symmetry

MDPI

*Article*

# Coherent States of Harmonic and Reversed Harmonic Oscillator

**Alexander Rauh**

Department of Physics, University of Oldenburg, Oldenburg D-26111, Germany; alexander.rauh@uni-oldenburg.de; Tel.:+49-441-798-3460

Academic Editor: Young Suh Kim
Received: 16 January 2016; Accepted: 3 June 2016; Published: 13 June 2016

**Abstract:** A one-dimensional wave function is assumed whose logarithm is a quadratic form in the configuration variable with time-dependent coefficients. This trial function allows for general time-dependent solutions both of the harmonic oscillator (HO) and the reversed harmonic oscillator (RO). For the HO, apart from the standard coherent states, a further class of solutions is derived with a time-dependent width parameter. The width of the corresponding probability density fluctuates, or "breathes" periodically with the oscillator frequency. In the case of the RO, one also obtains normalized wave packets which, however, show diffusion through exponential broadening with time. At the initial time, the integration constants give rise to complete sets of coherent states in the three cases considered. The results are applicable to the quantum mechanics of the Kepler-Coulomb problem when transformed to the model of a four-dimensional harmonic oscillator with a constraint. In the classical limit, as was shown recently, the wave packets of the RO basis generate the hyperbolic Kepler orbits, and, by means of analytic continuation, the elliptic orbits are also obtained quantum mechanically.

**Keywords:** inverted harmonic oscillator; harmonic trap; Kepler-Coulomb problem; Kustaanheimo-Stiefel transformation

## 1. Introduction

Coherent states of the harmonic oscillator (HO) were introduced already at the beginning of wave mechanics [1]. Much later, such states were recognized as being useful as a basis to describe radiation fields [2] and optical correlations [3]. The reversed harmonic oscillator (RO) refers to a model with repulsive harmonic forces, and was discussed in [4] in the context of irreversibility. Recently, in [5], which also communicates historical remarks, the RO was applied to describe nonlinear optical phenomena. As mentioned in [5], the term "inverted harmonic oscillator" (IO) originally refers to a model with negative kinetic and potential energy, as proposed in [6]. Nevertheless, most articles under the headline IO, actually consider the RO model, see, e.g., [7–9].

The RO model formally can be obtained by assuming a purely imaginary oscillator frequency. It is then not anymore possible to construct coherent states by means of creation and annihilation operators; for a text book introduction see [10]. In [9], the RO was generalized by the assumption of a time-dependent mass and frequency. The corresponding Schrödinger equation was solved by means of an algebraic method with the aim to describe quantum tunneling.

In the present study, emphasis is laid on the derivation of complete sets of coherent states both for the HO and the RO model, together with their time evolution. In the case of the HO, in addition to the standard coherent states, a further function set is found with a time-dependent width parameter. Both in the HO and RO case, the integration constants of the time-dependent solutions induce complete function sets which, at time $t = 0$, are isomorphic to the standard coherent states of the HO.

In Section 6, an application to the quantum mechanics of the Kepler-Coulomb problem will be briefly discussed. As has first been observed by Fock [11], the underlying four-dimensional rotation symmetry of the non-relativistic Hamiltonian of the hydrogen atom permits the transformation to the problem of four isotropic harmonic oscillators with a constraint; for applications see, e.g., [12–14]. The transformation proceeds conveniently by means of the Kustaanheimo-Stiefel transformation [15]. In [14], the elliptic Kepler orbits were derived in the classical limit on the basis of coherent HO states. By means of coherent RO states, the classical limit for hyperbolic Kepler orbits was achieved in [16,17], whereby the elliptic regime could be obtained by analytic continuation from the hyperbolic side. Recently, by means of the same basis, a first order quantum correction to Kepler's equation was derived in [18], whereby the smallness parameter was defined by the reciprocal angular momentum in units of $\hbar$.

