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On a Reduction Formula for a Kind of Double q -Integrals

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Abstract: Using the q -integral representation of Sears' nonterminating extension of the q -Saalschütz summation, we derive a reduction formula for a kind of double q -integrals. This reduction formula is used to derive a curious double q -integral formula, and also allows us to prove a general q -beta integral formula including the Askey–Wilson integral formula as a special case. Using this double q -integral formula and the theory of q -partial differential equations, we derive a general q -beta integral formula, which includes the Nassrallah–Rahman integral as a special case. Our evaluation does not require the orthogonality relation for the q -Hermite polynomials and the Askey–Wilson integral formula.

Keywords: q -series; q -beta integral; q -partial differential equation; double q -integral

1. A Double q -Integral Formula

Throughout the paper, we assume that $0 < q < 1$. For $a \in \mathbb{C}$, we define the q -shifted factorials as follows:

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 0, 1, 2, \dots,$$

and

$$(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n = \prod_{k=0}^{\infty} (1 - aq^k).$$

If n is an integer or ∞ , the multiple q -shifted factorials are defined as:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n.$$

The q -binomial coefficients are the q -analogs of the binomial coefficients, which are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Definition 1. For $x = \cos \theta$, we define $h(x; a)$ and $h(x; a_1, a_2, \dots, a_m)$ as follows:

$$h(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty = \prod_{k=0}^{\infty} (1 - 2q^k ax + q^{2k} a^2),$$

$$h(x; a_1, a_2, \dots, a_m) = h(x; a_1) h(x; a_2) \dots h(x; a_m).$$

Definition 2. For simplicity, we use $\Delta(u, v)$ to denote the theta function:

$$(1 - q)v(q, u/v, qv/u; q)_\infty.$$

As usual, the basic hypergeometric series or q -hypergeometric series ${}_r\phi_s$ is defined by:

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n.$$

Now, we introduce the definition of the Thomae–Jackson q -integral in q -calculus, which was introduced by Thomae [1] and Jackson [2].

Definition 3. Given a function $f(x)$, the Thomae–Jackson q -integral is defined by:

$$\int_a^b f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} [bf(bq^n) - af(aq^n)] q^n.$$

Using the q -integral notation, one can write some q -formulas in more compact forms.

On making use of the q -exponential operator method, we [3] proved the following proposition (some misprints have been corrected here):

Proposition 1. For $\max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|, |d/b|\} < 1$, we have:

$$\int_u^v \frac{(qx/u, qx/v, dx; q)_\infty}{(ax, bx, cx; q)_\infty} d_q x = \frac{\Delta(u, v)(abuv, bcuv, d/b; q)_\infty}{(au, av, bu, bv, cu, cv; q)_\infty} \times {}_3\phi_2 \left(\begin{matrix} bu, bv, abcuv/d \\ abuv, bcuv \end{matrix}; q, \frac{d}{b} \right).$$

When $d = abcuv$ in Proposition 1, upon noting that $(1; q)_k = \delta_{0,k}$, the ${}_3\phi_2$ series in the proposition reduces to 1 and thus the proposition becomes the Al-Salam–Verma formula, which is the q -integral representation of Sears' nonterminating extension of the q -Saalschütz summation [4], see also [5] (p. 52).

Proposition 2. If there are no zero factors in the denominator of the integral and $\max\{|au|, |av|, |bu|, |bv|, |cu|, |cv|\} < 1$, then we have:

$$\int_u^v \frac{(qx/u, qx/v, abcuvx; q)_\infty}{(ax, bx, cx; q)_\infty} d_q x = \frac{\Delta(u, v)(abuv, acuv, bcuv; q)_\infty}{(au, bu, cu, av, bv, cv; q)_\infty}.$$

When $c = 0$, this q -integral formula reduces to the following q -integral formula due to Andrews and Askey [6], which can be derived from Ramanujan ${}_1\psi_1$ summation.

Proposition 3. If there are no zero factors in the denominator of the integral and $\max\{|au|, |av|, |bu|, |bv|\} < 1$, then we have:

$$\int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x = \frac{\Delta(u, v)(abuv; q)_\infty}{(au, bu, av, bv; q)_\infty}.$$

The main purpose of this paper is to study double q -integrals. There are not many studies on this subject, and it is difficult to find the definition of the double q -integrals in the literature. We now give the definition of the double q -integral of a two-variable function $f(x, y)$ over the rectangular region.

Definition 4. If $f(x, y)$ is a two variable function, the double q -integral of f over $[a, b] \times [c, d]$ is formally defined as:

$$\begin{aligned} & \iint_{[a,b] \times [c,d]} f(x, y) d_q x d_q y \\ &= (1-q)^2 \sum_{m,n=0}^{\infty} (bdf(bq^m, dq^n) + acf(aq^m, cq^n)) q^{m+n} \\ & - (1-q)^2 \sum_{m,n=0}^{\infty} (bcf(bq^m, cq^n) + adf(aq^m, dq^n)) q^{m+n}. \end{aligned}$$

Proposition 4. If for any $x \in [a, b]$ and $y \in [c, d]$, the double series,

$$\sum_{m,n=0}^{\infty} f(xq^m, yq^n) q^{m+n}$$

is absolutely convergent, then it is equal to each of the two iterated q -integrals, namely,

$$\iint_{[a,b] \times [c,d]} f(x, y) d_q x d_q y = \int_a^b d_q x \int_c^d f(x, y) d_q y = \int_c^d d_q y \int_a^b f(x, y) d_q x.$$

Proof. If the conditions of the proposition are satisfied, then the q -double integral of $f(x, y)$ represents an absolutely convergent double series, and we can interchange the order of summation to complete the proof of Proposition 4. \square

In order to determine the absolute convergence of double series, one can use the ratio test for double series (see, for example, [7] (Corollary 7.35)):

Proposition 5. Let $(a_{k,l})$ be a double sequence of nonzero real numbers such that either $|a_{k,l+1}|/|a_{k,l}| \rightarrow a$ or $|a_{k+1,l}|/|a_{k,l}| \rightarrow \tilde{a}$ as $(k, l) \rightarrow (\infty, \infty)$, where $a, \tilde{a} \in \mathbb{R} \cup \infty$. If each row-series as well as each column-series corresponding to $\sum_{k,l} a_{k,l}$ is absolutely convergent and $a < 1$ or $\tilde{a} < 1$, then $\sum_{k,l} a_{k,l}$ is absolutely convergent.

