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On Solutions for Linear and Nonlinear Schrödinger Equations with Variable Coefficients: A Computational Approach

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Academic Editor: Young Suh Kim

Received: 2 March 2016; Accepted: 6 May 2016; Published: 28 May 2016

Abstract: In this work, after reviewing two different ways to solve Riccati systems, we are able to present an extensive list of families of integrable nonlinear Schrödinger (NLS) equations with variable coefficients. Using Riccati equations and similarity transformations, we are able to reduce them to the standard NLS models. Consequently, we can construct bright-, dark- and Peregrine-type soliton solutions for NLS with variable coefficients. As an important application of solutions for the Riccati equation with parameters, by means of computer algebra systems, it is shown that the parameters change the dynamics of the solutions. Finally, we test numerical approximations for the inhomogeneous paraxial wave equation by the Crank-Nicolson scheme with analytical solutions found using Riccati systems. These solutions include oscillating laser beams and Laguerre and Gaussian beams.

Keywords: generalized harmonic oscillator; paraxial wave equation; nonlinear schrödinger-type equations; riccati systems; solitons

PACS: J0101

1. Introduction

In modern nonlinear sciences, some of the most important models are the variable coefficient nonlinear Schrödinger-type ones. Applications include long distance optical communications, optical fibers and plasma physics, (see [1–25] and references therein).

In this paper, we first review a generalized pseudoconformal transformation introduced in [26] (lens transform in optics [27] see also [28]). As the first main result, we will use this generalized lens transformation to construct solutions of the general variable coefficient nonlinear Schrödinger equation (VCNLS):

$$i\psi_t = -a(t)\psi_{xx} + (b(t)x^2 - f(t)x + G(t))\psi - ic(t)x\psi_x - id(t)\psi + ig(t)\psi_x + h(t)|\psi|^{2s}\psi, \quad (1)$$

extending the results in [1]. If we make $a(t) = \Lambda/4\pi n_0$, Λ being the wavelength of the optical source generating the beam, and choose $c(t) = g(t) = 0$, then Equation (1) models a beam propagation inside of a planar graded-index nonlinear waveguide amplifier with quadratic refractive index represented by

$b(t)x^2 - f(t)x + G(t)$, and $h(t)$ represents a Kerr-type nonlinearity of the waveguide amplifier, while $d(t)$ represents the gain coefficient. If $b(t) > 0$ [11] (resp. $b(t) < 0$, see [13]) in the low-intensity limit, the graded-index waveguide acts as a linear defocusing (focusing) lens.

Depending on the selections of the coefficients in Equation (1), its applications vary in very specific problems (see [16] and references therein):

- Bose-Einstein condensates: $b(\cdot) \neq 0$, a, h constants and other coefficients are zero.
- Dispersion-managed optical fibers and soliton lasers [9], [14] and [15]: $a(\cdot)$, $h(\cdot)$, $d(\cdot) \neq 0$ are respectively dispersion, nonlinearity and amplification, and the other coefficients are zero. $a(\cdot)$ and $h(\cdot)$ can be periodic as well, see [29].
- Pulse dynamics in the dispersion-managed fibers [10]: $h(\cdot) \neq 0$, a is a constant and other coefficients are zero.

In this paper, to obtain the main results, we use a fundamental approach consisting of the use of similarity transformations and the solutions of Riccati systems with several parameters inspired by the work in [30]. Similarity trasformations have been a very popular strategy in nonlinear optics since the lens transform presented by Talanov [27]. Extensions of this approach have been presented in [26] and [28]. Applications include nonlinear optics, Bose-Einstein condensates, integrability of NLS and quantum mechanics, see for example [3], [31], [32] and [33], and references therein. E. Marhic in 1978 introduced (probably for the first time) a one-parameter $\{\alpha(0)\}$ family of solutions for the linear Schrödinger equation of the one-dimensional harmonic oscillator, where the use of an explicit formulation (classical Melher's formula [34]) for the propagator was fundamental. The solutions presented by E. Marhic constituted a generalization of the original Schrödinger wave packet with oscillating width.

In addition, in [35], a generalized Melher's formula for a general linear Schrödinger equation of the one-dimensional generalized harmonic oscillator of the form Equation (1) with $h(t) = 0$ was presented. For the latter case, in [36], [37] and [38], multiparameter solutions in the spirit of Marhic in [30] have been presented. The parameters for the Riccati system arose originally in the process of proving convergence to the initial data for the Cauchy initial value problem Equation (1) with $h(t) = 0$ and in the process of finding a general solution of a Riccati system [38] and [39]. In addition, Ermakov systems with solutions containing parameters [36] have been used successfully to construct solutions for the generalized harmonic oscillator with a hidden symmetry [37], and they have also been used to present Galilei transformation, pseudoconformal transformation and others in a unified manner, see [37]. More recently, they have been used in [40] to show spiral and breathing solutions and solutions with bending for the paraxial wave equation. In this paper, as the second main result, we introduce a family of Schrödinger equations presenting periodic soliton solutions by using multiparameter solutions for Riccati systems. Furthermore, as the third main result, we show that these parameters provide a control on the dynamics of solutions for equations of the form Equation (1). These results should deserve numerical and experimental studies.

This paper is organized as follows: In Section 2, by means of similarity transformations and using computer algebra systems, we show the existence of Peregrine, bright and dark solitons for the family Equation (1). Thanks to the computer algebra systems, we are able to find an extensive list of integrable VCNLS, in the sense that they can be reduced to the standard integrable NLS, see Table 1. In Section 3, we use different similarity transformations than those used in Section 3. The advantage of the presentation of this section is a multiparameter approach. These parameters provide us a control on the center axis of bright and dark soliton solutions. Again in this section, using Table 2 and by means of computer algebra systems, we show that we can produce a very extensive number of integrable VCNLS allowing soliton-type solutions. A supplementary Mathematica file is provided where it is evident how the variation of the parameters change the dynamics of the soliton solutions. In Section 4, we use a finite difference method to compare analytical solutions described in [41] (using similarity transformations) with numerical approximations for the paraxial wave equation (also known as linear Schrödinger equation with quadratic potential).

Table 1. Families of NLS with variable coefficients.

