, the methodology in Subsection 3.2 may be used with $(\widetilde{U}_1,\widetilde{V}_1),\ldots,(\widetilde{U}_n,\widetilde{V}_n)$, where $\widetilde{U}_j = U_j - 1/2$ and $\widetilde{V}_j = V_j - 1/2$.

The marginal distributions F_X and F_Y are generally unknown. In that case, it is suggested to work instead with $(\widehat{U}_1,\widehat{V}_1),\ldots,(\widehat{U}_n,\widehat{V}_n)$, where $(\widehat{U}_j,\widehat{V}_j) = (\widehat{F}_X(X_j), \widehat{F}_Y(Y_j))$ and $\widehat{F}_X$, $\widehat{F}_Y$ are the empirical distribution functions. However, doing so results in much more complicated limit distributions and calls for suitably-adapted multiplier methods. See the works by [24] on copula exchangeability and by [25] on copula reflected symmetry (called radial symmetry in that case) for details.

5. Monte Carlo Study of the Sampling Properties of the Tests

5.1. Parameters of the Simulations

This section explores the sample properties of the tests for the three null hypotheses considered in Sections 3 and 4, namely $\mathbb{H}_0^{\mathrm{univ}}$, $\mathbb{H}_0^{\mathrm{exch}}$ and $\mathbb{H}_0^{\mathrm{refl}}$. Specifically, the ability of the tests to keep their 5% nominal level under the null hypothesis and their power against alternative hypotheses will be investigated with the help of simulated datasets. The probability of rejection of the null hypothesis will be estimated from 1000 replicates under each scenario. The computation of p-values will be based on $B = 1000$ bootstrap samples using a version of the multiplier method called the Bayesian bootstrap. In that case, $\xi_1,\ldots,\xi_n$ are replaced by $(\gamma_j/\bar{\gamma}) - 1$, $j \in \{1,\ldots,n\}$, where $\gamma_1,\ldots,\gamma_n$ are independent and identically distributed from the exponential law with mean one; see [26] for details.

Many other choices are possible for the stochastic structure of the multiplier variables, but from the author's experience, it has little influence on the performance of the tests.

5.2. Size and Power of the Tests of Univariate Symmetry

This subsection investigates the properties of the tests based on $V_n^{\rm univ}$, $W_n^{\rm univ}$ and $V_n^{\rm univ}(\omega)$ for testing the null hypothesis of univariate symmetry $\mathbb{H}_0^{\rm univ} : X \stackrel{d}{=} -X$. The computation of $V_n^{\rm univ}(\omega)$ calls for the choice of a weight function ω. For the simulation results that will be presented, one considers $\omega_1^\lambda(t) = e^{-\lambda|t|}$ and $\omega_2^\lambda(t) = e^{-\lambda^2 t^2/2}$ for $\lambda \in \{1,2\}$. One can show that for $x_+ = x_1 + x_2$ and $x_- = x_1 - x_2$,

$$\psi^{\rm univ}_{\omega_1^\lambda}(x_1,x_2) \propto \frac{x_1 x_2}{(\lambda^2 + x_-^2)\,(\lambda^2 + x_+^2)} \quad \text{and} \quad \psi^{\rm univ}_{\omega_1^\lambda}(x_1,x_2) \propto \phi\left(\frac{x_-}{\lambda}\right) - \phi\left(\frac{x_+}{\lambda}\right)$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the density of the standard univariate normal distribution.

In order to investigate the ability of the tests to reject the null hypothesis of univariate symmetry around zero, one considers the general family of skew-asymmetric densities, as defined by [27]. Specifically, for a given symmetric density f and a given absolutely continuous distribution function G, such that G' is a symmetric density around zero, a skew-asymmetric density is defined for $\delta \in \mathbb{R}$ by $g_\delta(x) = 2f(x)\,G(\delta x)$. The case $\delta = 0$ corresponds to a situation under the null hypothesis. When f and G are respectively the density and the cumulative distribution function of the standard normal distribution, one recovers the skew-normal family as introduced by [28]. For the simulation results that are reported in Table 1, one also considers the skew-T distribution with three degrees of freedom and the skew-Cauchy distribution (which is indeed the skew-T with one degree of freedom). Since $g_\delta(x)/f(x) \leq 2$ for all $x \in \mathbb{R}$, datasets from g_δ can be generated using the rejection method; see [29] for more details. The idea is to simulate repeatedly X from f and U from the uniform distribution on $(0,1)$ until $U \leq g_\delta(X)/2\,f(X)$; then $X \sim g_\delta$.

Looking at Table 1, one can say that the six tests are very good at keeping their 5% nominal level under the null hypothesis, even when $n = 50$. An exception occurs for $V_n^{\rm univ}$ under the Cauchy distribution, where the test is too conservative. This behaviour is explained by the fact that the requirement $\mathrm{E}\{\psi^{\rm univ}(X_1,X_2)^2\} < \infty$ is not satisfied in that case. As expected, the power of these tests increases as a function of the sample size, as expected from their theoretical consistency. The power also increases as a function of the parameter δ that controls the level of asymmetry. Note that departures from $\mathbb{H}_0^{\rm univ}$ based on skew-Student and skew-Cauchy alternatives are more easily detected than those from the skew-normal distribution. Overall, the best tests are those based on $V_n^{\rm univ}$ and $W_n^{\rm univ}$, as well as on the characteristic function statistics $V_n^{\rm univ}(\omega_1^2)$ and $V_n^{\rm univ}(\omega_2^2)$.