As compared to the classical elliptic Kepler orbits, the derivation of hyperbolic orbits from quantum mechanics was accomplished quite recently [16,17]. For this achievement, it was crucial to devise a suitable time-dependent ansatz for the wave function, see (1) below, in order to construct coherent RO states. As it turns out, the wave function (1) contains also the usual coherent HO states, and, unexpectedly, a further set of coherent states, which we call type-II states. The latter are characterized by a time-dependent width parameter and are solutions of the time-dependent Schrödinger equation of the HO. Section 4 contains the derivation. Essentially, the type-II states offer a disposable width parameter which allows us, for instance, to describe arbitrarily narrowly peaked initial states together with their time evolution in a harmonic potential. In this paper, a unified derivation is presented of coherent states of the HO, RO, and type-II HO states. Furthermore, the connection of HO and RO with the quantum mechanics of the Kepler-Coulomb problem is briefly discussed in the context of the derivation of the classical Kepler orbits from quantum mechanics.

## 2. Introducing a Trial Wave Function

In order to solve the Schrödinger equation for the harmonic oscillator (HO) and the reversed oscillator (RO), a trial wave function of Gaussian type is assumed as follows

$$\psi(x,t) = C_0 \exp\left[C(t) + B(t)x - \Gamma(t)x^2\right], \quad x \in \mathbf{R}, \quad \mathrm{Real}(\Gamma) > 0, \tag{1}$$

where $C, B, \Gamma$ are complex functions of time $t$ and $C_0$ the time-independent normalization constant. When the Schrödinger operator $[\mathrm{i}\,\hbar\partial_t - H]$ is applied to $\psi$ for a Hamiltonian with harmonic potential, then the wave function $\psi$ is reproduced up to a factor which is a quadratic polynomial and must vanish identically in the configuration variable $x$:

$$0 = p_0(t) + p_1(t)x + p_2(t)x^2. \tag{2}$$

The conditions $p_0 = 0$, $p_1 = 0$, and $p_2 = 0$, give rise to three first-order differential equations for the functions $C(t)$, $B(t)$, and $\Gamma(t)$. In the following we examine two cases for the HO: type-I and type-II are characterized by a constant and time-dependent function $\Gamma$, respectively. In the case of the RO, only a time-dependent $\Gamma$ leads to a solution. By a suitable choice of the parameters, the ansatz (1) solves the time-dependent Schrödinger equation both for the HO and the RO Hamiltonian

$$H = p^2/(2m) + (m\omega^2/2)x^2 \quad \text{and} \quad H_\Omega = p^2/(2m) - (m\Omega^2/2)x^2, \quad \omega,\ \Omega > 0,$$

respectively.

## 3. Standard (Type-I) Coherent States of the HO

In the following, the time-dependent solutions are derived, within the trial function scheme, for the Hamiltonian

$$H = p^2/(2m) + (m\omega^2/2)x^2 = (\hbar\omega/2)\left[-\partial_\zeta^2 + \zeta^2\right], \tag{3}$$

where $\zeta = \alpha x$ is dimensionless with $\alpha^2 = m\omega/\hbar$. For later comparison, we list the standard definition of coherent states from the textbook [10], see Equations (4.72) and (4.75):

$$|z\rangle = \exp\left[-\frac{1}{2}zz^*\right]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle, \tag{4}$$

$$\psi_z(\zeta) = \pi^{-1/4}\exp\left[-\frac{1}{2}(zz^*+z^2)\right]\exp\left[-\frac{1}{2}\zeta^2+\sqrt{2}\zeta z\right],\ \zeta=\alpha x,\ \alpha^2=\frac{m\omega}{\hbar}, \tag{5}$$

where $\psi_z(\zeta) = \langle\zeta|z\rangle$, $|n\rangle$ denotes the n-th energy eigenvector, and the star superscript means complex conjugation. The time evolution gives rise to, see [10],

$$|z,t\rangle = \exp[-\mathrm{i}\,\omega t/2]\,|z\,\exp[-\mathrm{i}\,\omega t]\rangle, \tag{6}$$

$$\psi_z(\zeta,t) = \exp[-\mathrm{i}\,\omega t/2]\psi_{(z\,\exp[-\mathrm{i}\,\omega t])}(\zeta). \tag{7}$$