#	Variable Coefficient NLS	Solutions ($j = 1, 2, 3$)
1	$i\psi_t = l_0\psi_{xx} - \frac{bmt^{m-1}+b^2t^{2m}}{4l_0}x^2\psi$ $-ibt^m x\psi_x - \lambda l_0 e^{\frac{-bt^{m+1}}{m+1}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{-bt^{m+1}}{m+1}}}}e^{i\left(\frac{bt^m}{4}l_0x^2\right)}u_j(x,t)$
2	$i\psi_t = l_0\psi_{xx} - \frac{t^{-2}}{2l_0}x^2\psi$ $+i\frac{1}{t}x\psi_x - \lambda l_0 t\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{t}}e^{i\left(\frac{-1}{4t}l_0x^2\right)}u_j(x,t)$
3	$i\psi_t = l_0\psi_{xx} - \left(\frac{c^2}{4}l_0\right)x^2\psi$ $+icx\psi_x - \lambda l_0 e^{ct}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{ct}}}e^{i\left(\frac{-c}{4}l_0x^2\right)}u_j(x,t)$
4	$i\psi_t = l_0\psi_{xx} - \frac{b^2}{4l_0}t^kx^2\psi$ $+ibx\psi_x - \lambda l_0 e^{bt}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{bt}}}e^{i\left(\frac{-b}{4}l_0x^2\right)}u_j(x,t)$
5	$i\psi_t = l_0\psi_{xx} - \frac{abe^{bt}+a^2e^{2bt}}{4l_0}x^2\psi$ $-iae^{bt}x\psi_x - \lambda l_0 e^{\frac{a-ae^{bt}}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{a-ae^{bt}}{b}}}}e^{i\left(\frac{ae^{bt}}{4}l_0x^2\right)}u_j(x,t)$
6	$i\psi_t = l_0\psi_{xx} - \frac{1}{4l_0}x^2\psi$ $-icoth(t)x\psi_x - \lambda l_0 csch(t)\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{csch(t)}}e^{i\left(\frac{coth(t)}{4}l_0x^2\right)}u_j(x,t)$
7	$i\psi_t = l_0\psi_{xx} - \frac{1}{4l_0}x^2\psi$ $-itan(t)x\psi_x - \lambda l_0 cos(t)\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{cos(t)}}e^{i\left(\frac{tan(t)}{4}l_0x^2\right)}u_j(x,t)$
8	$i\psi_t = l_0\psi_{xx} - \frac{bt^{-1}+b^2ln^2(t)}{4l_0}x^2\psi$ $-ibln(t)x\psi_x - \lambda l_0 t^{-bt}e^{bt}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{-t^{-bt}e^{bt}}}e^{i\left(\frac{bln(t)}{4}l_0x^2\right)}u_j(x,t)$
9	$i\psi_t = l_0\psi_{xx} + \frac{1}{4l_0}x^2\psi + icot(-t)x\psi_x$ $-\lambda l_0 csc(t)\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{csc(t)}}e^{i\left(\frac{-cot(-t)}{4}l_0x^2\right)}u_j(x,t)$
10	$i\psi_t = l_0\psi_{xx} + \frac{1}{4l_0}x^2\psi - itan(-t)x\psi_x$ $-\lambda l_0 sec(t)\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{sec(t)}}e^{i\left(\frac{tan(-t)}{4}l_0x^2\right)}u_j(x,t)$
11	$i\psi_t = l_0\psi_{xx} - \frac{2abte^{bt^2}+a^2e^{2bt^2}}{4l_0}x^2\psi$ $-iae^{bt^2}x\psi_x - \lambda l_0 e^{\frac{-a}{2}\sqrt{\frac{\pi}{b}}erfi(\sqrt{b}t)}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{-a}{2}\sqrt{\frac{\pi}{b}}erfi(\sqrt{b}t)}}}e^{\frac{aebt^2}{4}l_0x^2}u_j(x,t)$
12	$i\psi_t = l_0\psi_{xx} + \frac{atanh^2(bt)(b-a)-ab}{4l_0}x^2\psi$ $-iatanh(bt)x\psi_x - \lambda l_0\|cosh(bt)\|^{\frac{a}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{\|cosh(bt)\|^{\frac{a}{b}}}}e^{i\left(\frac{atanh(bt)}{4}l_0x^2\right)}u_j(x,t)$
13	$i\psi_t = l_0\psi_{xx} + \frac{acoth^2(bt)(b-a)-ab}{4l_0}x^2\psi$ $-iacoth(bt)x\psi_x - \lambda l_0\|sinh(bt)\|^{\frac{a}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{\|sinh(bt)\|^{\frac{a}{b}}}}e^{i\left(\frac{acoth(bt)}{4}l_0x^2\right)}u_j(x,t)$
14	$i\psi_t = l_0\psi_{xx} - \left(\frac{a^2+absinh(bt)+a^2sinh^2(bt)}{4l_0}\right)x^2\psi$ $-iacosh(bt)x\psi_x - \lambda l_0 e^{\frac{-asinh(bt)}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{-asinh(bt)}{b}}}}e^{i\left(\frac{acosh(bt)}{4}l_0x^2\right)}u_j(x,t)$
15	$i\psi_t = l_0\psi_{xx} - \left(\frac{a^2+absin(bt)-a^2sin^2(bt)}{4l_0}\right)x^2\psi$ $+iacos(bt)x\psi_x - \lambda l_0 e^{\frac{asin(bt)}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{asin(bt)}{b}}}}e^{i\left(\frac{-acos(bt)}{4}l_0x^2\right)}u_j(x,t)$
16	$i\psi_t = l_0\psi_{xx} - \left(\frac{a^2+abcos(bt)-a^2cos^2(bt)}{4l_0}\right)x^2\psi$ $-iasin(bt)x\psi_x + \lambda l_0 e^{\frac{acos(bt)}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{e^{\frac{acos(bt)}{b}}}}e^{i\left(\frac{-asin(bt)}{4}l_0x^2\right)}u_j(x,t)$
17	$i\psi_t = l_0\psi_{xx} - \frac{atan^2(bt)(a+b)+ab}{4l_0}x^2\psi$ $-iatan(bt)x\psi_x - \lambda l_0\|cos(bt)\|^{\frac{a}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{\|cos(bt)\|^{\frac{a}{b}}}}e^{i\left(\frac{atan(bt)}{4}l_0x^2\right)}u_j(x,t)$
18	$i\psi_t = l_0\psi_{xx} - \frac{acot^2(bt)(a+b)+ab}{4l_0}x^2\psi$ $+iacot(bt)x\psi_x - \lambda l_0\|sin(bt)\|^{\frac{a}{b}}\|\psi\|^2\psi$	$\psi_j(x,t) = \frac{1}{\sqrt{\|sin(bt)\|^{\frac{a}{b}}}}e^{i\left(\frac{acot(bt)}{4}l_0x^2\right)}u_j(x,t)$

Table 2. Riccati equations used to generate the similarity transformations.