The principal result of this paper is the following general iterated q -integral formula:

Theorem 1. Suppose that $f(y)$ is a function of y which satisfies,

$$\lim_{n \rightarrow \infty} |f(yq^n)/f(yq^{n-1})| \leq 1.$$

Then, we have the iterated q -integral identity,

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_{\infty}}{(ax, bx; q)_{\infty}} d_q x \int_c^d \frac{f(y)(qy/c, qy/d, abuvxy; q)_{\infty}}{(xy; q)_{\infty}} d_q y \\ &= \frac{\Delta(u, v)(abuv; q)_{\infty}}{(au, av, bu, bv; q)_{\infty}} \int_c^d \frac{f(y)(qy/c, qy/d, auvy, buvy; q)_{\infty}}{(uy, vy; q)_{\infty}} d_q y. \end{aligned}$$

Proof. For simplicity, we use $I(u, v, c, d)$ to denote the following q -double integral:

$$\iint_{[u,v] \times [c,d]} \frac{f(y)(qx/u, qx/v, qy/c, qy/d, abuvxy; q)_{\infty}}{(ax, bx, xy; q)_{\infty}} d_q x d_q y.$$

Using the ratio test for double series in Proposition 5, we can prove that $I(u, v, c, d)$ is absolutely convergent.

Interchanging the order of the iterated integral on the left-hand side of the equation in Theorem 1, we find that:

$$\int_c^d (qy/c, qy/d; q)_\infty f(y) d_q y \int_u^v \frac{(qx/u, qx/v, abuvxy; q)_\infty}{(ax, bx, xy; q)_\infty} d_q x.$$

On replacing c by y in the Al-Salam–Verma integral in Proposition 2, we immediately find that

$$\int_u^v \frac{(qx/u, qx/v, abuvxy; q)_\infty}{(ax, bx, xy; q)_\infty} d_q x = \frac{\Delta(u, v)(abuv, auvy, buvy; q)_\infty}{(au, av, bu, bv, uy, vy; q)_\infty}.$$

Combining these two equations, we complete the proof of Theorem 1. \square

Theorem 1 can be used to derive some double q -integral evaluation formulas in Theorems 2–7, some of whose proofs are given in the later sections.

Theorem 2. For $\max_{s \in \{a, b, c, d\}} \{|su|, |sv|, |ctw|, |dtw|, |r/w|\} < 1$, we have:

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x \int_c^d \frac{(qy/c, qy/d, abuvxy, ry; q)_\infty}{(auvy, buvy, xy, wy; q)_\infty} d_q y \\ &= \frac{\Delta(u, v)\Delta(c, d)(abuv, cduw, cdvw, r/w; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, cw, dw; q)_\infty} \\ & \quad \times {}_3\phi_2 \left(\begin{matrix} cw, dw, cduvw/r \\ cduw, cdvw \end{matrix}; q, \frac{r}{w} \right). \end{aligned}$$

When $r = cduvw$, the ${}_3\phi_2$ series in the above equation reduces to 1, and hence we have the following curious double q -integral formula.

Proposition 6. For $\max\{|a|, |b|, |c|, |d|, |u|, |v|, |w|\} < 1$, we have:

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x \int_c^d \frac{(qy/c, qy/d, abuvxy, cduvwy; q)_\infty}{(auvy, buvy, xy, wy; q)_\infty} d_q y \\ &= \frac{\Delta(u, v)\Delta(c, d)(abuv, cduv, cduw, cdvw; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, cw, dw; q)_\infty}. \end{aligned}$$

Using Theorem 1, we can prove the following general q -beta integral formula that includes the Askey–Wilson integral as a special case.

Theorem 3. Suppose that $f(y)$ is a function of y that satisfies:

$$\lim_{n \rightarrow \infty} |f(yq^n)/f(yq^{n-1})| \leq 1.$$

Then, we have the following general q -beta integral formula:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \left(\int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_\infty}{h(\cos \theta; y)(ry; q)_\infty} d_q y \right) d\theta \\ &= \frac{2\pi}{(q, ab; q)_\infty} \int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_\infty}{(ay, by, ry; q)_\infty} d_q y. \end{aligned}$$

Using this theorem and the theory of q -partial differential equations developed recently by us, we can prove the following general q -beta integral formula, which includes the Nassrallah–Rahman integral as a special case. Our evaluation does not require the orthogonality relation for the q -Hermite polynomials and the Askey–Wilson integral formula.

Theorem 4. For $\{|a|, |b|, |c|, |d|, |u|, |abs/t|\} < 1$, we have the q -formula:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; w)}{h(\cos \theta; a, b, c, d, u)} {}_3\phi_2 \left(\begin{matrix} te^{i\theta}, te^{-i\theta}, t/s \\ at, bt \end{matrix}; q, \frac{abs}{t} \right) d\theta \\ &= \frac{2\pi(w/c, w/d; q)_\infty}{\Delta(c, d)(q, cd, au, bu, abs/t; q)_\infty} \\ & \times \int_c^d \frac{(qy/c, qy/d, wy, absuy/t; q)_\infty}{(ay, by, uy, wy/cd; q)_\infty} {}_3\phi_2 \left(\begin{matrix} t/s, t/u, t/y \\ at, bt \end{matrix}; q, \frac{absuy}{t} \right) d_q y. \end{aligned}$$

Upon taking $s = t$ in this theorem and upon noting that $(1; q)_k = \delta_{0,k}$, the two ${}_3\phi_2$ series in the above Theorem both have the value 1; thus, we obtain the following integral formula due to the Nassrallah and Rahman [5] (pp. 157–158).