The state $|z\rangle$ is minimal with respect to the position-momentum uncertainty product $\Delta x\,\Delta p$, and there exists the following completeness property, see [3],

$$\frac{1}{\pi}\int_0^{\infty}u\mathrm{d}u\int_0^{2\pi}\mathrm{d}\varphi\,|z\rangle\langle z| = \sum_n|n\rangle\langle n|,\quad z=u\exp[\mathrm{i}\,\varphi]. \tag{8}$$

The relation (8) follows immediately from the definition (4). An equivalent statement is

$$\frac{1}{\pi}\int_0^{\infty}u\mathrm{d}u\int_0^{2\pi}\mathrm{d}\varphi\,\langle\zeta_2|z\rangle\langle z|\zeta_1\rangle = \delta(\zeta_2-\zeta_1), \tag{9}$$

which corresponds to the completeness of the energy eigenfunctions of the harmonic oscillator. In Appendix B, we reproduce a proof of (9), which is appropriate, since the proof has to be extended to the modified coherent states in the type-II HO and the RO cases.

In terms of the scaled variables $\zeta$ and $\tau = t\omega$, the trial ansatz reads

$$\psi(\zeta,\tau) = C_0\exp\left[c(\tau)+\beta(\tau)\zeta-\gamma(\tau)\zeta^2/2\right], \tag{10}$$

where $c, \beta, \gamma$ are dimensionless functions of $\tau$, and the re-scaling factor of the probability density, $1/\sqrt{\alpha}$, is taken into the normalization constant $C_0$.

We assume that $\gamma = \gamma_0 = const$. Then, the polynomial (2) gives rise to the equations

$$\gamma_0^2 = 1,\quad \mathrm{i}\,\beta'(\tau) = \beta(\tau),\quad 2\mathrm{i}\,c'(t) = 1-\beta^2(t), \tag{11}$$

which implies that $\gamma_0 = 1$ is fixed. The further solutions emerge easily as

$$\beta(\tau) = C_2\exp[-\mathrm{i}\,\tau],\quad c(\tau) = -\mathrm{i}\,\tau/2-(C_2^2/4)\exp[-2\mathrm{i}\,\tau]+C_3, \tag{12}$$

where $C_2$ and $C_3$ are complex integration constants. A comparison with (5), at $t = 0$, suggests to set

$$C_2 = \sqrt{2}\,z,\quad C_3 = -(1/2)zz^*, \tag{13}$$

which specifies the functions $\beta$ and $c$ as follows

$$\beta(\tau) = \sqrt{2}\,(z\exp[-\mathrm{i}\,\tau)],\quad c(t) = -\mathrm{i}\,\tau/2-(1/2)\left[zz^*+(z\exp[-\mathrm{i}\,\tau])^2\right]. \tag{14}$$

The normalization integral with respect to $\zeta$ amounts to the condition

$$C_0^2\sqrt{\pi}\exp[zz^*] = 1; \tag{15}$$

hence (7) with (5) is reproduced.

## 4. Type-II Solutions of the Harmonic Oscillator

With $\gamma$ being a function of time, one obtains the following differential equations with prime denoting the derivative with respect to the scaled time $\tau$:

$$\mathrm{i}\,\gamma' = \gamma^2 - 1, \quad \mathrm{i}\,\beta' = \gamma\,\beta; \quad 2\mathrm{i}\,c' = \gamma - \beta^2. \tag{16}$$

The solution for $\gamma$ is

$$\gamma(\tau) = \frac{\exp(2\mathrm{i}\,\tau) - C_1}{\exp(2\mathrm{i}\,\tau) + C_1} \quad C_1 = \frac{1-\gamma_0}{1+\gamma_0}. \quad \gamma_0 = \gamma(0). \tag{17}$$

Splitting $\gamma$ into its real and imaginary parts, one can write

$$\begin{aligned} \gamma(\tau) &= \gamma_R + \mathrm{i}\,\gamma_I; \quad \gamma_R = (1-C_1^2)N_1^{-1}, \quad \gamma_I = 2\,C_1 N_1^{-1}\sin(2\tau), \\ N_1(\tau) &= 1 + C_1^2 + 2C_1\cos(2\tau) = 4(1+\gamma_0)^{-2}\left[1 + (\gamma_0^2 - 1)\sin^2(\tau)\right]. \end{aligned} \tag{18}$$

In order that the wave function is square integrable, $\gamma_R$ has to be positive, which implies that

$$C_1^2 < 1 \quad \text{or} \quad \gamma_0 > 0. \tag{19}$$

The initial value $\gamma(t=0) \equiv \gamma_0 > 0$ emerges as a disposable parameter.