#	Riccati Equation	Similarity Transformation from Table 1
1	$y'_x = ax^n y^2 + bmx^{m-1} - ab^2x^{n+2m}$	1
2	$(ax^n + b)y'_x = by^2 + ax^{n-2}$	2
3	$y'_x = ax^n y^2 + bx^m y + bcx^m - ac^2x^n$	3
4	$y'_x = ax^n y^2 + bx^m y + ckx^{k-1} - bcx^{m+k} - ac^2x^{n+2k}$	1
5	$xy'_x = ax^n y^2 + my - ab^2x^{n+2m}$	3
6	$(ax^n + bx^m + c)y'_x = \alpha x^k y^2 + \beta x^s y - \alpha b^2 x^k + \beta b x^s$	4
7	$y'_x = be^{\mu x} y^2 + ace^{cx} - a^2 be^{(\mu+2c)x}$	5
8	$y'_x = ae^{\mu x} y^2 + cy - ab^2 e^{(\mu+2c)x}$	3
9	$y'_x = ae^{cx} y^2 + bnx^{n-1} - ab^2 e^{cx} x^{2n}$	1
10	$y'_x = ax^n y^2 + bce^{cx} - ab^2 x^n e^{2cx}$	8
11	$y'_x = ax^n y^2 + cy - ab^2 x^n e^{2cx}$	3
12	$y'_x = \left[a\sinh^2(cx) - c\right] y^2 - a\sinh^2(cx) + c - a$	6
13	$2y'_x = [a - b + a\cosh(bx)]\, y^2 + a + b - a\cosh(bx)$	7
14	$y'_x = a(\ln x)^n y^2 + bmx^{m-1} - ab^2 x^{2m} (\ln x)^n$	1
15	$xy'_x = ax^n y^2 + b - ab^2 x^n \ln^2 x$	8
16	$y'_x = \left[b + a\sin^2(bx)\right] y^2 + b - a + a\sin^2(bx)$	9
17	$2y'_x = [b + a + a\cos(bx)]\, y^2 + b - a + a\cos(bx)$	10
18	$y'_x = \left[b + a\cos^2(bx)\right] y^2 + b - a + a\cos^2(bx)$	10
19	$y'_x = c(\arcsin x)^n y^2 + ay + ab - b^2 c(\arcsin x)^n$	3
20	$y'_x = a(\arcsin x)^n y^2 + \beta mx^{m-1} - a\beta^2 x^{2m} (\arcsin x)^n$	1
21	$y'_x = c(\arccos x)^n y^2 + ay + ab - b^2 c(\arccos x)^n$	3
22	$y'_x = a(\arccos x)^n y^2 + \beta mx^{m-1} - a\beta^2 x^{2m} (\arccos x)^n$	1
23	$y'_x = c(\arctan x)^n y^2 + ay + ab - b^2 c(\arctan x)^n$	3
24	$y'_x = a(\arctan x)^n y^2 + bmx^{m-1} - ab^2 x^{2m} (\arctan x)^n$	1
25	$y'_x = c(\operatorname{arccot} x)^n y^2 + ay + ab - b^2 c(\operatorname{arccot} x)^n$	3
26	$y'_x = a(\operatorname{arccot} x)^n y^2 + bmx^{m-1} - ab^2 x^{2m} (\operatorname{arccot} x)^n$	1
27	$y'_x = fy^2 + ay - ab - b^2 f$	3
28	$y'_x = fy^2 + anx^{n-1} - a^2 x^{2n} f$	1
29	$y'_x = fy^2 + gy - a^2 f - ag$	3
30	$y'_x = fy^2 + gy + anx^{n-1} - ax^n g - a^2 f x^{2n}$	1
31	$y'_x = fy^2 - ax^n gy + anx^{n-1} - a^2 x^{2n} (g - f)$	1
32	$y'_x = fy^2 + abe^{bx} - a^2 e^{2bx} f$	5
33	$y'_x = fy^2 + gy + abe^{bx} - ae^{bx} g - a^2 e^{2bx} f$	5
34	$y'_x = fy^2 - ae^{bx} gy + abe^{bx} + a^2 e^{2bx} (g - f)$	5
35	$y'_x = fy^2 + 2abxe^{bx^2} - a^2 fe^{2bx^2}$	11
36	$y'_x = fy^2 - a\tanh^2(bx)(af + b) + ab$	12
37	$y'_x = fy^2 - a\coth^2(bx)(af + b) + ab$	13
38	$y'_x = fy^2 - a^2 f + ab\sinh(bx) - a^2 f\sinh^2(bx)$	14
39	$y'_x = fy^2 - a^2 f + ab\sin(bx) + a^2 f\sin^2(bx)$	15
40	$y'_x = fy^2 - a^2 f + ab\cos(bx) + a^2 f\cos^2(bx)$	16
41	$y'_x = fy^2 - a\tan^2(bx)(af - b) + ab$	17
42	$y'_x = fy^2 - a\cot^2(bx)(af - b) + ab$	18

2. Soliton Solutions for VCNLS through Riccati Equations and Similarity Transformations

In this section, by means of a similarity transformation introduced in [42], and using computer algebra systems, we show the existence of Peregrine, bright and dark solitons for the family Equation (1). Thanks to the computer algebra systems, we are able to find an extensive list of integrable variable coefficient nonlinear Schrödinger equations (see Table 1). For similar work and applications to Bose-Einstein condensates, we refer the reader to [1]

Lemma 1. *([42]) Suppose that $h(t) = -l_0\lambda\mu(t)$ with $\lambda \in \mathbb{R}$, $l_0 = \pm 1$ and that $c(t)$, $\alpha(t)$, $\delta(t)$, $\kappa(t)$, $\mu(t)$ and $g(t)$ satisfy the equations:*

$$\alpha(t) = l_0\frac{c(t)}{4}, \quad \delta(t) = -l_0\frac{g(t)}{2}, \quad h(t) = -l_0\lambda\mu(t), \tag{2}$$

$$\kappa(t) = \kappa(0) - \frac{l_0}{4}\int_0^t g^2(z)dz, \tag{3}$$

$$\mu(t) = \mu(0)exp\left(\int_0^t (2d(z) - c(z))dz\right) \quad \mu(0) \neq 0, \tag{4}$$

$$g(t) = g(0) - 2l_0 exp\left(-\int_0^t c(z)dz\right)\int_0^t exp\left(\int_0^z c(y)dy\right)f(z)dz. \tag{5}$$

Then,

$$\psi(t,x) = \frac{1}{\sqrt{\mu(t)}}e^{i(\alpha(t)x^2+\delta(t)x+\kappa(t))}u(t,x) \tag{6}$$

is a solution to the Cauchy problem for the nonautonomous Schrödinger equation

$$i\psi_t - l_0\psi_{xx} - b(t)x^2\psi + ic(t)x\psi_x + id(t)\psi + f(t)x\psi - ig(t)\psi_x - h(t)|\psi|^2\psi = 0, \tag{7}$$

$$\psi(0,x) = \psi_0(x), \tag{8}$$

if and only if $u(t,x)$ is a solution of the Cauchy problem for the standard Schrödinger equation

$$iu_t - l_0u_{xx} + l_0\lambda|u|^2u = 0, \tag{9}$$

with initial data

$$u(0,x) = \sqrt{\mu(0)}e^{-i(\alpha(0)x^2+\delta(0)x+\kappa(0))}\psi_0(x). \tag{10}$$

Now, we proceed to use Lemma 1 to discuss how we can construct NLS with variable coefficients equations that can be reduced to the standard NLS and therefore be solved explicitly. We start recalling that

$$u_1(t,x) = A\exp(2iA^2t)\left(\frac{3+16iA^2t-16A^4t^2-4A^2x^2}{1+16A^4t^2+4A^2x^2}\right), \quad A \in \mathbb{R} \tag{11}$$

is a solution for ($l_0 = -1$ and $\lambda = -2$)

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad t,x \in \mathbb{R}. \tag{12}$$