5.3. Size and Power of the Tests of Exchangeability

The test statistics $W_n^{\rm exch}$ and $V_n^{\rm exch}(\Omega)$ are investigated here for testing the null hypothesis $\mathbb{H}_0^{\rm exch}$ of exchangeability. Two weight functions are considered for $V_n^{\rm exch}(\Omega)$, namely:

$$\Omega_1^\lambda(s,t) = e^{-\lambda(|s|+|t|)} \quad \text{and} \quad \Omega_2^\lambda(s,t) = e^{-\lambda^2(s^2+t^2)/2}$$

One can show that:

$$\begin{aligned}
\psi^{\rm exch}_{\Omega_1^\lambda}\{(x_1,y_1),(x_2,y_2)\} &\propto \frac{1}{\{\lambda^2 + (x_1-x_2)^2\}\{\lambda^2 + (y_1-y_2)^2\}} \\
&\quad - \frac{1}{\{\lambda^2 + (x_1-y_2)^2\}\{\lambda^2 + (y_1-x_2)^2\}} \\
\psi^{\rm exch}_{\Omega_2^\lambda}\{(x_1,y_1),(x_2,y_2)\} &\propto \phi\left(\frac{x_1-x_2}{\lambda}\right)\phi\left(\frac{y_1-y_2}{\lambda}\right) - \phi\left(\frac{x_1-y_2}{\lambda}\right)\phi\left(\frac{y_1-x_2}{\lambda}\right)
\end{aligned}$$

Table 1. Probability of the rejection of the null hypothesis of univariate symmetry, as estimated from 1000 replicates, for the tests based on V_n^{univ}, W_n^{univ} and $V_n^{\text{univ}}(\omega)$ under skew-normal, skew-T and skew-Cauchy alternatives.

Law	δ	n	V_n^{univ}	W_n^{univ}	$V_n^{\text{univ}}(\omega_1^1)$	$V_n^{\text{univ}}(\omega_1^2)$	$V_n^{\text{univ}}(\omega_2^1)$	$V_n^{\text{univ}}(\omega_2^2)$
Skew-Normal	0	50	5.8	6.3	6.4	6.8	6.1	6.1
		100	5.2	5.0	5.5	5.6	5.1	5.5
		200	5.3	4.7	5.7	5.4	6.2	5.6
	0.1	50	9.5	11.1	8.6	9.4	9.3	10.8
		100	13.6	14.9	12.1	12.7	13.9	14.5
		200	21.3	20.5	17.7	18.9	19.9	21.4
	0.25	50	28.8	30.2	25.7	26.6	28.5	29.5
		100	47.8	50.9	40.6	41.7	45.8	48.3
		200	79.1	77.7	71.3	72.8	76.5	79.6
	0.5	50	74.6	74.4	67.9	70.3	73.0	75.2
		100	96.3	95.3	93.5	94.2	95.7	96.2
		200	100.0	100.0	99.8	99.8	99.8	100.0
Skew-T	0	50	4.4	7.2	4.4	5.0	4.7	5.2
		100	4.2	7.1	4.8	5.1	5.1	5.3
		200	4.5	5.7	5.4	5.4	4.5	4.4
	0.1	50	11.6	14.3	8.3	8.5	10.5	10.6
		100	17.6	19.5	11.7	11.8	13.9	15.2
		200	30.9	29.0	18.9	19.0	22.5	24.3
	0.25	50	43.1	39.4	27.5	27.7	33.4	36.2
		100	66.4	66.2	43.1	43.5	55.5	59.5
		200	94.0	92.3	78.9	78.5	86.9	89.7
	0.5	50	85.5	85.1	66.8	67.6	77.5	80.7
		100	98.9	99.0	94.4	94.4	97.6	98.2
		200	100.0	100.0	99.9	99.9	100.0	100.0
Skew-Cauchy	0	50	3.3	6.2	5.7	5.9	5.3	5.6
		100	3.4	6.4	5.1	5.0	5.4	5.2
		200	2.9	6.2	5.6	5.4	6.1	6.3
	0.1	50	23.9	28.5	9.8	9.6	11.7	11.9
		100	47.0	46.6	11.8	12.1	16.4	16.8
		200	76.8	73.4	17.5	16.5	29.0	29.6
	0.25	50	56.9	67.1	24.7	23.9	35.1	35.6
		100	83.3	91.1	45.4	42.7	61.1	61.5
		200	93.8	99.6	76.1	71.5	89.8	90.2
	0.5	50	81.3	93.5	57.8	56.4	74.0	75.0
		100	94.1	99.7	89.4	87.8	96.0	96.3
		200	97.8	100.0	99.6	99.5	100.0	100.0

As enlightened in Subsection 4.3, the hypothesis of exchangeability of a pair (X, Y) requires that $X \stackrel{d}{=} Y$ and that $(U, V) \stackrel{d}{=} (V, U)$. For the simulation results that will be presented, one assumes a $\mathbb{N}(0, 1)$ distribution for both X and Y, so that the asymmetry will be controlled solely by the form of the copula. Here, one considers a general class of asymmetric bivariate distributions of the form:

$$H_{D,\delta}(x, y) = \{\Phi(x)\}^{\delta} D \left\{ \{\Phi(x)\}^{1-\delta}, \Phi(y) \right\}$$

where Φ is the cumulative distribution function of the $\mathbb{N}(0,1)$ law and D is a symmetric copula, *i.e.*, $D(u,v) = D(v,u)$ for all $(u,v) \in [0,1]^2$. The special case $\delta = 0$ corresponds to a scenario under the null hypothesis of exchangeability. This construction is based on a proposal by [30]. For the results in Table 2, the copula D belongs either to the normal or the Gumbel–Hougaard family of symmetric models, *i.e.*,

$$D(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \phi_\varrho(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{and} \quad D(u,v) = \exp\left\{ -\left|\log uv\right|^{1/(1-\theta)} \right\}$$

where ϕ_ϱ is the bivariate standard normal density with correlation $\varrho \in [-1,1]$ and $\theta \in [0,1]$. These parameters are taken so that they match a Kendall's tau of 0.75, *i.e.*, $\varrho = 0.924$ and $\theta = 0.75$. The values of the asymmetry parameter are $\delta \in \{0, 0.25, 0.50, 0.75\}$.

Table 2. Probability of the rejection of the null hypothesis of exchangeability, as estimated from 1000 replicates, for the tests based on W_n^{exch} and $V_n^{\mathrm{exch}}(\Omega)$ under the copula-based distribution $H_{D,\delta}$.