Theorem 5. For $\max\{|a|, |b|, |c|, |d|, |u|\} < 1$, we have:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; w)}{h(\cos \theta; a, b, c, d, u)} d\theta \\ &= \frac{2\pi(w/c, w/d; q)_\infty}{\Delta(c, d)(q, ab, au, bu, cd; q)_\infty} \int_c^d \frac{(qy/c, qy/d, wy, abuy; q)_\infty}{(ay, by, uy, wy/cd; q)_\infty} d_q y. \end{aligned}$$

The proof of this formula given in [5] (pp. 157–158) needs to know the Askey–Wilson integral formula in advance.

Theorem 6. (Nassrallah–Rahman Integral). For $\max\{|a|, |b|, |c|, |d|, |u|\} < 1$, we have:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcd u)}{h(\cos \theta; a, b, c, d, u)} d\theta \\ &= \frac{2\pi(abcu, abcd, abdu, acdu, bcdu; q)_\infty}{(q, ab, ac, ad, au, bc, bd, bu, cd, cu, du; q)_\infty}. \end{aligned}$$

Proof. Setting $w = abcd u$ in Theorem 5, we immediately deduce that:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; abcd u)}{h(\cos \theta; a, b, c, d, u)} d\theta \\ &= \frac{2\pi(abcu, abdu; q)_\infty}{\Delta(c, d)(q, ab, au, bu, cd; q)_\infty} \int_c^d \frac{(qy/c, qy/d, abcd u y; q)_\infty}{(ay, by, uy; q)_\infty} d_q y. \end{aligned}$$

Using the Al-Salam–Verma integral in Proposition 2, we immediately have:

$$\int_c^d \frac{(qy/c, qy/d, abcd u y; q)_\infty}{(ay, by, uy; q)_\infty} d_q y = \frac{\Delta(c, d)(abcd, acdu, bcdu; q)}{(ac, ad, bc, bd, cu, du; q)_\infty}.$$

Combining these two equations, we complete the proof of the theorem. \square

Putting $t = u$, then the ${}_3\phi_2$ series on the right-hand side of the equation in Theorem 4 reduces to 1. Hence, we arrive at the following proposition.

Theorem 7. For $\{|a|, |b|, |c|, |d|, |u|, |abs/u|\} < 1$, we have the q -formula:

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; w)}{h(\cos \theta; a, b, c, d, u)} {}_3\phi_2 \left(\begin{matrix} te^{i\theta}, te^{-i\theta}, u/s \\ at, bt \end{matrix}; q, \frac{abs}{u} \right) d\theta \\ &= \frac{2\pi(w/c, w/d; q)_\infty}{\Delta(c, d)(q, cd, au, bu, abs/u; q)_\infty} \\ & \times \int_c^d \frac{(qy/c, qy/d, wy, absy; q)_\infty}{(ay, by, uy, wy/cd; q)_\infty} d_q y. \end{aligned}$$

The remainder of the paper is organized as follows: Section 2 is devoted to the proof of Theorem 2, and Theorem 3 is proved in Section 3. Theorem 4 is established in Section 4. In Section 5, we give a general double q -integral formula.

2. The Proof of Theorem 2

Proof. If $f(y) = (ry; q)_\infty / (auvy, buvy, wy; q)_\infty$, then, it is easily seen that:

$$\frac{f(q^n y)}{f(q^{n-1} y)} = \frac{(1 - auvyq^{n-1})(1 - buvyq^{n-1})(1 - wyq^{n-1})}{(1 - ryq^{n-1})}.$$

It follows that $\lim_{n \rightarrow \infty} |f(yq^n) / f(yq^{n-1})| = 1$. Thus, we can choose:

$$f(y) = (ry; q)_\infty / (auvy, buvy, wy; q)_\infty$$

in Theorem 1 to obtain:

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x \int_c^d \frac{(qy/c, qy/d, abuvxy, ry; q)_\infty}{(auvy, buvy, wy, xy; q)_\infty} d_q y \\ &= \frac{\Delta(u, v)(abuv; q)_\infty}{(au, av, bu, bv; q)_\infty} \int_c^d \frac{(qy/c, qy/d, ry; q)_\infty}{(uy, wy, vy; q)_\infty} d_q y. \end{aligned}$$

If we use I to denote the q -integral in the right-hand side of the above equation, then, appealing to Proposition 1, we find that:

$$I = \frac{\Delta(c, d)(cduw, cdvw, r/w; q)_\infty}{(cu, du, cv, dv, cw, dw; q)_\infty} {}_3\phi_2 \left(\begin{matrix} cw, dw, cduvw/r \\ cdw, cdvw \end{matrix}; q, \frac{r}{w} \right).$$

Combining these two equations, we complete the proof of Theorem 2. \square

3. The Proof of Theorem 3 and the Askey–Wilson Integral

Proof. If $f(y)$ is replaced by $f(y)(wy; q)_\infty / (auvy, buvy, ry; q)_\infty$ in Theorem 1, then, we deduce that:

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_\infty}{(ax, bx; q)_\infty} d_q x \int_c^d \frac{f(y)(qy/c, qy/d, abuvxy, wy; q)_\infty}{(auvy, buvy, xy, ry; q)_\infty} d_q y \\ &= \frac{\Delta(u, v)(abuv; q)_\infty}{(au, av, bu, bv; q)_\infty} \int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_\infty}{(uy, vy, ry; q)_\infty} d_q y. \end{aligned} \tag{1}$$

Noting the definition of $\Delta(u, v)$ in Definition 2 and by a direct computation, we deduce that:

$$(e^{i\theta} - e^{-i\theta})\Delta(e^{i\theta}, e^{-i\theta}) = (1 - q)(q; q)_\infty h(\cos 2\theta; 1).$$