In addition,

$$u_2(\xi,\tau) = A\tanh(A\xi)e^{-2iA^2\tau} \tag{13}$$

is a solution of ($l_0 = -1$ and $\lambda = 2$)

$$iu_\tau + u_{\xi\xi} - 2|u|^2u = 0, \tag{14}$$

and

$$u_3(\tau,\xi) = \sqrt{v}\,\mathrm{sech}(\sqrt{v}\xi)\exp(-iv\tau), \quad v > 0 \tag{15}$$

is a solution of ($l_0 = 1$ and $\lambda = -2$),

$$iu_\tau - u_{\xi\xi} - 2|u|^2 u = 0. \tag{16}$$

Example 1. Consider the NLS:

$$i\psi_t + \psi_{xx} - \frac{c^2}{4}x^2\psi - icx\psi_x \pm 2e^{ct}\,|\psi|^2\,\psi = 0. \tag{17}$$

Our intention is to construct a similarity transformation from Equation (17) to standard NLS Equation (9) by means of Lemma 1. Using the latter, we obtain

$$b(t) = \frac{c^2}{4}, \qquad c(t) = c, \qquad \mu(t) = e^{ct},$$

and

$$\alpha(t) = -\frac{c}{4}, \qquad h(t) = \pm 2e^{ct}.$$

Therefore,

$$\psi(x,t) = \frac{e^{-i\frac{c}{4}x^2}}{\sqrt{e^{ct}}}u_j(x,t), \qquad j = 1,2$$

is a solution of the form Equation (6), and $u_j(x,t)$ are given by Equations (12) and (13).

Example 2. Consider the NLS:

$$i\psi_t + \psi_{xx} - \frac{1}{2t^2}x^2\psi - i\frac{1}{t}x\psi_x \pm 2t|\psi|^2\psi = 0. \tag{18}$$

By Lemma 1, a Riccati equation associated to the similarity transformation is given by

$$\frac{dc}{dt} + c(t)^2 - 2t^{-2} = 0, \tag{19}$$

and we obtain the functions

$$b(t) = \frac{1}{2t^2}, \qquad c(t) = -\frac{1}{t}, \qquad \mu(t) = t,$$

$$\alpha(t) = -\frac{1}{4t}, \qquad h_1(t) = -2t, \qquad h_2(t) = 2t.$$

Using $u_j(x,t)$, $j = 1$ and 2, given by Equations (12) and (13), we get the solutions

$$\psi_j(x,t) = \frac{e^{-i\frac{1}{4t}x^2}}{\sqrt{t}}u_i(x,t). \tag{20}$$

Table 1 shows integrable variable coefficient NLS and the corresponding similarity transformation to constant coefficient NLS. Table 2 lists some Riccati equations that can be used to generate these transformations.

Example 3. If we consider the following family (m and B are parameters) of variable coefficient NLS,

$$i\psi_t + \psi_{xx} - \frac{Bmt^{m-1} + Bt^{2m}}{4}x^2\psi + iBt^m x\psi_x + \gamma e^{-\frac{Bt^{m+1}}{m+1}}|\psi|^2\psi = 0, \tag{21}$$

by means of the Riccati equation

$$y_t = At^n y^2 + Bmt^{m-1} - AB^2 t^{n+2m}, \tag{22}$$

and Lemma 1, we can construct soliton-like solutions for Equation (21). For this example, we restrict ourselves to taking $A = -1$ and $n = 0$. Furthermore, taking in Lemma 1 $l_0 = -1$, $\lambda = -2$, $a(t) = 1$, $b(t) = \frac{Bmt^{m-1}+Bt^{2m}}{4}$, $c(t) = Bt^m$, $\mu(t) = e^{-\frac{Bt^{m+1}}{m+1}}$, $h(t) = -2e^{-\frac{Bt^{m+1}}{m+1}}$, and $\alpha(t) = -Bt^m/4$, soliton-like solutions to the Equation (21) are given by

$$\psi_j(x,t) = e^{i\frac{-Bx^2t^m}{4}} e^{\frac{Bt^{m+1}}{2(m+1)}} u_j(x,t), \tag{23}$$

where using $u_j(x,t)$, $j = 1$ and 2, given by Equations (12) and (15), we get the solutions. It is important to notice that if we consider $B = 0$ in Equation (21) we obtain standard NLS models.

3. Riccati Systems with Parameters and Similarity Transformations

In this section, we use different similarity trasformations than those used in Section 2, but they have been presented previously [26], [35], [39] and [42]. The advantage of the presentation of this section is a multiparameter approach. These parameters provide us with a control on the center axis of bright and dark soliton solutions. Again in this section, using Table 2, and by means of computer algebra systems, we show that we can produce a very extensive number of integrable VCNLS allowing soliton-type solutions. The transformations will require:

$$\frac{d\alpha}{dt} + b(t) + 2c(t)\alpha + 4a(t)\alpha^2 = 0, \tag{24}$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t)\alpha(t))\beta = 0, \tag{25}$$

$$\frac{d\gamma}{dt} + l_0 a(t)\beta^2(t) = 0, \quad l_0 = \pm 1, \tag{26}$$

$$\frac{d\delta}{dt} + (c(t) + 4a(t)\alpha(t))\delta = f(t) + 2\alpha(t)g(t), \tag{27}$$

$$\frac{d\varepsilon}{dt} = (g(t) - 2a(t)\delta(t))\beta(t), \tag{28}$$

$$\frac{d\kappa}{dt} = g(t)\delta(t) - a(t)\delta^2(t). \tag{29}$$

Considering the standard substitution

$$\alpha(t) = \frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{30}$$

it follows that the Riccati Equation (24) becomes

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0, \tag{31}$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{32}$$

We will refer to Equation (31) as the characteristic equation of the Riccati system. Here, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$ and $g(t)$ are real value functions depending only on the variable t. A solution of the

Riccati system Equations (24)–(29) with multiparameters is given by the following expressions (with the respective inclusion of the parameter l_0) [26], [35] and [39]:

$$\mu(t) = 2\mu(0)\,\mu_0(t)\,(\alpha(0) + \gamma_0(t)), \tag{33}$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \tag{34}$$

$$\beta(t) = -\frac{\beta(0)\,\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\,\mu(0)}{\mu(t)} w(t), \tag{35}$$

$$\gamma(t) = l_0\gamma(0) - \frac{l_0\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}, \quad l_0 = \pm 1, \tag{36}$$