Copula D	δ	n	W_n^{exch}	$V_n^{\mathrm{exch}}(\Omega_1^{1/4})$	$V_n^{\mathrm{exch}}(\Omega_2^{1/4})$	$V_n^{\mathrm{exch}}(\Omega_1^{1/2})$	$V_n^{\mathrm{exch}}(\Omega_2^{1/2})$
Normal	0	50	2.3	1.5	2.5	3.1	4.2
		100	4.6	3.6	4.2	4.3	5.3
		200	4.5	2.9	3.7	2.8	3.7
	0.25	50	9.8	8.4	16.7	15.6	22.4
		100	16.7	32.1	44.0	37.9	48.1
		200	37.5	73.2	82.4	76.7	85.5
	0.5	50	13.8	18.7	26.3	29.6	37.8
		100	29.3	57.4	66.8	60.1	69.5
		200	48.1	94.6	97.0	94.0	98.7
	0.75	50	9.3	4.6	7.3	10.8	16.1
		100	12.7	18.8	19.4	23.9	33.4
		200	24.0	48.1	54.7	54.4	66.4
Gumbel–Hougaard	0	50	2.3	1.5	3.2	3.9	4.2
		100	4.4	2.5	3.7	4.3	6.1
		200	4.1	5.0	6.0	5.4	5.2
	0.25	50	11.2	11.7	20.0	16.7	25.6
		100	23.1	47.2	60.7	47.6	57.5
		200	41.6	90.0	94.8	87.9	93.2
	0.5	50	17.0	26.7	39.7	32.9	45.0
		100	29.6	70.8	85.5	72.1	83.9
		200	56.4	98.1	99.0	98.0	99.1
	0.75	50	11.6	10.5	14.2	18.6	27.0
		100	20.5	31.6	38.7	38.3	49.1
		200	25.2	64.8	74.4	68.2	80.8

In light of simulations not presented here, the values $\lambda \in \{1/4, 1/2\}$ offer the best performance for the test statistics $V_n^{\mathrm{exch}}(\Omega_1^\lambda)$ and $V_n^{\mathrm{exch}}(\Omega_2^\lambda)$. From the entries in Table 2, one can see that the five tests are rather good at keeping their size under $\mathbb{H}_0^{\mathrm{exch}}$, having in mind the fact that the multiplier method is valid asymptotically as $n \to \infty$. As expected, the power of the tests increases with the sample size. Here, the level of asymmetry is not necessarily monotone in δ. Indeed, the highest level of asymmetry occurs for values of δ around 0.5 when it is measured for example by the index introduced by [31]; the simulation results concord with this fact, where the highest power are observed when $\delta = 0.5$. Here, the test based on the empirical distribution function statistic W_n^{exch} is significantly less powerful than those based on the empirical characteristic function; a similar feature has been documented by [32] when testing for copula symmetry. The best tests overall are those based on $V_n^{\text{exch}}(\Omega_2^\lambda)$. Finally, note that asymmetries based on the Gumbel–Hougaard copula are better detected than those based on the normal copula.

5.4. Size and Power of the Tests of Reflected Symmetry

For the same weight functions Ω_1^λ and Ω_2^λ considered in the preceding subsection for testing exchangeability, one can show that:

$$\begin{aligned}
\psi^{\text{refl}}_{\Omega_1^\lambda}\{(x_1,y_1),(x_2,y_2)\} &\propto \frac{x_1\,x_2\,(\lambda^2+y_1^2+y_2^2)+y_1\,y_2\,(\lambda^2+x_1^2+x_2^2)}{(\lambda^2+x_-^2)(\lambda^2+x_+^2)(\lambda^2+y_-^2)(\lambda^2+y_+^2)}\\
\psi^{\text{refl}}_{\Omega_2^\lambda}\{(x_1,y_1),(x_2,y_2)\} &\propto \phi\left(\frac{x_-}{\lambda}\right)\phi\left(\frac{y_-}{\lambda}\right)-\phi\left(\frac{x_+}{\lambda}\right)\phi\left(\frac{y_+}{\lambda}\right)
\end{aligned}$$

where $x_- = x_1 - x_2$, $x_+ = x_1 + x_2$, $y_- = y_1 - y_2$ and $y_+ = y_1 + y_2$.

Following [33], reflected asymmetric bivariate densities can be built from a generalization of skew asymmetric univariate densities. Specifically, consider a density f, such that $f(x,y) = f(-x,-y)$, and a one-dimensional distribution function G, such that its density G' is symmetric around zero. Then, $g_\delta(x,y) = 2\,f(x,y)\,G\{\delta(x+y)\}$ is a skew asymmetric bivariate density. In the special case when $f = \phi_\varrho$ and $G = \Phi$ is the cumulative distribution function of the $\mathbb{N}(0,1)$ distribution, one recovers the so-called skew-normal distribution with correlation coefficient $\varrho \in [-1,1]$, namely:

$$g_\delta^{\text{N}}(x,y) = 2\,\phi_\varrho(x,y)\,\Phi\{\delta(x+y)\}$$

For the results in Table 3, $\varrho \in \{1/3, 2/3\}$ and $\delta \in \{0, 0.25, 0.5\}$. Results not presented here with $\delta = 0.75$ show that the power is one, even for a sample size as low as $n = 50$. Here, similar comments as for the tests of exchangeability apply for the ability of the tests to keep their nominal level and for their power as n increases. Comparing to the results in Table 2, however, one sees that the estimated probabilities of rejection are higher here. It can be explained, at least in part, by the fact that the asymmetry in the bivariate skew asymmetric model g_δ affects both the marginal distributions and the copula. Here, reflected asymmetry increases as a function of δ, resulting in power results that increase with δ. Overall, the test based on W_n^{refl} performs well under all of the scenarios that were considered. The characteristic function statistics are also doing well, the best being $V_n^{\text{refl}}(\Omega_1^2)$. Finally, note that the power is higher when $\varrho = 1/3$ compared to $\varrho = 2/3$.