Keeping this in mind and replacing (u, v) by $(e^{i\theta}, e^{-i\theta})$ in (1), we conclude that:

$$\begin{aligned} & (e^{i\theta} - e^{-i\theta}) \int_{e^{i\theta}}^{e^{-i\theta}} (qxe^{i\theta}, qxe^{-i\theta}; q)_{\infty} I(x) d_q x \\ &= \frac{(1-q)(q, ab; q)_{\infty} h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_{\infty}}{h(\cos \theta; y)(ry; q)_{\infty}} d_q y, \end{aligned} \quad (2)$$

where

$$I(x) := \frac{1}{(ax, bx; q)_{\infty}} \int_c^d \frac{f(y)(qy/c, qy/d, abxy, wy; q)_{\infty}}{(ay, by, xy, ry; q)_{\infty}} d_q y.$$

It is easily seen that $I(x)$ is analytic near $x = 0$. Thus, there exists a sequence $\{a_k\}_{k=0}^{\infty}$ independent of x such that:

$$I(x) = a_0 + \sum_{k=1}^{\infty} a_k x^k.$$

By setting $x = 0$ in the above equation, we immediately conclude that:

$$a_0 = \int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_{\infty}}{(ay, by, ry; q)_{\infty}} d_q y. \quad (3)$$

Using the definition of the q -integral, we find that the left-hand side of (2) equals:

$$\begin{aligned} & (1-q)(1-e^{-2i\theta}) \sum_{n=0}^{\infty} (q^{n+1}; q)_{\infty} (q^{n+1}e^{-2i\theta}; q)_{\infty} I(q^n e^{-i\theta}) q^n \\ &+ (1-q)(1-e^{2i\theta}) \sum_{n=0}^{\infty} (q^{n+1}; q)_{\infty} (q^{n+1}e^{2i\theta}; q)_{\infty} I(q^n e^{i\theta}) q^n. \end{aligned}$$

Inspecting the first series in the above equation, we see that this series can be expanded in terms of the negative powers of $\{e^{-ki\theta}\}_{k=0}^{\infty}$, and the constant term of the Fourier expansion of this series is $(1-q)a_0$, since

$$\sum_{n=0}^{\infty} (q^{n+1}; q)_{\infty} q^n = \sum_{n=0}^{\infty} \frac{(q; q)_{\infty}}{(q; q)_n} q^n = \frac{(q; q)_{\infty}}{(q; q)_{\infty}} = 1$$

by the binomial theorem (see, for example [5] (1.3.15)). Thus, there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ independent of θ such that the first series equals

$$(1-q)a_0 + \sum_{k=1}^{\infty} \alpha_k e^{-ik\theta}.$$

On replacing θ by $-\theta$, we immediately find that the second series is equal to

$$(1-q)a_0 + \sum_{k=1}^{\infty} \alpha_k e^{ik\theta}.$$

It follows that:

$$\begin{aligned} & (e^{i\theta} - e^{-i\theta}) \int_{e^{i\theta}}^{e^{-i\theta}} (qxe^{i\theta}, qxe^{-i\theta}; q)_{\infty} I(x) d_q x \\ &= 2(1-q)a_0 + 2 \sum_{k=1}^{\infty} \alpha_k \cos k\theta. \end{aligned}$$

Comparing this equation with (2), we are led to the Fourier series expansion

$$\begin{aligned} & 2(1-q)a_0 + 2 \sum_{k=1}^{\infty} \alpha_k \cos k\theta \\ &= \frac{(1-q)(q, ab; q)_{\infty} h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \int_c^d \frac{f(y)(qy/c, qy/d, wy; q)_{\infty}}{h(\cos \theta; y)(ry; q)_{\infty}} d_q y. \end{aligned}$$

If we integrate this equation with respect to θ over $[-\pi, \pi]$, using the well known fact

$$\int_{-\pi}^{\pi} (\cos k\theta) d\theta = 2\pi \delta_{k,0},$$

and noting that the integrand is an even function of θ , we immediately deduce that

$$\frac{2\pi a_0}{(q, ab; q)_{\infty}} = \int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \int_c^d \left(\frac{f(y)(qy/c, qy/d, wy; q)_{\infty}}{h(\cos \theta; y)(ry; q)_{\infty}} d_q y \right) d\theta$$

Substituting (3) into the left-hand side of this equation, we complete the proof of Theorem 3. \square

Theorem 3 can be used to give a very simple derivation of the Askey–Wilson integral [8].

Theorem 8. If $\max\{|a|, |b|, |c|, |d|\} < 1$, then we have:

$$\int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} d\theta = \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}.$$

Proof. Choosing $f(y) = (ry; q)_{\infty} / (wy; q)_{\infty}$ in Theorem 3, we conclude that

$$\begin{aligned} & \int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \left(\int_c^d \frac{(qy/c, qy/d; q)_{\infty}}{h(\cos \theta; y)} d_q y \right) d\theta \\ &= \frac{2\pi}{(q, ab; q)_{\infty}} \int_c^d \frac{(qy/c, qy/d; q)_{\infty}}{(ay, by; q)_{\infty}} d_q y. \end{aligned}$$

Appealing to the Andrews–Askey integral in Proposition 3, we arrive at

$$\begin{aligned} \int_c^d \frac{(qy/c, qy/d; q)_{\infty}}{h(\cos \theta; y)} d_q y &= \frac{\Delta(c, d)(cd; q)_{\infty}}{h(\cos \theta; c, d)}, \\ \int_c^d \frac{(qy/c, qy/d; q)_{\infty}}{(ay, by; q)_{\infty}} d_q y &= \frac{\Delta(c, d)(abcd; q)_{\infty}}{(ac, ad, bc, bd; q)_{\infty}}. \end{aligned}$$

Combining these three equations, we complete the proof of Theorem 8. \square

For other proofs of this integral formula, see [9–18].