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)\,(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{37}$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)\,(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \tag{38}$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))}, \tag{39}$$

subject to the initial arbitrary conditions $\mu(0)$, $\alpha(0)$, $\beta(0) \neq 0$, $\gamma(0)$, $\delta(0)$, $\varepsilon(0)$ and $\kappa(0)$. α_0, β_0, γ_0, δ_0, ε_0 and κ_0 are given explicitly by:

$$\alpha_0(t) = \frac{1}{4a(t)}\frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{40}$$

$$\beta_0(t) = -\frac{w(t)}{\mu_0(t)}, \quad w(t) = \exp\left(-\int_0^t (c(s) - 2d(s))\,ds\right), \tag{41}$$

$$\gamma_0(t) = \frac{d(0)}{2a(0)} + \frac{1}{2\mu_1(0)}\frac{\mu_1(t)}{\mu_0(t)}, \tag{42}$$

$$\delta_0(t) = \frac{w(t)}{\mu_0(t)}\int_0^t \left[\left(f(s) - \frac{d(s)}{a(s)}g(s)\right)\mu_0(s) + \frac{g(s)}{2a(s)}\mu_0'(s)\right]\frac{ds}{w(s)}, \tag{43}$$

$$\begin{aligned}\varepsilon_0(t) \quad &= \quad -\frac{2a(t)\,w(t)}{\mu_0'(t)}\delta_0(t) + 8\int_0^t \frac{a(s)\,\sigma(s)\,w(s)}{(\mu_0'(s))^2}(\mu_0(s)\,\delta_0(s))\,ds \\ &\quad +2\int_0^t \frac{a(s)\,w(s)}{\mu_0'(s)}\left[f(s) - \frac{d(s)}{a(s)}g(s)\right]ds,\end{aligned} \tag{44}$$

$$\begin{aligned}\kappa_0(t) \quad &= \quad \frac{a(t)\,\mu_0(t)}{\mu_0'(t)}\delta_0^2(t) - 4\int_0^t \frac{a(s)\,\sigma(s)}{(\mu_0'(s))^2}(\mu_0(s)\,\delta_0(s))^2\,ds \\ &\quad -2\int_0^t \frac{a(s)}{\mu_0'(s)}(\mu_0(s)\,\delta_0(s))\left[f(s) - \frac{d(s)}{a(s)}g(s)\right]ds,\end{aligned} \tag{45}$$

with $\delta_0(0) = g_0(0)\,/\,(2a(0))$, $\varepsilon_0(0) = -\delta_0(0)$, $\kappa_0(0) = 0$. Here, μ_0 and μ_1 represent the fundamental solution of the characteristic equation subject to the initial conditions $\mu_0(0) = 0$, $\mu_0'(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu_1'(0) = 0$.

Using the system Equations (34)–(39), in [26], a generalized lens transformation is presented. Next, we recall this result (here we use a slight perturbation introducing the parameter $l_0 = \pm 1$ in order to use Peregrine type soliton solutions):

Lemma 2 ($l_0 = 1$, [26]). *Assume that $h(t) = \lambda a(t)\beta^2(t)\mu(t)$ with $\lambda \in \mathbb{R}$. Then, the substitution*

$$\psi(t,x) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2+\delta(t)x+\kappa(t))} u(\tau,\xi), \tag{46}$$

where $\xi = \beta(t)\,x + \varepsilon(t)$ and $\tau = \gamma(t)$, transforms the equation

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi$$

into the standard Schrödinger equation

$$iu_\tau - l_0 u_{\xi\xi} + l_0\lambda|u|^2 u = 0, \qquad l_0 = \pm 1, \tag{47}$$

as long as α, β, γ, δ, ε and κ satisfy the Riccati system Equations (24)–(29) and also Equation (30).

Example 4. Consider the NLS:

$$i\psi_t = \psi_{xx} - \frac{x^2}{4}\psi + h(0)\,\mathrm{sech}(t)|\psi|^2\psi. \tag{48}$$

It has the associated characteristic equation $\mu'' + a\mu = 0$, and, using this, we will obtain the functions:

$$\alpha(t) = \frac{\coth(t)}{4} - \frac{1}{2}\mathrm{csch}(t)\,\mathrm{sech}(t), \quad \delta(t) = -\,\mathrm{sech}(t), \tag{49}$$

$$\kappa(t) = 1 - \frac{\tanh(t)}{2}, \quad \mu(t) = \cosh(t), \tag{50}$$

$$h(t) = h(0)\,\mathrm{sech}(t), \quad \beta(t) = \frac{1}{\cosh(t)}, \tag{51}$$

$$\varepsilon(t) = -1 + \tanh(t), \quad \gamma(t) = 1 - \frac{\tanh(t)}{2}. \tag{52}$$

Then, we can construct solution of the form

$$\psi_j(t,x) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2+\delta(t)x+\kappa(t))} u_j\left(1 - \frac{\tanh(t)}{2}, \frac{x}{\cosh(t)} - 1 + \tanh(t)\right), \tag{53}$$

with u_j, $j = 1$ and 2, given by Equations (12) and (13).

Example 5. Consider the NLS:

$$i\psi_t(x,t) = \psi_{xx}(x,t) + \frac{h(0)\beta(0)^2\mu(0)}{1+\alpha(0)2c_2 t}|\psi(x,t)|^2\psi(x,t).$$

It has the characteristic equation $\mu'' + a\mu = 0$, and, using this, we will obtain the functions:

$$\alpha(t) = \frac{1}{4t} - \frac{1}{2+\alpha(0)4c_2^2t^2}, \quad \delta(t) = \frac{\delta(0)}{1+\alpha(0)2c_2 t}, \tag{54}$$

$$\kappa(t) = \kappa(0) - \frac{\delta(0)^2 c_2 t}{2+4\alpha(0)c_2 t}, \quad h(t) = \frac{h(0)\beta(0)^2\mu(0)}{1+\alpha(0)2c_2 t}, \tag{55}$$

$$\mu(t) = (1+\alpha(0)2c_2 t)\mu(0), \quad \beta(t) = \frac{\beta(0)}{1+\alpha(0)2c_2 t},$$

$$\gamma(t) = \gamma(0) - \frac{\beta(0)^2 c_2 t}{2 + 4\alpha(0)c_2 t}, \quad \epsilon(t) = \epsilon(0) - \frac{\beta(0)\delta(0)c_2 t}{1 + 2\alpha(0)c_2 t}.$$

Then, we can construct a solution of the form

$$\begin{aligned} \psi_j(t,x) &= \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \\ & u_j\left(\gamma(0) - \frac{\beta(0)^2 c_2 t}{2 + 4\alpha(0)c_2 t}, \frac{\beta(0)x}{1 + \alpha(0)2c_2 t} + \epsilon(0) - \frac{\beta(0)\delta(0)c_2 t}{1 + 2\alpha(0)c_2 t}\right), \end{aligned} \tag{56}$$

with u_j, $j = 1$ and 2, Equations (12) and (13).

Following Table 2 of Riccati equations, we can use Equation (24) and Lemma 2 to construct an extensive list of integrable variable coefficient nonlinear Schrödinger equations.