The probability density, $P = |\psi(\zeta,\tau)|^2$, is characterized by a width of order of magnitude $d = 1/\sqrt{\gamma_R}$:

$$d(\tau) = \sqrt{\left[1 + (\gamma_0^2 - 1)\sin^2(\tau)\right]/\gamma_0}. \tag{20}$$

Obviously, the width fluctuates, or "breathes", periodically with time. Of course, this is not a breathing mode as observed in systems of confined interacting particles, see [19,20], e.g.,

Integration of the $\beta$ equation leads to

$$\beta = C_2\exp(\mathrm{i}\,\tau)\left[\exp(2\mathrm{i}\,\tau) + C_1\right]^{-1} = C_2 N_1^{-1}\left[\exp(-\mathrm{i}\,\tau) + \exp(\mathrm{i}\,\tau)\,C_1\right]. \tag{21}$$

Later on, the complex integration constant $C_2 \equiv A_2 + \mathrm{i}\,B_2$ will serve as a state label. The third differential equation of (16) amounts to

$$c(\tau) = \mathrm{i}\,\tau/2 - C_2^2\left[4\left(\exp(2\mathrm{i}\,\tau) + C_1\right)\right]^{-1} - (1/2)\ln\left(\sqrt{\exp(2\mathrm{i}\,\tau) + C_1}\right) + C_3. \tag{22}$$

By reasons explained in Appendix A, we dispose of the integration constant $C_3$ as follows

$$C_3 = -(1+\gamma_0)(8\gamma_0)^{-1}(A_2^2 + \gamma_0 B_2^2), \quad C_2 = A_2 + \mathrm{i}\,B_2. \tag{23}$$

In Appendix A, the probability density $P$ is derived in the following form

$$P(\xi,\tau) = \frac{C_0^2}{\sqrt{N_1}}\exp\left[-\gamma_R\left(\xi - \beta_R/\gamma_R\right)^2\right], \tag{24}$$

where the time-dependent functions $\gamma_R$ and $N_1$ are defined through (17) and (18), and $\beta_R$ comes out as

$$\beta_R(\tau) = (1/8)(1+\gamma_0)^{-1}N_1^{-1}\left[A_2\cos(\tau) + B_2\sin(\tau)\right]. \tag{25}$$

The complex integration constant $C_2$ corresponds to the familiar complex quantum number $z$ in the case of the standard coherent states; hence, the real numbers $A_2$, $B_2$ characterize different states. The normalization constant $C_0$ obeys the following condition, see Appendix A,

$$1 = (1/2)C_0^2\sqrt{\pi/\gamma_0}(1+\gamma_0). \tag{26}$$

### 4.1. Completeness of Type-II States

Combining the above results, we write the time-dependent wave function as follows

$$\psi(\xi,\tau) = \frac{C_0}{\sqrt{\exp(2\mathrm{i}\,\tau)+C_1}}\exp\left[C_3 - \frac{C_2^2\,(\exp(-2\mathrm{i}\,\tau)+C_1)}{4N_1} + \beta(\tau)\xi - \gamma(\tau)\xi^2/2\right], \tag{27}$$

where $\gamma$, $\beta$, and $C_3$ are defined in (18), (21), and (23), respectively. Let us consider $\psi$ at zero time:

$$\psi(\xi,0) = \frac{C_0}{\sqrt{1+C_1}}\exp\left[C_3 - \frac{C_2^2}{4(1+C_1)} + C_2(1+\gamma_0)\,\xi/2 - \gamma_0\xi^2/2\right]. \tag{28}$$