Theorem 9. For $\max\{|a|, |b|, |c|, |d|\} < 1$, we have the q -beta integral formula:

$$\begin{aligned} & \int_0^{\pi} \frac{h(\cos 2\theta; 1)h(\cos \theta; w)}{h(\cos \theta; a, b, c, d)} d\theta \\ &= \frac{2\pi(w/c, w/d; q)_{\infty}}{\Delta(c, d)(q, ab, cd; q)_{\infty}} \int_c^d \frac{(qy/c, qy/d, wy; q)_{\infty}}{(ay, by, wy/cd; q)_{\infty}} d_q y. \end{aligned}$$

Proof. Choosing $f(y) = (ry; q)_\infty / (wy/cd; q)_\infty$ in Theorem 3, we arrive at

$$\begin{aligned} & \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} \left(\int_c^d \frac{(qy/c, qy/d, wy; q)_\infty}{h(\cos \theta; y)(wy/cd; q)_\infty} d_q y \right) d\theta \\ &= \frac{2\pi}{(q, ab; q)_\infty} \int_c^d \frac{(qy/c, qy/d, wy; q)_\infty}{(ay, by, wy/cd; q)_\infty} d_q y. \end{aligned}$$

On using the Al-Salam–Verma integral in Proposition 2, we conclude that

$$\int_c^d \frac{(qy/c, qy/d, wy; q)_\infty}{h(\cos \theta; y)(wy/cd; q)_\infty} d_q y = \frac{\Delta(c, d)h(\cos \theta; w)(cd; q)_\infty}{h(\cos \theta; c, d)(w/c, w/d; q)_\infty}.$$

Combining the above two equations, we complete the proof of Theorem 9. \square

4. The Rogers–Szegő Polynomials and the Proof Theorem 4

4.1. The Rogers–Szegő Polynomials and q -Hermite Polynomials

The Rogers–Szegő polynomials play an important role in the theory of orthogonal polynomials, which are defined by (see, for example, [19] (Definition 1.2)):

$$h_n(a, b|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}.$$

By multiplying two copies of the q -binomial theorem (see, for example, [5] (p. 8, Equation (1.3.2))), we readily find that

$$\sum_{n=0}^{\infty} h_n(a, b|q) \frac{t^n}{(q; q)_n} = \frac{1}{(at, bt; q)_\infty}, \quad |at| < 1, |bt| < 1. \quad (4)$$

The continuous q -Hermite polynomials $H_n(\cos \theta|q)$ is defined by:

$$H_n(\cos \theta|q) := h_n(e^{-i\theta}, e^{i\theta}|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}.$$

Using (4), one can easily find the following generating function for the q -Hermite polynomials [16] (Equation (5.3)):

$$\sum_{n=0}^{\infty} H_n(\cos \theta|q) \frac{t^n}{(q; q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \frac{1}{h(\cos \theta; t)}, \quad \text{for } |t| < 1. \quad (5)$$

Proposition 7. For $|t| < 1$, the following series converges uniformly on $0 \leq \theta \leq \pi$:

$$\sum_{n=0}^{\infty} H_n(\cos \theta|q) \frac{t^n}{(q; q)_n},$$

and we also have

$$\frac{1}{|(te^{i\theta}, te^{-i\theta}; q)_\infty|} \leq \frac{1}{(|t; q)_\infty^2}.$$

Proof. It is easily seen that $|H_n(x|q)| \leq H_n(1|q)$ for $-1 \leq x \leq 1$, and the series in (5) converges at $x = 1$. Thus, the series in (5) converges for $|t| < 1$ uniformly on $\theta \in [-\pi, \pi]$.

Using $|H_n(x|q)| \leq H_n(1|q)$ for $-1 \leq x \leq 1$, we easily find that for any $|t| < 1$,

$$\left| \frac{1}{(te^{i\theta}; te^{-i\theta}; q)_\infty} \right| \leq \sum_{n=0}^{\infty} H_n(1|q) \frac{|t|^n}{(q; q)_n} = \frac{1}{(|t|; q)_\infty^2}.$$

□

Proposition 8. If k is an non-negative integer, then, for $\max\{|a|, |b|, |c|, |d|, |u|\} < 1$, $A_k(\theta)$ is bounded on $[-\pi, \pi]$, where $A_k(\theta)$ is given by:

$$A_k(\theta) = \frac{h(\cos 2\theta; 1)h(\cos \theta; w)(te^{i\theta}; te^{-i\theta}; q)_k}{h(\cos \theta; a, b, c, d, u)}.$$

Proof. On applying the inequality in Proposition 7, we immediately deduce that

$$\left| \frac{1}{h(\cos \theta; a, b, c, d, u)} \right| \leq \frac{1}{(|a|, |b|, |c|, |d|, |u|; q)_\infty^2}.$$

Using the triangle inequality, we easily find that $|h(\cos 2\theta; 1)| \leq 4(-q; q)_\infty^2$, and

$$|h(\cos \theta; w)| \leq (-|w|; q)_\infty^2, \quad |(te^{i\theta}; te^{-i\theta}; q)_k| \leq (-|t|; q)_\infty^2. \quad (6)$$

It follows that $A_k(\theta)$ is bounded on $[-\pi, \pi]$. This completes the proof of the proposition. □

Proposition 9. If k is an non-negative integer, then, for $\max\{|a|, |b|, |c|, |d|, |u|\} < 1$, $\int_0^\pi A_k(\theta) d\theta$ is an analytic function of a, b, c, d, u .