6. Unification into a General Framework

The hypotheses considered so far can be treated somewhat simultaneously by taking a general group of transformations. To this end, take a random vector $\mathbf{X} = (X_1, \ldots, X_p)$ in $\mathbb{R}^p$ with joint distribution function $F(\mathbf{x}) = \mathrm{P}(\mathbf{X} \leq \mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_p)$ and p-variate characteristic function $C(\mathbf{t}) = \mathrm{E}(e^{i\mathbf{t}^\top \mathbf{X}})$, $\mathbf{t} = (t_1, \ldots, t_p)$. Then, let $\mathcal{M} \in \mathbb{R}^{p\times p}$ be a symmetric matrix, such that $\mathcal{M}\mathcal{M} = I_p$ and consider testing the null hypothesis $\mathbb{H}_0^{\mathcal{M}} : \mathbf{X} \stackrel{d}{=} \mathcal{M}\mathbf{X}$ against $\mathbb{H}_1^{\mathcal{M}} : \mathbf{X} \stackrel{d}{\neq} \mathcal{M}\mathbf{X}$. When $p = 1$ and $\mathcal{M} = -1$, one recovers the univariate symmetry encountered in Section 3. In the case $p = 2$, the exchangeability and reflected symmetry hypotheses treated in Section 4 correspond respectively to:

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Letting $F_{\mathcal{M}}(\mathbf{x}) = \mathrm{P}(\mathcal{M}\mathbf{X} \leq \mathbf{x})$ and upon noting that $C_{\mathcal{M}}(\mathbf{t}) = \mathrm{E}(e^{i\mathbf{t}^\top \mathcal{M}\mathbf{X}}) = C(\mathcal{M}^\top \mathbf{t})$, the null hypothesis $\mathbb{H}_0^{\mathcal{M}} : \mathbf{X} \stackrel{d}{=} \mathcal{M}\mathbf{X}$ can be written equivalently as:

$$\begin{aligned}
&\mathbb{H}_0^{\mathcal{M}} : F(\mathbf{x}) = F_{\mathcal{M}}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^p;\\
&\mathbb{H}_0^{\mathcal{M}} : C(\mathbf{t}) = C(\mathcal{M}^\top \mathbf{t}) \quad \forall \mathbf{t} \in \mathbb{R}^p.
\end{aligned}$$

Table 3. Probability of the rejection of the null hypothesis of reflected symmetry, as estimated from 1000 replicates, for the tests based on W_n^{refl} and $V_n^{\mathrm{refl}}(\Omega)$ under the skew-normal distribution.

ϱ	δ	n	W_n^{refl}	$V_n^{\mathrm{refl}}(\Omega_1^1)$	$V_n^{\mathrm{refl}}(\Omega_2^1)$	$V_n^{\mathrm{refl}}(\Omega_1^2)$	$V_n^{\mathrm{refl}}(\Omega_2^2)$
1/3	0	50	8.3	7.5	7.1	8.6	7.1
		100	6.0	4.6	4.6	4.5	4.6
		200	3.3	5.4	4.8	3.6	4.9
	0.25	50	35.4	28.1	28.5	35.1	28.5
		100	65.1	55.0	55.9	63.3	55.8
		200	92.2	83.1	84.4	89.6	84.3
	0.5	50	94.5	89.9	91.1	94.6	90.9
		100	100.0	99.1	99.2	99.5	99.1
		200	100.0	100.0	100.0	100.0	100.0
2/3	0	50	6.7	4.8	5.4	6.2	5.4
		100	6.8	6.3	6.7	6.1	6.7
		200	5.3	4.3	4.1	4.2	4.1
	0.25	50	30.4	28.1	28.1	33.8	28.1
		100	58.7	47.5	48.2	57.3	48.2
		200	84.2	75.6	76.7	82.4	76.9
	0.5	50	88.9	83.3	84.2	89.7	84.1
		100	99.6	98.4	98.6	99.5	98.5
		200	100.0	100.0	100.0	100.0	100.0

From a sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of independent copies of $\mathbf{X}$, define the empirical versions of F and C respectively by:

$$F_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\left(\mathbf{X}_j \leq \mathbf{x}\right) \quad \text{and} \quad C_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^{n} e^{i\mathbf{t}^\top \mathbf{X}_j}$$

A Cramér–von Mises statistic based on the sample distribution function is:

$$W_n^{\mathcal{M}} = n\int_{\mathbb{R}^p} \left\{F_n(\mathbf{x}) - F_{n,\mathcal{M}}(\mathbf{x})\right\}^2 \mathrm{d}F_n(\mathbf{x})$$

where $F_{n,\mathcal{M}}$ is the distribution function of $\mathcal{M}\mathbf{X}_1, \ldots, \mathcal{M}\mathbf{X}_n$. Taking Ω to be a nonnegative integrable weight function defined on $\mathbb{R}^p$, a characteristic-function statistic is:

$$V_n^{\mathcal{M}}(\Omega) = n\int_{\mathbb{R}^p} \left|C_n(\mathbf{t}) - C_n(\mathcal{M}^\top \mathbf{t})\right|^2 \Omega(\mathbf{t})\,\mathrm{d}\mathbf{t}$$

From computations similar to those in Lemmas 3–5, one can show that:

$$W_n^{\mathcal{M}} = \frac{1}{n^2}\sum_{j,j',k=1}^{n} \phi^{\mathcal{M}}(\mathbf{X}_j, \mathbf{X}_{j'}, \mathbf{X}_k) \quad \text{and} \quad V_n^{\mathcal{M}}(\Omega) = \frac{1}{n}\sum_{j,j'=1}^{n} \psi_\Omega^{\mathcal{M}}(\mathbf{X}_j, \mathbf{X}_{j'})$$

where for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^p$,

$$\begin{aligned} \phi^{\mathcal{M}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \;=\;& \mathbb{I}\left\{\mathbf{x}_3 \geq \max(\mathbf{x}_1, \mathbf{x}_2)\right\} - \mathbb{I}\left\{\mathbf{x}_3 \geq \max(\mathbf{x}_1, \mathcal{M}\mathbf{x}_2)\right\} \\ & - \mathbb{I}\left\{\mathbf{x}_3 \geq \max(\mathcal{M}\mathbf{x}_1, \mathbf{x}_2)\right\} + \mathbb{I}\left\{\mathbf{x}_3 \geq \max(\mathcal{M}\mathbf{x}_1, \mathcal{M}\mathbf{x}_2)\right\} \end{aligned}$$

and for $\widetilde{\psi}_\Omega^{\mathcal{M}}(\mathbf{x}) = \int_{\mathbb{R}^p} \cos(\mathbf{t}^\top \mathbf{x})\,\Omega(\mathbf{t})\,\mathrm{d}\mathbf{t}$,

$$\psi_\Omega^{\mathcal{M}}(\mathbf{x}_1, \mathbf{x}_2) = \widetilde{\psi}_\Omega^{\mathcal{M}}(\mathbf{x}_1 - \mathbf{x}_2) - \widetilde{\psi}_\Omega^{\mathcal{M}}(\mathbf{x}_1 - \mathcal{M}\mathbf{x}_2) - \widetilde{\psi}_\Omega^{\mathcal{M}}(\mathcal{M}\mathbf{x}_1 - \mathbf{x}_2) + \widetilde{\psi}_\Omega^{\mathcal{M}}(\mathcal{M}\mathbf{x}_1 - \mathcal{M}\mathbf{x}_2)$$