In (28), we set $\xi = \tilde{\xi}/\sqrt{\gamma_0}$ to write

$$\psi(\tilde{\xi},0) = \frac{C_0\gamma_0^{-1/4}}{\sqrt{1+C_1}}\exp\left[C_3 - \frac{C_2^2}{4(1+C_1)} + C_2(1+\gamma_0)/\sqrt{\gamma_0}\,\tilde{\xi}/2 - \tilde{\xi}^2/2\right]. \tag{29}$$

Now we substitute the complex variable $z$ for the integration constant $C_2$ as follows

$$C_2\frac{1+\gamma_0}{2\sqrt{\gamma_0}} = \sqrt{2}\,z \tag{30}$$

and obtain

$$\psi(\tilde{\xi},0) = \frac{C_0}{\sqrt{1+C_1}}\exp\left[C_3 - z^2\frac{\gamma_0}{1+\gamma_0} + \sqrt{2}\,z\,\tilde{\xi} - \tilde{\xi}^2/2\right]. \tag{31}$$

In $C_3$, given in (23), we make the following replacements which are induced by (30):

$$A_2 \to \kappa(z+z^*), \quad B_2 \to -\mathrm{i}\,\kappa(z-z^*), \quad \kappa = \sqrt{2\gamma_0}/(1+\gamma_0). \tag{32}$$

There occur some nice cancelations, and one obtains

$$\psi_z(\tilde{\xi}) = \frac{C_0\gamma_0^{-1/4}}{\sqrt{1+C_1}}\exp\left[-\frac{1}{2}\left(zz^*+z^2\right) + \mathrm{i}\,D + \sqrt{2}\,z\,\tilde{\xi} - \tilde{\xi}^2/2\right], \quad D = \frac{1-\gamma_0}{2(1+\gamma_0)}\mathrm{Im}(z^2). \tag{33}$$

Comparison with (5) shows that the wave function (33) has the same structure apart from the purely imaginary phase $\mathrm{i}\,D$. The latter drops out in the completeness proof, see (A15) in Appendix B. As a consequence, the states (33) form a complete set of states with respect to the state label $z$.

At $\tau = 0$, the states (33) differ from the standard coherent states (5) by the state dependent phase $D$, through the variables $\xi$ and $\tilde{\xi}$ which denote the differently scaled space variable $x$, and also through the different definition of the quantum number $z$, which for simplicity was denoted by the same symbol in (30). Essentially, type-I and type-II states differ by their time evolution and width parameter $\gamma_0$ which is equal to $\alpha^2 = m\omega/\hbar$ and to an arbitrary positive number, respectively.

### 4.2. Mean Values and Uncertainty Product

In the following, we list mean values for the time-dependent states (27) including the position momentum uncertainty product $\Delta_{xp}$. They are periodic in time with the oscillator angular frequency $\omega \equiv 2\pi/T$. The uncertainty product is minimal at the discrete times $t_n = (1/4)nT$, $n = 0, 1, \ldots$. For comparison, the traditional coherent states are always minimal [10]. We use the abbreviations $(\Delta_x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta_v)^2 = \langle v^2 \rangle - \langle v \rangle^2$ for the mean square deviations of position and velocity, respectively.

$$\langle x(\tau) \rangle = (1/\alpha)(1+\gamma_0)(2\gamma_0)^{-1} \left[A_2 \cos(\tau) + B_2 \gamma_0 \sin(\tau)\right]; \tag{34}$$

$$\langle v(\tau) \rangle = \hbar\alpha(2m\gamma_0)^{-1} \left[-A_2 \sin(\tau) + \gamma_0 B_2 \cos(\tau)\right]; \tag{35}$$

$$(\Delta_x)^2 = (4\alpha^2\gamma_0)^{-1} \left[1 + \gamma_0^2 + (1-\gamma_0^2)\cos(2\tau)\right]; \tag{36}$$

$$(\Delta_v)^2 = \hbar^2\alpha^2(4m^2\gamma_0)^{-1} \left[1 + \gamma_0^2 + (\gamma_0^2 - 1)\cos(2\tau)\right]; \tag{37}$$

$$\langle H \rangle = \hbar\omega(8\gamma_0^2)^{-1} \left[(1+\gamma_0)^2 \left(A_2^2 + \gamma_0^2 B_2^2\right) + 2\gamma_0(1+\gamma_0^2)\right]. \tag{38}$$