Proof. From Proposition 7, we know that, for $|a| < 1$, the following series is convergent uniformly on $0 \leq \theta \leq \pi$:

$$\sum_{n=0}^{\infty} H_n(\cos \theta | q) \frac{a^n}{(q; q)_n} = \frac{1}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}.$$

Thus, for $|a| < 1$, the integrand of the integral in Proposition 9 can be expanded into the following series, which is convergent uniformly on $0 \leq \theta \leq \pi$:

$$\sum_{n=0}^{\infty} \frac{a^n H_n(\cos \theta | q) h(\cos 2\theta; 1) h(\cos \theta; w) (te^{i\theta}; te^{-i\theta}; q)_k}{h(\cos \theta; b, c, d, u) (q; q)_n}.$$

On replacing the integrand of the integral in Proposition 9 by this series and then integrating term by term, one can see that the resulting series is of the form

$$\sum_{n=0}^{\infty} \frac{a^n}{(q; q)_n} \int_0^\pi \frac{H_n(\cos \theta | q) h(\cos 2\theta; 1) h(\cos \theta; w) (te^{i\theta}; te^{-i\theta}; q)_k}{h(\cos \theta; b, c, d, u)} d\theta. \quad (7)$$

Using the triangle inequality and inequalities in Proposition 7 and (6), we find that

$$\left| \sum_{n=0}^{\infty} \frac{a^n}{(q; q)_n} \int_0^\pi \frac{H_n(\cos \theta | q) h(\cos 2\theta; 1) h(\cos \theta; w) (te^{i\theta}; te^{-i\theta}; q)_k}{h(\cos \theta; b, c, d, u)} d\theta \right| \leq \frac{4\pi(-q, -|w|, -|t|; q)_\infty^2}{(|a|, |b|, |c|, |d|, |u|; q)_\infty^2}.$$

It follows that the infinite series in (7) is uniformly and absolutely convergent. Thus, the integral in Proposition 9 is an analytic function of a for $|a| < 1$. By symmetry, we know that the integral is also analytic for b, c, d, u . □

Proposition 10. For $\max\{|a|, |b|, |c|, |d|, |u|, |abs/t|\} < 1$, the following integral is an analytic function of a and b :

$$\int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos \theta; w)}{h(\cos \theta; a, b, c, d, u)} {}_3\phi_2 \left(\begin{matrix} te^{i\theta}, te^{-i\theta}, t/s \\ at, bt \end{matrix}; q, \frac{abs}{t} \right) d\theta.$$

Proof. Using the definition of $A_k(\theta)$ in Proposition 8, we find that the integrand of the integral in Proposition 10 can be written as:

$$\sum_{k=0}^{\infty} \frac{(t/s; q)_k}{(q, at, bt; q)_k} \left(\frac{abs}{t} \right)^k A_k(\theta). \quad (8)$$

By the ratio test, we easily find that the following series is absolutely convergent for $|abs/t| < 1$:

$$\sum_{k=0}^{\infty} \frac{(t/s; q)_k}{(q, at, bt; q)_k} \left(\frac{abs}{t} \right)^k.$$

Thus, the series in (8) is absolutely and uniformly convergent on $[-\pi, \pi]$. Substituting the series in (8) into the integral in Proposition 10 and then integrating term by term, we obtain

$$\sum_{k=0}^{\infty} \frac{(t/s; q)_k}{(q, at, bt; q)_k} \left(\frac{abs}{t} \right)^k \int_0^\pi A_k(\theta) d\theta.$$

The above series is uniformly convergent and every term is an analytic function of a and b , so the series converges to an analytic function of a and b . This completes the proof of the proposition. \square

Using the same argument used in Proposition 10, we can prove the following proposition.

Proposition 11. The following q -integral is an analytic function of a and b at $(0, 0) \in \mathbb{C}^2$:

$$\int_c^d \frac{(qy/c, qy/d, wy, absuy/t; q)_\infty}{(ay, by, uy, wy/cd; q)_\infty} {}_3\phi_2 \left(\begin{matrix} t/s, t/u, t/y \\ at, bt \end{matrix}; q, \frac{absuy}{t} \right) d_q y.$$

4.2. q -Partial Differential Equations

For any function $f(x)$ of one variable, the q -derivative of $f(x)$ with respect to x , is defined as:

$$\mathcal{D}_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{x},$$

and we further define $\mathcal{D}_{q,x}^0\{f\} = f$, and for $n \geq 1$, $\mathcal{D}_{q,x}^n\{f\} = \mathcal{D}_{q,x}\{\mathcal{D}_{q,x}^{n-1}\{f\}\}$.

Now, we give the definitions of the q -partial derivative and the q -partial differential equations.

Definition 5. A q -partial derivative of a function of several variables is its q -derivative with respect to one of those variables, regarding other variables as constants. The q -partial derivative of a function f with respect to the variable x is denoted by $\partial_{q,x}\{f\}$.

Definition 6. A q -partial differential equation is an equation that contains unknown multivariable functions and their q -partial derivatives.

It turns out that the q -partial differential equation methods are useful for deriving q -formulas [16,19–21]. The following useful expansion theorem for q -series can be found in [19] (Proposition 1.6).

Theorem 10. If $f(x, y)$ is a two-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then f can be expanded in terms of $h_n(x, y|q)$ if and only if f satisfies the q -partial differential equation $\partial_{q,x}\{f\} = \partial_{q,y}\{f\}$.

Proposition 12. If we use $L(a, b, u, v, s, t)$ to denote

$$\frac{(av, bv, abstu/v; q)_\infty}{(as, at, au, bs, bt, bu; q)_\infty} {}_3\phi_2 \left(\begin{matrix} v/s, v/t, v/u \\ av, bv \end{matrix}; q, \frac{abstu}{v} \right),$$

then $L(a, b, u, v, s, t)$ satisfies the q -partial differential equation $\partial_{q,a}\{L\} = \partial_{q,b}\{L\}$.