Since $\phi^{\mathcal{M}}(\mathbf{x}_1, \mathcal{M}\mathbf{x}_2, \mathbf{x}_3) = -\phi^{\mathcal{M}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, it follows that $\mathrm{E}\{\phi^{\mathcal{M}}(\mathbf{x}_1, \mathbf{X}_2, \mathbf{x}_3)\} = 0$ under $\mathbb{H}_0^{\mathcal{M}} : \mathbf{X} \stackrel{d}{=} \mathcal{M}\mathbf{X}$. Since in addition, $\phi^{\mathcal{M}}$ is symmetric with respect to its first two components, the asymptotic distribution of $W_n^{\mathcal{M}}$ under the null hypothesis can be deduced from Proposition 2. One also has $\mathrm{E}\{\psi_\Omega^{\mathcal{M}}(\mathbf{x}_1, \mathbf{X}_2)\} = 0$, and then, $V_n^{\mathcal{M}}(\Omega)$ is a first-order degenerate V-statistic with bivariate kernel $\psi_\Omega^{\mathcal{M}}$ whose asymptotic distribution follows from Proposition 1. The multiplier versions of these statistics follow from the formulas in Equation (4).

To close this section, note that many symmetry hypotheses are related to a group of transformations rather than to a single transformation matrix $\mathcal{M}$. This situation has been considered by [34] from a distribution function point-of-view using a bootstrap method for the computation of p-values. In order to handle this case under the framework of the current paper, let $\mathcal{G}$ be a set of $p \times p$ symmetric matrices and consider the null hypothesis $\mathbb{H}_0^{\mathcal{G}} : \mathbf{X} \stackrel{d}{=} \mathcal{M}\mathbf{X}$ *for all* $\mathcal{M} \in \mathcal{G}$. For example, spherical symmetry corresponds to $\mathcal{G}$ being the set of all orthogonal transformations in $\mathbb{R}^p$, while multivariate exchangeability occurs when $\mathcal{G}$ is the set of all permutation matrices in $\mathbb{R}^p$.

The key here is to work with a combination matrix $\mathcal{L} \in \mathbb{R}^{q \times |\mathcal{G}|}$, such that for $\mathbf{z} \in \mathbb{R}^{|\mathcal{G}|}$, $\mathcal{L}\,\mathbf{z} = \mathbf{0}_q \in \mathbb{R}^q$ if and only if $\mathbf{z}$ is a constant vector. Then, define $\mathbf{F}_{\mathcal{G}} = (F_{\mathcal{M}_1}, \ldots, F_{\mathcal{M}_{|\mathcal{G}|}})$ and $\boldsymbol{C}_{\mathcal{G}} = (C_{\mathcal{M}_1}, \ldots, C_{\mathcal{M}_{|\mathcal{G}|}})$ and note that under the null hypothesis $\mathbb{H}_0^{\mathcal{G}}$, $\mathbf{F}_{\mathcal{G}}$ and $\boldsymbol{C}_{\mathcal{G}}$ are $|\mathcal{G}|$-dimensional vectors of identical functions in $\mathbb{R}^p$. With this in hand, the null hypothesis can be re-written either as $\mathbb{H}_0^{\mathcal{G}} : \mathcal{L}\,\mathbf{F}_{\mathcal{G}}(\mathbf{x}) = \mathbf{0}_q \;\; \forall \mathbf{x} \in \mathbb{R}^p$ or $\mathbb{H}_0^{\mathcal{G}} : \mathcal{L}\,\boldsymbol{C}_{\mathcal{G}}(\mathbf{t}) = \mathbf{0}_q \;\; \forall \mathbf{t} \in \mathbb{R}^p$. Hence, letting $\mathbf{F}_{n,\mathcal{G}} = (F_{n,\mathcal{M}_1}, \ldots, F_{n,\mathcal{M}_{|\mathcal{G}|}})$ and $\boldsymbol{C}_{n,\mathcal{G}} = (C_{n,\mathcal{M}_1}, \ldots, C_{n,\mathcal{M}_{|\mathcal{G}|}})$, with $C_{n,\mathcal{M}_j}(\mathbf{t}) = C_n(\mathcal{M}_j^\top \mathbf{t})$, test statistics are given by:

$$W_n^{\mathcal{G}} = n \int_{\mathbb{R}^p} \{\mathcal{L}\,\mathbf{F}_{n,\mathcal{G}}(\mathbf{x})\}^2 \,\mathrm{d}F_n(\mathbf{x}) \quad \text{and} \quad V_n^{\mathcal{G}}(\Omega) = n \int_{\mathbb{R}^p} |\mathcal{L}\,\boldsymbol{C}_{n,\mathcal{G}}(\mathbf{t})|^2 \,\Omega(\mathbf{t})\,\mathrm{d}\mathbf{t}$$

It can be shown that $W_n^{\mathcal{G}}$ is of the form required in Proposition 2, while $V_n^{\mathcal{G}}(\Omega)$ is a V-statistic with a bivariate kernel having a first-order degeneracy, hence falling under the requirements of Proposition 1.

Acknowledgments: The author gratefully acknowledges the valuable comments of two referees that led to an improved version of this manuscript. This research was supported by an individual grant from the Natural Sciences and Engineering Research Council of Canada (NSERC).