It is noticed that the mean square deviations do not depend on the state label $(A_2, B_2)$. The uncertainty product follows immediately from (36) and (37) as

$$\Delta_{xp} := (\Delta_x)^2(\Delta_p)^2 \equiv m^2(\Delta x)^2(\Delta v)^2 = \frac{\hbar^2}{16\gamma_0^2} \left[(1+\gamma_0^2)^2 - (1-\gamma_0^2)^2 \cos^2(2\tau)\right]. \tag{39}$$

In the special case $\gamma_0 = 1$, the product is always minimal. As a matter of fact, $\gamma_0 = 1$ is the type-I case of Section 3.

By (38), the mean energy does not depend on time and is positive definite, as it must be. The limit to the standard case with $\gamma_0 = 1$, gives the known result

$$\langle H \rangle_{\gamma_0=1} = \hbar\omega(zz^* + 1/2). \tag{40}$$

and the state with $z = 0$ is the ground state of the HO with zero point energy $\hbar\omega/2$.

## 5. Wave Packet Solutions for the RO

For convenience, we will keep the same symbols for the trial functions $\gamma(\tau)$, $\beta(\tau)$, and $c(\tau)$. Setting $\omega = \mathrm{i}\,\Omega$ with $\Omega > 0$, implies that $\alpha^2 = -m\Omega/\hbar$. In the coherent state (5), the exponential part, $-\zeta^2/2 \equiv -(m\omega/\hbar)x^2/2$, is then replaced by $+(m\Omega/\hbar)x^2/2$, which precludes normalization.

We introduce $1/\alpha_\Omega$ as the new length parameter and define the dimensionless magnitudes

$$\zeta = \alpha_\Omega x, \quad \tau = t\,\Omega, \quad \text{with} \quad \alpha_\Omega^2 = m\Omega/\hbar. \tag{41}$$

The Schrödinger equation, with the ansatz (10), has to be solved for the RO Hamiltonian

$$H_\Omega = p^2/(2m) - m\Omega^2/2\,x^2 = -\hbar\Omega/2 \left[\partial_\zeta^2 + \zeta^2\right]. \tag{42}$$

From (2), the following differential equations result:

$$\mathrm{i}\,\gamma'(\tau) = 1 + \gamma^2(\tau), \quad \mathrm{i}\,\beta'(\tau) = \gamma(\tau)\beta(\tau), \quad 2\mathrm{i}\,c'(\tau) = \gamma(\tau) - \beta^2(\tau), \tag{43}$$

where, as compared with the HO case in (16), only the equation for $\gamma$ differs. Beginning with $\gamma$, one successively obtains the following solutions

$$\begin{aligned} \gamma(\tau) &= -\mathrm{i}\tanh(\tau + \mathrm{i}\, C_1), &(44)\\ \beta(\tau) &= C_2/\cosh(\tau + \mathrm{i}\, C_1), &(45)\\ c(\tau) &= C_3 - (1/2)\ln(\cosh(\tau + \mathrm{i}\, C_1)) + (\mathrm{i}/2)C_2^2 \tanh(\tau + \mathrm{i}\, C_1), &(46)\end{aligned}$$

where $C_1, C_2, C_3$ are integration constants. We assume that

$$\gamma_0 \equiv \gamma(0) = \tan(C_1) > 0, \quad 0 < C_1 < \pi/2, \tag{47}$$

which implies that

$$\cos(C_1) = (1+\gamma_0^2)^{-1/2}, \quad \sin(C_1) = \gamma_0(1+\gamma_0^2)^{-1/2}. \tag{48}$$

In order to decompose the functions $c(\tau), \beta(\tau), \gamma(\tau)$ into their real and imaginary parts, we take over the following abbreviations from [16]

$$f(\tau) = \cosh(\tau) - \mathrm{i}\,\gamma_0 \sinh(\tau), \quad h(\tau) = [f f^*]^{-1}. \tag{49}$$