Proof. If we set $v = cst$, then the q -integral formula in [19] (Proposition 13.8) becomes

$$\int_s^t \frac{(qx/s, qx/t, abux; q)_\infty}{(ax, bx, vx/st; q)_\infty} d_q x = \frac{\Delta(s, t)(av, bv, abstu/v; q)_\infty}{(as, bs, at, bt, v/s, v/t; q)_\infty} \times {}_3\phi_2 \left(\begin{matrix} v/s, v/t, v/u \\ av, bv \end{matrix}; q, \frac{abstu}{v} \right).$$

It follows that

$$L(a, b, u, v, s, t) = \frac{(v/s, v/t; q)_\infty}{\Delta(s, t)(au, bu; q)_\infty} \int_s^t \frac{(qx/s, qx/t, abux; q)_\infty}{(ax, bx, vx/st; q)_\infty} d_q x.$$

By a direct computation, we find that

$$\begin{aligned} \partial_{q,a}\{L\} &= \partial_{q,b}\{L\} \\ &= \frac{(v/s, v/t; q)_\infty}{\Delta(s, t)(au, bu; q)_\infty} \int_s^t \frac{(x + u - (a + b)ux)(qx/s, qx/t, abux; q)_\infty}{(ax, bx, vx/st; q)_\infty} d_q x. \end{aligned}$$

This completes the proof of Proposition 12. \square

4.3. The Proof of Theorem 4

Proof. From Propositions 10 and 11, we know that both sides of the equation in Theorem 4 are all analytic functions of a and b at $(0, 0) \in \mathbb{C}^2$.

Using the definition of $L(a, b, u, v, s, t)$ in Proposition 12 and a simple calculation, we find that

$$\begin{aligned} &L(a, b, s, t, e^{i\theta}, e^{-i\theta}) \\ &= \frac{(at, bt, abs/t; q)_\infty}{(as, bs; q)_\infty h(\cos \theta; a, b)} {}_3\phi_2 \left(\begin{matrix} te^{i\theta}, te^{-i\theta}, t/s \\ at, bt \end{matrix}; q, \frac{abs}{t} \right) \\ &L(a, b, u, t, s, y) \\ &= \frac{(at, bt, absyu/t; q)_\infty}{(as, ay, au, bs, by, bu; q)_\infty} {}_3\phi_2 \left(\begin{matrix} t/s, t/y, t/u \\ at, bt \end{matrix}; q, \frac{absuy}{t} \right). \end{aligned}$$

Using these two equations, the q -integral formula in Theorem 4 can be rewritten as:

$$\begin{aligned} &\int_0^\pi \frac{L(a, b, s, t, e^{i\theta}, e^{-i\theta}) h(\cos 2\theta; 1) h(\cos \theta; w)}{h(\cos \theta; c, d, u)} d\theta \\ &= \frac{2\pi(w/c, w/d; q)_\infty}{\Delta(c, d)(q, cd; q)_\infty} \int_c^d L(a, b, u, t, s, y) \frac{(qy/c, qy/d, wy; q)_\infty}{(uy, wy/cd; q)_\infty} d_q y. \end{aligned} \tag{9}$$

If we use $f(a, b)$ to denote the left-hand side of (9), then it satisfies the partial q -differential equation $\partial_{q,a}\{f\} = \partial_{q,b}\{f\}$ by Proposition 12. Thus, there exists a sequence $\{\alpha_n\}$ independent of a and b such that

$$f(a, b) = \sum_{n=0}^{\infty} \alpha_n h_n(a, b|q).$$

Setting $b = 0$ in this equation, using the obvious fact $h_n(a, 0|q) = a^n$ and the identity

$$L(a, 0, s, t, e^{i\theta}, e^{-i\theta}) = \frac{(at; q)_{\infty}}{(as; q)_{\infty} h(\cos \theta; a)}$$

in the resulting equation, we immediately conclude that

$$f(a, 0) = \frac{(at; q)_{\infty}}{(as; q)_{\infty}} \int_0^{\pi} \frac{h(\cos 2\theta; 1) h(\cos \theta; w)}{h(\cos \theta; a, c, d, u)} d\theta = \sum_{n=0}^{\infty} \alpha_n a^n. \quad (10)$$

If we use $g(a, b)$ to denote the right-hand side of (9), then it satisfies the partial q -differential equation $\partial_{q,a}\{g\} = \partial_{q,b}\{g\}$ by Proposition 12. Thus, there exists a sequence $\{\beta_n\}$ independent of a and b such that

$$g(a, b) = \sum_{n=0}^{\infty} \beta_n h_n(a, b|q).$$

Putting $b = 0$ in this equation, substituting $h_n(a, 0|q) = a^n$ and the identity

$$L(a, 0, u, t, s, y) = \frac{(at; q)_{\infty}}{(as, ay, au; q)_{\infty}}$$

in the resulting equation, we are led to the expansion formula

$$\frac{2\pi(w/c, w/d, at; q)_{\infty}}{\Delta(c, d)(q, au, cd, as; q)_{\infty}} \int_c^d \frac{(qy/c, qy/d, wy; q)_{\infty}}{(ay, uy, wy/cd; q)_{\infty}} d_q y = \sum_{n=0}^{\infty} \beta_n a^n.$$

On replacing b by u in the equation in Theorem 9, we conclude that

$$\begin{aligned} & \int_0^{\pi} \frac{h(\cos 2\theta; 1) h(\cos \theta; w) d\theta}{h(\cos \theta; a, u, c, d)} \\ &= \frac{2\pi(w/c, w/d; q)_{\infty}}{\Delta(c, d)(q, au, cd; q)_{\infty}} \int_c^d \frac{(qy/c, qy/d, wy; q)_{\infty}}{(ay, uy, wy/cd; q)_{\infty}} d_q y. \end{aligned}$$

Combining these two equations, we deduce that

$$\frac{(at; q)_{\infty}}{(as; q)_{\infty}} \int_0^{\pi} \frac{h(\cos 2\theta; 1) h(\cos \theta; w) d\theta}{h(\cos \theta; a, u, c, d)} = \sum_{n=0}^{\infty} \beta_n a^n.$$

Combining this equation with (10), we obtain the power series identity

$$\sum_{n=0}^{\infty} \alpha_n a^n = \sum_{n=0}^{\infty} \beta_n a^n.$$

It follows that $\alpha_n = \beta_n$ for $n = 0, 1, \dots$, which completes the proof of Theorem 4. \square

5. Conclusions

Using the same method used in the proof of Theorem 4, we can also prove the following double q -integral formula.