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Proofs of Proposition 2, Lemma 3, Lemma 4 and Lemma 5

Appendix A.1. Proof of Proposition 2

First, define the symmetric kernel:

$$\widetilde{\phi}(x_1, x_2, x_3) = \frac{\phi(x_1, x_2, x_3) + \phi(x_1, x_3, x_2) + \phi(x_2, x_3, x_1)}{3}$$

and note that:

$$W_n = \frac{1}{n^2} \sum_{j,j',k=1}^{n} \widetilde{\phi}\left(X_j, X_{j'}', X_k\right)$$

The fact that $\mathrm{E}\{\phi(x_1, X_2, x_3)\} = 0$ entails $\mathrm{E}\{\widetilde{\phi}(x_1, X_2, X_3)\} = 0$ for all $x_1 \in \mathbb{R}^p$, so W_n is a first-order degenerate V-statistic. In that case, it follows from Example 2, p. 185, in [2] and the H-decomposition of U-statistics (see, e.g., Section 3.3.2, p. 78 in [2]) that for $\widetilde{\Phi}(x_1, x_2) = 3\,\mathrm{E}\{\widetilde{\phi}(x_1, x_2, X_3)\}$,

$$W_n = \mathrm{E}\left\{\widetilde{\Phi}\left(X_1, X_1\right)\right\} + n\,U_n^{(2)} + o_{\mathrm{P}}(1)$$

where:

$$U_n^{(2)} = \binom{n}{2}^{-1} \sum_{j<j'=1}^{\infty} \widetilde{\Phi}\left(X_j, X_{j'}\right)$$

The representation of W_n in Equation (3) follows by noting that:

$$\begin{aligned}
\widetilde{\Phi}(x_1, x_2) &= \mathrm{E}\left\{\phi(x_1, x_2, X_3) + \phi(x_1, X_3, x_2) + \phi(x_2, X_3, x_1)\right\} \\
&= \mathrm{E}\left\{\phi(x_1, x_2, X_3)\right\} \\
&= \Phi(x_1, x_2)
\end{aligned}$$

Finally, the asymptotic representation of W_n is a consequence of Proposition 1 with $\psi = \Phi$.

Appendix A.2. Proof of Lemma 3

First, note that:

$$V_n^{\text{univ}} = 2\,n \int_0^\infty \left\{F_n(-x) + F_n(x^-) - 1\right\}^2 \mathrm{d}x$$

When $x > 0$, one has $\mathbb{I}(X_j \leq -x) + \mathbb{I}(X_j < x) - 1 = -\operatorname{sign}(X_j)\,\mathbb{I}(x \leq |X_j|)$, so that:

$$\begin{aligned}
\left\{F_n(-x) + F_n(x^-) - 1\right\}^2 &= \left\{-\frac{1}{n}\sum_{j=1}^{n} \operatorname{sign}(X_j)\,\mathbb{I}(x \leq |X_j|)\right\}^2 \\
&= \frac{1}{n^2}\sum_{j,j'=1}^{n} \operatorname{sign}(X_j)\operatorname{sign}(X_{j'})\,\mathbb{I}\left\{x \leq \min(|X_j|, |X_{j'}|)\right\}
\end{aligned}$$

It then follows that:

$$V_n^{\text{univ}} = \frac{1}{n}\sum_{j,j'=1}^{n} 2\operatorname{sign}(X_j)\operatorname{sign}(X_{j'})\ \min\left(|X_j|, |X_{j'}|\right) = \frac{1}{n}\sum_{j,j'=1}^{n} \psi^{\text{univ}}\left(X_j, X_{j'}\right)$$

Upon noting that $\mathrm{d}F_n(x)$ puts mass $1/n$ at $X_1, \ldots, X_n$, one also has:

$$\begin{aligned}
W_n^{\text{univ}} &= \sum_{k=1}^{n} \left\{F_n(-X_k) + F_n(X_k^-) - 1\right\}^2 \\
&= \sum_{k=1}^{n} \left[\frac{1}{n}\sum_{j=1}^{n} \left\{\mathbb{I}(X_j \leq -X_k) + \mathbb{I}(X_j < X_k) - 1\right\}\right]^2 \\
&= \sum_{k=1}^{n} \left[\frac{1}{n}\sum_{j=1}^{n} \left\{\mathbb{I}(X_k \leq -X_j) - \mathbb{I}(X_k \leq X_j)\right\}\right]^2 \\
&= \frac{1}{n^2}\sum_{j,j',k=1}^{n} \left\{\mathbb{I}(X_k \leq -X_j) - \mathbb{I}(X_k \leq X_j)\right\}\left\{\mathbb{I}(X_k \leq -X_{j'}) - \mathbb{I}(X_k \leq X_{j'})\right\} \\
&= \frac{1}{n^2}\sum_{j,j',k=1}^{n} \Big[\mathbb{I}\left\{X_k \leq \min(-X_j, -X_{j'})\right\} - \mathbb{I}\left\{X_k \leq \min(X_j, -X_{j'})\right\} \\
&\qquad - \mathbb{I}\left\{X_k \leq \min(-X_j, X_{j'})\right\} + \mathbb{I}\left\{X_k \leq \min(X_j, X_{j'})\right\}\Big] \\
&= \frac{1}{n^2}\sum_{j,j',k=1}^{n} \phi^{\text{univ}}(X_j, X_j', X_k)
\end{aligned}$$

For $V_n^{\text{univ}}(\omega)$, the fact that:

$$c_n(t) - c_n(-t) = i\left\{\frac{2}{n}\sum_{j=1}^{n}\sin(tX_j)\right\}$$

entails:

$$|c_n(t) - c_n(-t)|^2 = \frac{4}{n^2}\left\{\sum_{j=1}^{n}\sin(tX_j)\right\}^2 = \frac{4}{n^2}\sum_{j,j'=1}^{n}\sin(tX_j)\,\sin(tX_{j'})$$

Integrating this last expression with respect to ω yields:

$$V_n^{\text{univ}}(\omega) = \frac{1}{n}\sum_{j,j'=1}^{n} 4\int_{\mathbb{R}}\sin(tX_j)\,\sin(tX_{j'})\,\omega(t)\,\mathrm{d}t = \frac{1}{n}\sum_{j,j'=1}^{n}\psi_{\omega}^{\text{univ}}(X_j, X_{j'})$$