After the decompositions $\beta = \beta_R + \mathrm{i}\,\beta_I$, $\gamma = \gamma_R + \mathrm{i}\,\gamma_I$, $C_2 = A_2 + \mathrm{i}\, B_2$, we infer from (44) to (46):

$$\begin{aligned} \gamma_R &= h(\tau)\gamma_0, \quad \gamma_I = -\,(h(\tau)/2)\,(1+\gamma_0^2)\sinh(2\tau); &(50)\\ \beta_R &= h(\tau)\sqrt{1+\gamma_0^2}\,[A_2\cosh(\tau) + \gamma_0 B_2 \sinh(\tau)]\,, &\\ \beta_I &= h(\tau)\sqrt{1+\gamma_0^2}\,[B_2\cosh(\tau) - \gamma_0 A_2 \sinh(\tau)]\,; &(51)\\ \exp[c(\tau)] &= [\cosh(\tau + \mathrm{i}\, C_1)]^{-1/2} \exp\left[C_3 - C_2^2\gamma(\tau)/2\right]. &(52)\end{aligned}$$

According to (50), $\gamma_R$ is larger zero, which makes the wave function (10) a normalizable wave packet. The probability density reads:

$$P(\zeta, \tau) \quad = \quad C_0^2 \exp\left[c + c^* + 2\beta_R\zeta - \gamma_R\zeta^2\right]. \tag{53}$$

Integration with respect to $\zeta$ leads to the normalization condition

$$1 = C_0^2\sqrt{\pi/\gamma_R}\exp\left[c(\tau) + c^*(\tau) + \beta_R^2/\gamma_R\right]. \tag{54}$$

The normalization constant $C_0$ was determined in [16] for real constants $C_2$. With $C_2 = A_2 + \mathrm{i}\, B_2$, we dispose of the integration constant $C_3$ as

$$C_3 = -(1/2)(A_2^2/\gamma_0 + B_2^2\gamma_0) \tag{55}$$

to obtain in a straightforward manner

$$C_0^2 = \sqrt{\pi(\gamma_0^{-1} + \gamma_0)}, \tag{56}$$

which is a time independent condition as it must be.

With the aid of elementary trigonometric manipulations and the normalization constant $C_0$ given in (56), the wave function can be written as follows

$$\psi(\zeta, \tau) = (\gamma_0/\pi)^{1/4}\,\sqrt{h(\tau) f(\tau)}\exp\left[C_3 - (1/2)C_2^2\gamma(\tau) + \beta(\tau)\zeta - \gamma(\tau)\zeta^2/2\right]. \tag{57}$$

### 5.1. Coherent States of the RO

As before, let us consider the wave function at time $t = 0$, where in particular $h = f = 1$:

$$\psi(\zeta, 0) \equiv \psi(\zeta, \tau = 0) = (\gamma_0/\pi)^{1/4} \exp\left[C_3 - 1/2 C_2^2 \gamma_0 + C_2 \sqrt{1+\gamma_0^2}\,\zeta - \gamma_0 \zeta^2/2\right]. \quad (58)$$

After the re-scaling $\zeta \to \tilde{\zeta}$ with $\tilde{\zeta} = \sqrt{\gamma_0}\,\zeta$, one obtains

$$\Psi(\tilde{\zeta}, 0) = \pi^{-1/4} \exp\left[C_3 - 1/2 C_2^2 \gamma_0 + C_2 \sqrt{(1+\gamma_0^2)/\gamma_0}\,\tilde{\zeta} - \tilde{\zeta}^2/2\right]. \quad (59)$$

In view of the standard HO wave function (5), we replace the integration constant $C_2$ by $z$:

$$C_2 \sqrt{(1+\gamma_0^2)/\gamma_0} = \sqrt{2}\, z \quad (60)$$

and obtain

$$\Psi_z(\tilde{\zeta}) = \pi^{-1/4} \exp\left[C_3 - \gamma_0^2 z^2/(1+\gamma_0^2) + \sqrt{2}\, z\, \tilde{\zeta} - \tilde{\zeta}^2/2\right]. \quad (61)$$