4. Crank-Nicolson Scheme for Linear Schrödinger Equation with Variable Coefficients Depending on Space

In addition, in [35], a generalized Melher's formula for a general linear Schrödinger equation of the one-dimensional generalized harmonic oscillator of the form Equation (1) with $h(t) = 0$ was presented. As a particular case, if $b = \lambda \frac{\omega^2}{2}$; $f = b$, $\omega > 0$, $\lambda \in \{-1, 0, 1\}$, $c = g = 0$, then the evolution operator is given explicitly by the following formula (note—this formula is a consequence of Mehler's formula for Hermite polynomials):

$$\psi(x,t) = U_V(t)f := \frac{1}{\sqrt{2i\pi\mu_j(t)}} \int_{\mathbb{R}^n} e^{iS_V(x,y,t)} f(y)dy, \tag{57}$$

where

$$\begin{aligned} S_V(x,y,t) &= \frac{1}{\mu_j(t)}\left(\frac{x_j^2 + y_j^2}{2} l_j(t) - x_j y_j\right), \\ \{\mu_j(t), l_j(t)\} &= \begin{cases} \left\{\frac{\sinh(\omega_j t)}{\omega_j}, \cosh(\omega_j t)\right\}, & \text{if } \lambda_j = -1 \\ \{t, 1\}, & \text{if } \lambda_j = 0 \\ \left\{\frac{\sin(\omega_j t)}{\omega_j}, \cos(\omega_j t)\right\}, & \text{if } \lambda_j = +1 \end{cases}. \end{aligned} \tag{58}$$

Using Riccati-Ermakov systems in [41], it was shown how computer algebra systems can be used to derive the multiparameter formulas (33)–(45). This multi-parameter study was used also to study solutions for the inhomogeneous paraxial wave equation in a linear and quadratic approximation including oscillating laser beams in a parabolic waveguide, spiral light beams, and more families of propagation-invariant laser modes in weakly varying media. However, the analytical method is restricted to solve Riccati equations exactly as the ones presented in Table 2. In this section, we use a finite differences method to compare analytical solutions described in [41] with numerical approximations. We aim (in future research) to extend numerical schemes to solve more general cases that the analytical method exposed cannot. Particularly, we will pursue to solve equations of the general form:

$$i\psi_t = -\Delta\psi + V(\mathbf{x},t)\psi, \tag{59}$$

using polynomial approximations in two variables for the potential function $V(\mathbf{x},t)$ ($V(\mathbf{x},t) \approx b(t)(x_1^2 + x_2^2) + f(t)x_1 + g(t)x_2 + h(t)$). For this purpose, it is necessary to analyze stability of different methods applied to this equation.

We also will be interested in extending this process to nonlinear Schrödinger-type equations with potential terms dependent on time, such as

$$i\psi_t = -\Delta\psi + V(\mathbf{x},t)\psi + s|\psi|^2\psi. \tag{60}$$

In this section, we show that the Crank-Nicolson scheme seems to be the best method to deal with reconstructing numerically the analytical solutions presented in [41].

Numerical methods arise as an alternative when it is difficult to find analytical solutions of the Schrödinger equation. Despite numerical schemes not providing explicit solutions to the problem, they do yield approaches to the real solutions which allow us to obtain some relevant properties of the problem. Most of the simplest and often-used methods are those based on finite differences.

In this section, the Crank-Nicolson scheme is used for linear Schrödinger equation in the case of coefficients depending only on the space variable because it is absolutely stable and the matrix of the associate system does not vary for each iteration.

A rectangular mesh $(\mathbf{x}_m, t_n)$ is introduced in order to discretize a bounded domain $\Omega \times [0,T]$ in space and time. In addition, τ and $\mathbf{h}$ represent the size of the time step and the size of space step, respectively. $\mathbf{x}_m$ and $\mathbf{h}$ are in $\mathbb{R}$ if one-dimensional space is considered; otherwise, they are in $\mathbb{R}^2$.

The discretization is given by the matrix system

$$\left(I + \frac{ia\tau}{2h^2}\Delta + \frac{i\tau}{2}V(\mathbf{x})\right)\psi^{n+1} = \left(I - \frac{ia\tau}{2h^2}\Delta - \frac{i\tau}{2}V(\mathbf{x})\right)\psi^{n}, \tag{61}$$

where I is the identity matrix, Δ is the discrete representation of the Laplacian operator in space, and $V(\mathbf{x})$ is the diagonal matrix that represents the operator of the external potential depending on $\mathbf{x}$.

The paraxial wave equation (also known as harmonic oscillator)

$$2i\psi_t + \Delta\psi - \mathbf{r}^2\psi = 0, \tag{62}$$

where $\mathbf{r} = x$ for $\mathbf{x} \in \mathbb{R}$ or $\mathbf{r} = \sqrt{x_1^2 + x_2^2}$ for $\mathbf{x} \in \mathbb{R}^2$, describes the wave function for a laser beam [40].

One solution for this equation can be presented as Hermite-Gaussian modes on a rectangular domain:

$$\begin{aligned}
\psi_{nm}(\mathbf{x},t) &= A_{nm}\frac{exp\left[i(\kappa_1+\kappa_2)+2i(n+m+1)\gamma\right]}{\sqrt{2^{n+m}n!m!\pi}}\beta \\
&\times\ exp\left[i(\alpha\mathbf{r}^2+\delta_1 x_1+\delta_2 x_2)-(\beta x_1+\varepsilon_1)^2/2-(\beta x_2+\varepsilon_2)^2/2\right] \\
&\times\ H_n(\beta x_1+\varepsilon_1)H_m(\beta x_2+\varepsilon_2),
\end{aligned} \tag{63}$$

where $H_n(x)$ is the n-th order Hermite polynomial in the variable x, see [40] and [41].

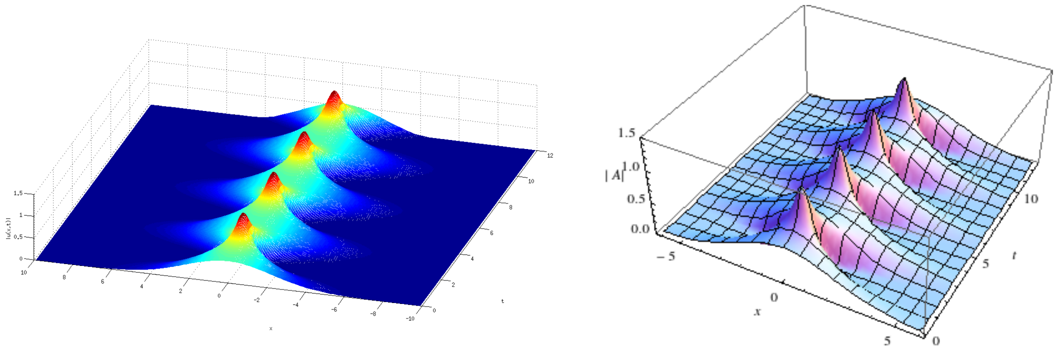
In addition, some solutions of the paraxial equation may be expressed by means of Laguerre–Gaussian modes in the case of cylindrical domains (see [43]):

$$\begin{aligned}
\psi_n^m(\mathbf{x},t) &= A_n^m\sqrt{\frac{n!}{\pi(n+m)!}}\beta \\
&\times\ exp\left[i(\alpha\mathbf{r}^2+\delta_1 x_1+\delta_2 x_2+\kappa_1+\kappa_2)-(\beta x_1+\varepsilon_1)^2/2-(\beta x_2+\varepsilon_2)^2/2\right] \\
&\times\ exp\left[i(2n+m+1)\gamma\right](\beta(x_1\pm ix_2)+\varepsilon_1\pm i\varepsilon_2)^m \\
&\times\ L_n^m((\beta x_1+\varepsilon_1)^2+(\beta x_2+\varepsilon_2)^2),
\end{aligned} \tag{64}$$

with $L_n^m(x)$ being the n-th order Laguerre polynomial with parameter m in the variable x.