Theorem 11. If we use $\Phi(x, y)$ to denote the following function of x and y :

$$\frac{(abuvxy, rbuvxy, cdvwx; q)_{\infty}}{(auvy, buvy, xy, wy, ruvy; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} buvy, bx, xy \\ abuvxy, rbuvxy \end{matrix}; q, aruv \right),$$

then, we have the double q -integral formula

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_{\infty}}{(ax, bx, rx; q)_{\infty}} d_q x \int_c^d (qy/c, qy/d; q)_{\infty} \Phi(x, y) d_q y \\ &= \frac{\Delta(c, d) \Delta(u, v) (abuv, cdv, cdv, cdv, rbuv; q)_{\infty}}{(au, av, bu, bv, cu, cv, du, dv, ru, rv, cw, dw; q)_{\infty}}. \end{aligned}$$

Next, we will use Proposition 12 and Theorem 10 to give a proof of the theorem.

Proof. Using the definition of $L(a, b, u, v, s, t)$ in Proposition 12, we easily find that

$$\begin{aligned} & L(a, r, x, buvxy, uv, buv) \\ &= \frac{(abuvxy, rbuvxy, aruv; q)_{\infty}}{(ax, rx, abuv, rbuv, auvy, ruvy; q)_{\infty}} {}_3\phi_2 \left(\begin{matrix} buvy, bx, xy \\ abuvxy, rbuvxy \end{matrix}; q, aruv \right). \end{aligned} \quad (11)$$

For the sake of brevity, we temporarily use $I(a, r, x, y)$ to denote the expression

$$\frac{(cdvwx; q)_{\infty}}{(buvy, xy, wy; q)_{\infty}} L(a, r, x, buvxy, uv, buv).$$

Using this expression and (11), the q -integral formula in Theorem 11 can be rewritten as:

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_{\infty}}{(bx; q)_{\infty}} d_q x \int_c^d (qy/c, qy/d; q)_{\infty} I(a, r, x, y) d_q y \\ &= \frac{\Delta(c, d) \Delta(u, v) (cdv, cdv, cdv, aruv; q)_{\infty}}{(au, av, bu, bv, cu, cv, du, dv, ru, rv, cw, dw; q)_{\infty}}. \end{aligned} \quad (12)$$

If we use $f(a, r)$ to denote the right-hand of the above equation, then, using the same method as that used in the proof of Proposition 10, we can prove that $f(a, r)$ is analytic at $(0, 0) \in \mathbb{C}^2$. By Proposition 12, one can show that $f(a, r)$ satisfies the partial q -differential equation $\partial_{q,a}\{f\} = \partial_{q,r}\{f\}$. Hence, by Theorem 10, there exists a sequence $\{\lambda_n\}$ independent of a and r such that

$$f(a, r) = \sum_{n=0}^{\infty} \lambda_n h_n(a, r|q).$$

Setting $r = 0$ in this equation, substituting the equations $h_n(a, 0|q) = a^n$ and

$$L(a, 0, x, buvxy, uv, buv) = \frac{(abuvxy; q)_{\infty}}{(ax, auvy, abuv; q)_{\infty}}$$

in the resulting equation, we conclude that

$$\begin{aligned} & \int_u^v \frac{(qx/u, qx/v; q)_{\infty}}{(ax, bx; q)_{\infty}} d_q x \int_c^d \frac{(qy/c, qy/d, abuvxy, cdvwx; q)_{\infty}}{(auvy, buvy, xy, wy; q)_{\infty}} d_q y \\ &= (abuv; q)_{\infty} \sum_{n=0}^{\infty} \lambda_n a^n. \end{aligned}$$

Using Proposition 6, we know that the left-hand side of the above equation equals

$$\frac{\Delta(u, v)\Delta(c, d)(abuv, cdv, cdw, cdvw; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, cw, dw; q)_\infty}.$$

It follows that

$$\sum_{n=0}^{\infty} \lambda_n a^n = \frac{\Delta(u, v)\Delta(c, d)(cdv, cdw, cdvw; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, cw, dw; q)_\infty}. \quad (13)$$

If we use $g(a, r)$ to denote the right-hand side of (12), then, by a direct computation, we find that

$$\partial_{q,a}\{g(a, r)\} = \partial_{q,r}\{g(a, r)\} = \frac{u + v - (a + r)uv}{1 - aruv} g(a, r).$$

Thus, by Theorem 10, there exists a sequence $\{\delta_n\}$ independent of a and r such that

$$g(a, r) = \sum_{n=0}^{\infty} \delta_n h_n(a, r|q).$$

Setting $r = 0$ in this equation and using the fact $h_n(a, 0|q) = a^n$, we arrive at

$$g(a, 0) = \sum_{n=0}^{\infty} \delta_n a^n = \frac{\Delta(u, v)\Delta(c, d)(cdv, cdw, cdvw; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv, cw, dw; q)_\infty}. \quad (14)$$

Comparing Equations (13) and (14), we have the power series identity

$$\sum_{n=0}^{\infty} \lambda_n a^n = \sum_{n=0}^{\infty} \delta_n a^n.$$

Hence, we find for $n = 0, 1, \dots$, that $\lambda_n = \delta_n$. This completes the proof of Theorem 11. \square

Remark 1. Proceedings through the same steps used to derive Theorem 3 from Theorem 1, setting $u = e^{i\theta}$ and $v = e^{-i\theta}$ in the equation in Theorem 11, and then integrating both sides of the resulting equation with respect to θ over $[-\pi, \pi]$, we can give a new proof of Theorem 5.

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