Appendix A.3. Proof of Lemma 4

For the test statistic W_n^{exch}, one has:

$$H_n(X_k, Y_k) - H_n(Y_k, X_k) = \frac{1}{n}\sum_{j=1}^{n}\left\{\mathbb{I}\left(X_k \geq X_j, Y_k \geq Y_j\right) - \mathbb{I}\left(X_k \geq Y_j, Y_k \geq X_j\right)\right\}$$

so that:

$$\begin{aligned}\{H_n(X_k, Y_k) - H_n(Y_k, X_k)\}^2 &= \frac{1}{n^2}\sum_{j,j'=1}^{n}\mathbb{I}\left\{X_k \geq \max(X_j, X_{j'}), Y_k \geq \max(Y_j, Y_{j'})\right\}\\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n}\mathbb{I}\left\{X_k \geq \max(X_j, Y_{j'}), Y_k \geq \max(Y_j, X_{j'})\right\}\\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n}\mathbb{I}\left\{X_k \geq \max(Y_j, X_{j'}), Y_k \geq \max(X_j, Y_{j'})\right\}\\ &\quad + \frac{1}{n^2}\sum_{j,j'=1}^{n}\mathbb{I}\left\{X_k \geq \max(Y_j, Y_{j'}), Y_k \geq \max(X_j, X_{j'})\right\}\end{aligned}$$

It is then a straightforward exercise to show that:

$$W_n^{\text{exch}} = \sum_{k=1}^{n}\{H_n(X_k, Y_k) - H_n(Y_k, X_k)\}^2 = \frac{1}{n^2}\sum_{j,j',k=1}^{n}\phi^{\text{exch}}\left\{(X_j, Y_j), (X_{j'}, Y_{j'}), (X_k, Y_k)\right\}$$

For the test statistic $V_n^{\text{exch}}(\Omega)$, first note that:

$$\begin{aligned}C_n(s,t) - C_n(t,s) &= \frac{1}{n}\sum_{j=1}^{n}\{\cos(s\,X_j + t\,Y_j) - \cos(t\,X_j + s\,Y_j)\}\\ &\quad + i\,\frac{1}{n}\sum_{j=1}^{n}\{\sin(s\,X_j + t\,Y_j) - \sin(t\,X_j + s\,Y_j)\}\end{aligned}$$

Hence,

$$\begin{aligned}|C_n(s,t) - C_n(t,s)|^2 &= \frac{1}{n^2}\sum_{j,j'=1}^{n}\{\cos(s\,X_j + t\,Y_j) - \cos(t\,X_j + s\,Y_j)\}\left\{\cos(s\,X_{j'} + t\,Y_{j'}) - \cos(t\,X_{j'} + s\,Y_{j'})\right\}\\ &\quad + \frac{1}{n^2}\sum_{j,j'=1}^{n}\{\sin(s\,X_j + t\,Y_j) - \sin(t\,X_j + s\,Y_j)\}\left\{\sin(s\,X_{j'} + t\,Y_{j'}) - \sin(t\,X_{j'} + s\,Y_{j'})\right\}\end{aligned}$$

Using the trigonometric identity $\cos a \cos b + \sin a \sin b = \cos(a-b)$, one obtains after straightforward computations that:

$$\begin{aligned}|C_n(s,t) - C_n(t,s)|^2 &= \frac{1}{n^2}\sum_{j,j'=1}^{n} \cos\left\{s(X_j - X_{j'}) + t(Y_j - Y_{j'})\right\} \\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n} \cos\left\{s(X_j - Y_{j'}) + t(Y_j - X_{j'})\right\} \\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n} \cos\left\{s(Y_j - X_{j'}) + t(X_j - Y_{j'})\right\} \\ &\quad + \frac{1}{n^2}\sum_{j,j'=1}^{n} \cos\left\{s(Y_j - Y_{j'}) + t(X_j - X_{j'})\right\}\end{aligned}$$

Integrating this last expression with respect to $\Omega(s,t)$ yields:

$$V_n^{\mathrm{exch}}(\Omega) = \frac{1}{n}\sum_{j,j'=1}^{n} \psi_\Omega^{\mathrm{exch}}\left\{(X_j, Y_j), (X_{j'}, Y_{j'})\right\}$$

Appendix A.4. Proof of Lemma 5

Proceeding as in the proof of Lemma 4, note that:

$$\begin{aligned}\{H_n(X_k, Y_k) - \bar{H}_n(-X_k, -Y_k)\}^2 &= \left[\frac{1}{n}\sum_{j=1}^{n}\{\mathbb{I}(X_k \geq X_j, Y_k \geq Y_j) - \mathbb{I}(X_k \geq -X_j, Y_k \geq -Y_j)\}\right]^2 \\ &= \frac{1}{n^2}\sum_{j,j'=1}^{n} \mathbb{I}\left\{X_k \geq \max(X_j, X_{j'}), Y_k \geq \max(Y_j, Y_{j'})\right\} \\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n} \mathbb{I}\left\{X_k \geq \max(X_j, -X_{j'}), Y_k \geq \max(Y_j, -Y_{j'})\right\} \\ &\quad - \frac{1}{n^2}\sum_{j,j'=1}^{n} \mathbb{I}\left\{X_k \geq \max(-X_j, X_{j'}), Y_k \geq \max(-Y_j, Y_{j'})\right\} \\ &\quad + \frac{1}{n^2}\sum_{j,j'=1}^{n} \mathbb{I}\left\{X_k \geq \max(-X_j, -X_{j'}), Y_k \geq \max(-Y_j, -Y_{j'})\right\}\end{aligned}$$

It then follows that:

$$W_n^{\mathrm{refl}} = \sum_{k=1}^{n}\{H_n(X_k, Y_k) - \bar{H}_n(-X_k, -Y_k)\}^2 = \frac{1}{n^2}\sum_{j,j',k=1}^{n} \phi^{\mathrm{refl}}\left\{(X_j, Y_j), (X_{j'}, Y_{j'}), (X_k, Y_k)\right\}$$