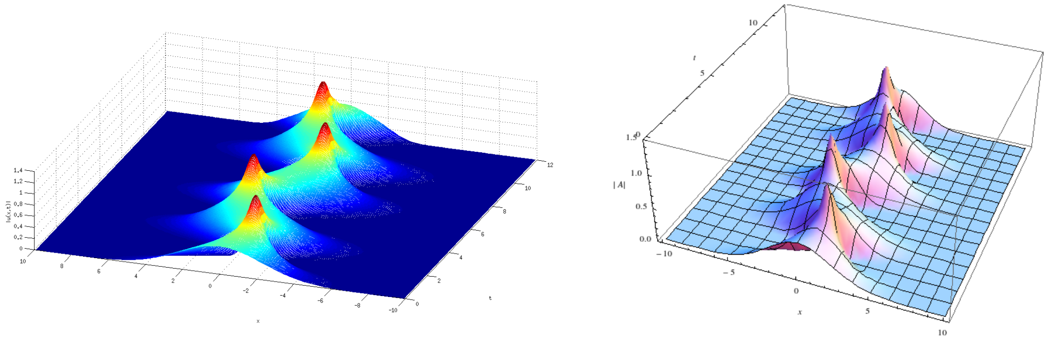
α, β, γ, δ_1, δ_2, ε_1, ε_2, κ_1 and κ_2 given by Equations (34)–(39) for both Hermite-Gaussian and Laguerre-Gaussian modes.

Figures 1 and 2 show two examples of solutions of the one-dimensional paraxial equation with $\Omega = [-10, 10]$ and $T = 12$. The step sizes are $\tau = \frac{10}{200}$ and $h = \frac{10}{200}$.



(a) **(b)**

Figure 1. (**a**) corresponding approximation for the one-dimensional Hermite-Gaussian beam with $t = 10$. The initial condition is $\sqrt{\frac{2}{3\sqrt{\pi}}}e^{\left(\frac{2}{3}x\right)^2/2}$; (**b**) the exact solution for the one-dimensional Hermite-Gaussian beam with $t = 10$, $A_n = 1$, $\mu_0 = 1$, $\alpha_0 = 0$, $\beta_0 = \frac{4}{9}$, $n_0 = 0$, $\delta_0 = 0$, $\gamma_0 = 0$, $\epsilon_0 = 0$, $\kappa_0 = 0$.



(a) **(b)**

Figure 2. (**a**) corresponding approximation for the one-dimensional Hermite-Gaussian beam with $t = 10$. The initial condition is $\sqrt{\frac{2}{3\sqrt{\pi}}}e^{\left(\frac{2}{3}x\right)^2/2+ix}$; (**b**) the exact solution for the one-dimensional Hermite-Gaussian beam with $t = 10$, $A_n = 1$, $\mu_0 = 1$, $\alpha_0 = 0$, $\beta_0 = \frac{4}{9}$, $n_0 = 0$, $\delta_0 = 1$, $\gamma_0 = 0$, $\epsilon_0 = 0$, $\kappa_0 = 0$.

Figure 3 shows four profiles of two-dimensional Hermite-Gaussian beams considering $\Omega = [-6, 6] \times [-6, 6]$ and $T = 10$. The corresponding step sizes are $\tau = \frac{10}{40}$ and $\mathbf{h} = \left(\frac{12}{48}, \frac{12}{48}\right)$.

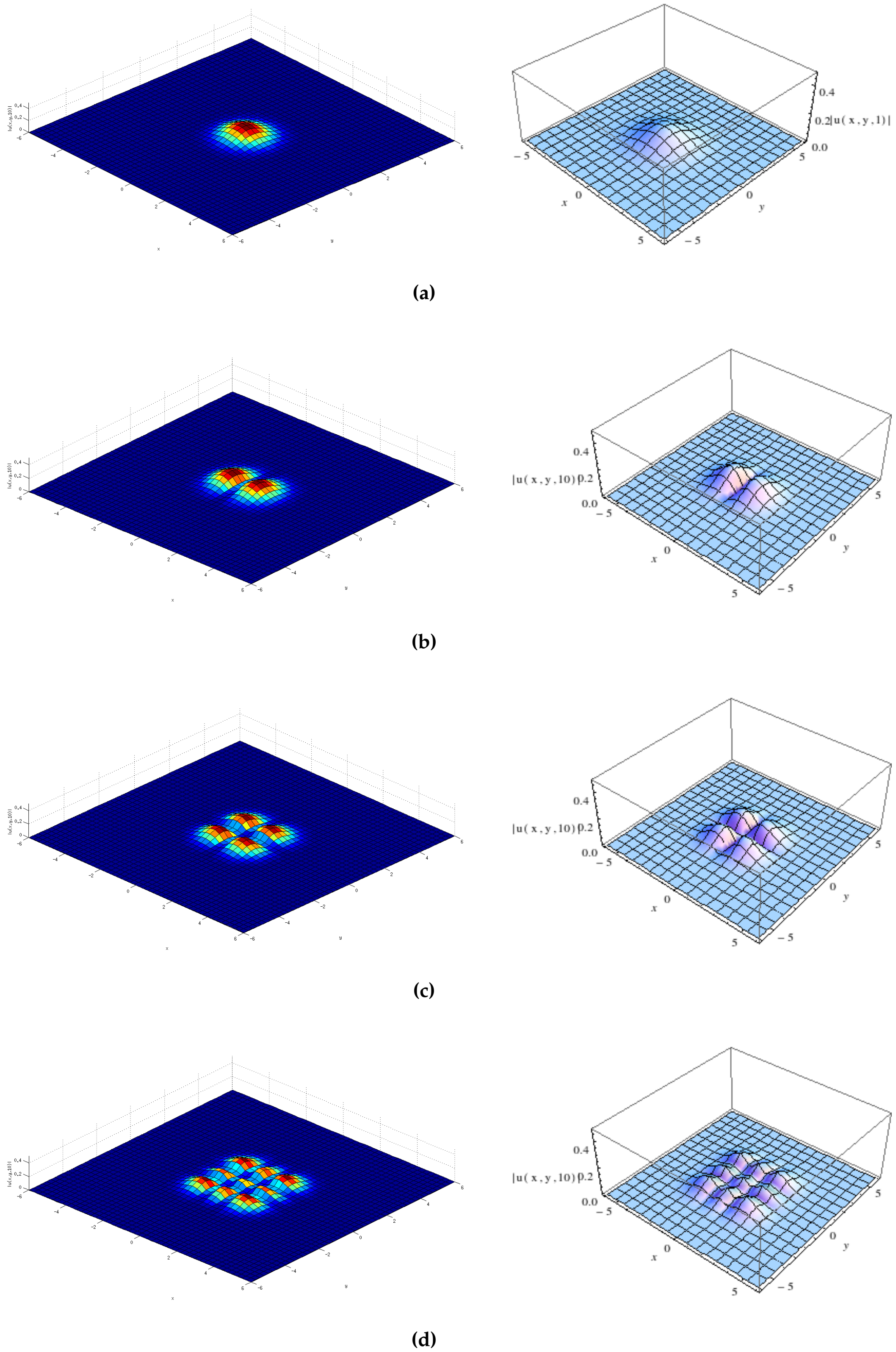
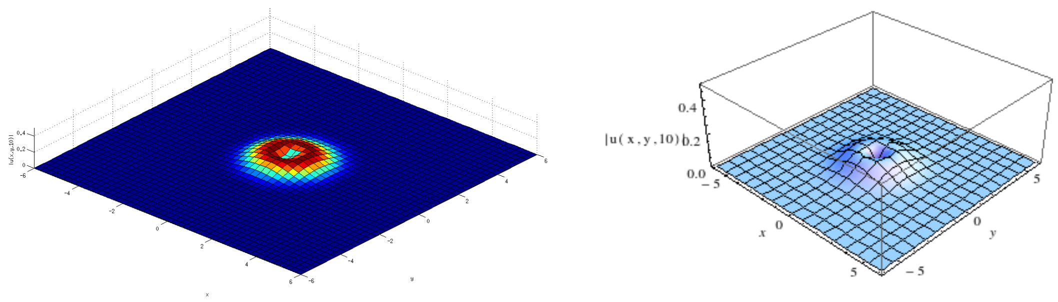
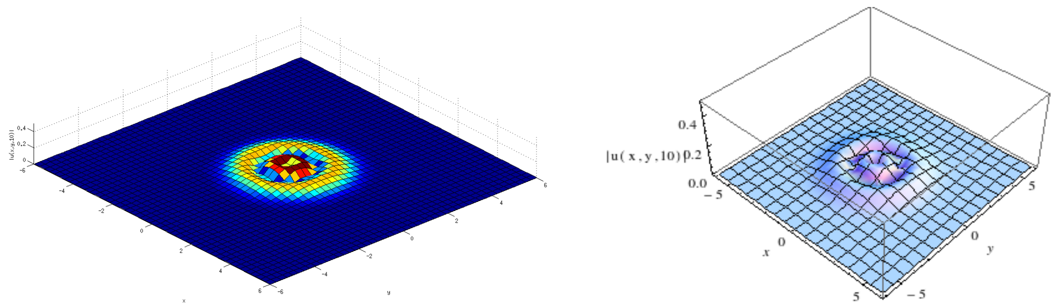


Figure 3. (**Left**): corresponding approximations for the two-dimensional Hermite-Gaussian beams with $t = 10$. The initial conditions are (**a**) $\frac{1}{\sqrt{8\pi}}e^{-\left(x^2+y^2\right)}$; (**b**) $\frac{1}{\sqrt{2\pi}}e^{-\left(x^2+y^2\right)}x$; (**c**) $\sqrt{\frac{2}{\pi}}e^{-\left(x^2+y^2\right)}xy$; (**d**) $\frac{1}{4\sqrt{32\pi}}e^{-\left(x^2+y^2\right)}\left(8x^2-2\right)\left(8y^2-2\right)$. (**Right**): the exact solutions for the two-dimensional Hermite-Gaussian beams with $t = 10$ and parameters $A_{nm} = \frac{1}{4}$, $\alpha_0 = 0$, $\beta_0 = \sqrt{2}$, $\delta_{0,1} = 1$, $\gamma_{0,1} = 0$, $\epsilon_{0,1} = 0$, $\kappa_{0,1} = 0$. For (**a**) $n = 0$ and $m = 0$, for (**b**) $n = 1$ and $m = 0$, for (**c**) $n = 1$ and $m = 1$, for (**d**) $n = 2$ and $m = 2$.

Figure 4 shows two profiles of two-dimensional Laguerre–Gaussian beams considering $\Omega = [-6, 6] \times [-6, 6]$ and $T = 10$. The corresponding step sizes are $\tau = \frac{10}{40}$ and $\mathbf{h} = \left(\frac{12}{48}, \frac{12}{48}\right)$.



(a)



(b)

Figure 4. (**Left**): corresponding approximations for the two-dimensional Laguerre–Gaussian beams with $t = 10$. The initial conditions are (**a**) $\frac{1}{\sqrt{4\pi}} e^{-\left(x^2+y^2\right)} (x + iy)$; (**b**) $\frac{1}{\sqrt{2\pi}} e^{-\left(x^2+y^2\right)} (x + iy) \left(1 - x^2 - y^2\right)$. (**Right**): the exact solutions for the two-dimensional Laguerre–Gaussian beams with $t = 10$ and parameters $A_n^m = \frac{1}{4}$, $\alpha_0 = 0$, $\beta_0 = \sqrt{2}$, $\delta_{0,1} = 1$, $\gamma_{0,1} = 0$, $\epsilon_{0,1} = 0$, $\kappa_{0,1} = 0$.

5. Conclusions

Rajendran *et al.*in [1] used similarity transformations introduced in [28] to show a list of integrable NLS equations with variable coefficients. In this work, we have extended this list, using similarity transformations introduced by Suslov in [26], and presenting a more extensive list of families of integrable nonlinear Schrödinger (NLS) equations with variable coefficients (see Table 1 as a primary list. In both approaches, the Riccati equation plays a fundamental role. The reader can observe that, using computer algebra systems, the parameters (see Equations (33)–(39)) provide a change of the dynamics of the solutions; the Mathematica files are provided as a supplement for the readers. Finally, we have tested numerical approximations for the inhomogeneous paraxial wave equation by the Crank-Nicolson scheme with analytical solutions. These solutions include oscillating laser beams and Laguerre and Gaussian beams. The explicit solutions have been found previously thanks to explicit solutions of Riccati-Ermakov systems [41].

Supplementary Materials: The following are available online at http://www.mdpi.com/2073-8994/8/5/38/s1, Mathematica supplement file.

Acknowledgments: The authors were partially funded by the Mathematical American Association through NSF (grant DMS-1359016) and NSA (grant DMS-1359016). Also, the authors are thankful for the funding received from the Department of Mathematics and Statistical Sciences and the College of Liberal Arts and Sciences at University of Puerto Rico, Mayagüez. E. S. is funded by the Simons Foundation Grant # 316295 and by the

National Science Foundation Grant DMS-1440664. E.S is also thankful for the start up funds and the "Faculty Development Funding Program Award" received from the School of Mathematics and Statistical Sciences and the College of Sciences at University of Texas, Rio Grande Valley.

Author Contributions: The original results presented in this paper are the outcome of a research collaboration started during the Summer 2015 and continuous until Spring 2016. Similarly, the selection of the examples, tables, graphics and extended bibliography is the result of a continuous long interaction between the authors.

Conflicts of Interest: The authors declare no conflict of interest.

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