For $V_n^{\mathrm{refl}}(\Omega)$, the fact that:

$$C_n(s,t) - C_n(-s,-t) = i\left\{\frac{2}{n}\sum_{j=1}^{n}\sin\left(sX_j + tY_j\right)\right\}$$

entails:

$$|C_n(s,t) - C_n(-s,-t)|^2 = \frac{4}{n^2}\sum_{j,j'=1}^{n}\sin\left(sX_j + tY_j\right)\,\sin\left(sX_{j'} + tY_{j'}\right)$$

Integrating this expression with respect to $\Omega(s,t)$ yields:

$$V_n^{\text{refl}}(\Omega) = \frac{1}{n} \sum_{j,j'=1}^{n} \psi_\Omega^{\text{refl}} \left\{ (X_j, Y_j), (X_{j'}, Y_{j'}) \right\}$$

References

1. Hušková, M. Hypothesis of symmetry. In *Nonparametric Methods Volume 4 of Handbook of Statistic*; North-Holland: Amsterdam, The Netherland, 1984; pp. 63–78.
2. Lee, A.J. *U-Statistics: Theory and Practice. Statistics: Textbooks and Monographs 110*; Marcel Dekker Inc.: New York, NY, USA, 1990.
3. Koroljuk, V.S.; Borovskich, Y.V. *Theory of U-statistics*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1994.
4. Dehling, H.; Mikosch, T. Random quadratic forms and the bootstrap for *U*-statistics. *J. Multivariate Anal.* **1994**, *51*, 392–413.
5. Hill, D.L.; Rao, P.V. Tests of symmetry based on Cramér-von Mises statistics. *Biometrika* **1977**, *64*, 489–494.
6. Bhattacharya, P.K.; Gastwirth, J.L.; Wright, A.L. Two modified Wilcoxon tests for symmetry about an unknown location parameter. *Biometrika* **1982**, *69*, 377–382.
7. Aki, S. On nonparametric tests for symmetry. *Ann. Inst. Stat. Math.* **1987**, *39*, 457–472.
8. Henze, N.; Klar, B.; Meintanis, S.G. Invariant tests for symmetry about an unspecified point based on the empirical characteristic function. *J. Multivar. Anal.* **2003**, *87*, 275–297.
9. Ngatchou-Wandji, J. Testing for symmetry in multivariate distributions. *Stat. Methodol.* **2009**, *6*, 230–250.
10. Ahmad, I.A.; Li, Q. Testing symmetry of an unknown density function by kernel method. *J. Nonparametr. Stat.* **1997**, *7*, 279–293.
11. Henderson, D.J.; Parmeter, C.F. A consistent bootstrap procedure for nonparametric symmetry tests. *Econom. Lett.* **2015**, *131*, 78–82.
12. Fang, Y.; Li, Q.; Wu, X.;Zhang, D. A data-driven smooth test of symmetry. *J. Econom.* **2015**, *188*, 490–501.
13. Rothman, E.D.; Woodroofe, M. A Cramér-von Mises type statistic for testing symmetry. *Ann. Math. Stat.* **1972**, *43*, 2035–2038.
14. Orlov, A.I. Testing the symmetry of a distribution. *Teor. Verojatnost. i Primemen.* **1972**, *17*, 372–377.
15. Bell, C.B.; Haller, H.S. Bivariate symmetry tests: Parametric and nonparametric. *Ann. Math. Stat.* **1969**, *40*, 259–269.
16. Hollander, M. A nonparametric test for bivariate symmetry. *Biometrika* **1971**, *58*, 203–212.
17. Hilton, J.F. A new asymptotic distribution for Hollander's bivariate symmetry statistic. *Comput. Stat. Data Anal.* **2000**, *32*, 455–463.
18. Koziol, J.A. A test for bivariate symmetry based on the empirical distribution function. *Comm. Stat. Theory Methods* **1979**, *8*, 207–221.
19. Kepner, J.L.; Randles, R.H. Comparison of tests for bivariate symmetry versus location and/or scale alternatives. *Comm. Stat. Theory Methods* **1984**, *13*, 915–930.
20. Heathcote, C.R.; Rachev, S.T.; Cheng, B. Testing multivariate symmetry. *J. Multivariate Anal.* **1995**, *54*, 91–112.
21. Hilton, J.F.; Gee, L. The size and power of the exact bivariate symmetry test. *Comput. Stat. Data Anal.* **1997**, *26*, 53–69.
22. Sklar, A. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Stat. Univ. Paris* **1959**, *8*, 229–231.
23. Nelsen, R.B. *An introduction to copulas*. Springer: New York, NY, USA, 2006.
24. Genest, C.; Nešlehová, J.; Quessy, J.-F. Tests of symmetry for bivariate copulas. *Ann. Inst. Stat. Math.* **2012**, *64*, 811–834.
25. Genest,C.; Nešlehová, J. G. On tests of radial symmetry for bivariate copulas. *Statistical Papers* **2014**, *55*, 1107–1119.
26. Kosorok, M. *Introduction to empirical processes and semiparametric inference*. Springer: New York, NY, USA, 2008.

27. Azzalini, A. Further results on a class of distributions which includes the normal ones. *Statistica* **1986**, *46*, 199–208.
28. Azzalini, A. A class of distributions which includes the normal ones. *Scand. J. Stat.* **1985**, *12*, 171–178.
29. Luc, D. *Nonuniform random variate generation*. Springer-Verlag: New York, NY, USA, 1986.
30. Khoudraji, A. *Contributions à l'étude des copules et à la modélisation de valeurs extrêmes bivariées*. Ph.D. Thesis, Université Laval, Quebec City, QC, Canada, 1995.
31. Nelsen, R.B. Extremes of nonexchangeability. *Stat. Papers* **2007**, *48*, 329–336.
32. Quessy, J.-F. A general framework for testing homogeneity hypotheses about copulas. *Electron. J. Stat.* **2016**, *10*, 1064–1097.
33. Azzalini, A. The skew-normal distribution and related multivariate families. *Scand. J. Stat.* **2005**, *32*, 159–188.
34. Sakhanenko, L. Testing group symmetry of a multivariate distribution. *Symmetry* **2009**, *1*, 180–200.