In $C_3$, given in (55), the relation (60) gives rise to the substitutions

$$A_2 \to \kappa_1 (z + z^*), \quad B_2 \to -\mathrm{i}\, \kappa_1 (z - z^*), \quad \kappa_1 = (1/2)\sqrt{2\gamma_0/(1+\gamma_0^2)}, \quad (62)$$

and hence to

$$C_3 = \left[4(1+\gamma_0^2)\right]^{-1} \left[(\gamma_0^2 - 1)(z^2 + z^* z^*) - 2(1+\gamma_0^2) z z^*\right]. \quad (63)$$

After some elementary re-arrangements, one finds

$$\Psi_z(\tilde{\zeta}) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{1}{2}(z z^* + z^2) + \mathrm{i}\, D_1 + \sqrt{2}\, z\, \tilde{\zeta} - \frac{\tilde{\zeta}^2}{2}\right], \quad D_1 = \frac{1-\gamma_0^2}{2(1+\gamma_0^2)} \mathrm{Im}(z^2). \quad (64)$$

Apart from the purely imaginary phase $\mathrm{i}\, D_1$, the wave functions $\Psi_z$ are the same as the standard coherent states (5). Since in the completeness proof the $D_1$ phase drops out, see (A15) in Appendix B, the states $\Psi_z$ form a complete function set.

### 5.2. Mean Values

With the aid of Mathematica [21], we get the following mean values for position $x$, velocity $v$, their mean square deviations $(\Delta x)^2$, $(\Delta v)^2$, and the mean energy $\langle H_\Omega \rangle$:

$$\langle x \rangle = (\alpha_\Omega)^{-1} \sqrt{1+\gamma_0^{-2}}\,[A_2 \cosh(\tau) + \gamma_0 B_2 \sinh(\tau)]; \quad (65)$$

$$(\Delta x)^2 = \left(2\alpha_\Omega^2 \gamma_0\right)^{-1} \left[\cosh^2(\tau) + \gamma_0^2 \sinh^2(\tau)\right]; \quad (66)$$

$$\langle v \rangle = (\hbar \alpha_\Omega/m)\sqrt{1+\gamma_0^{-2}}\;[A_2 \sinh(\tau) + \gamma_0 B_2 \cosh(\tau)]; \quad (67)$$

$$(\Delta v)^2 = (\hbar \alpha_\Omega/(2m))^2 \gamma_0^{-1} \left[\gamma_0^2 - 1 + (1+\gamma_0^2)\cosh(2\tau)\right]; \quad (68)$$

$$\langle H_\Omega \rangle = \hbar\Omega (4\gamma_0)^{-1} \left[\gamma_0^2 - 1 + 2(\gamma_0 + \gamma_0^{-1})\left(\gamma_0^2 B_2^2 - A_2^2\right)\right]. \quad (69)$$

The mean energy does not depend on time, as it must be. With the aid of (62), the mean energy could also be expressed in terms of the complex state label $z$. Since $A_2$ and $B_2$ are arbitrary real numbers, the mean energy can have any positive or negative value. From (66) and (68) one infers the position-momentum uncertainty product $\Delta_{xp}$ as

$$\Delta_{xp}^2(\tau) = \hbar^2/(8\gamma_0^2)\left[\cosh^2(\tau) + \gamma_0^2\sinh(\tau)\right]\left[\gamma_0^2 - 1 + (1+\gamma_0^2)\cosh(2\tau)\right]. \tag{70}$$

This product obeys the inequality

$$\Delta_{xp}^2(\tau) > \Delta_{xp}^2(0) = \hbar^2/4, \quad \tau > 0. \tag{71}$$

Obviously, the uncertainty product is minimal at $\tau = 0$, which means for the coherent states (64). By (66), the wave packets broaden exponentially with time.

## 6. Application to the Kepler-Coulomb Problem

The connection of the non-relativistic Hamiltonian for the hydrogen atom with the model of a four-dimensional oscillator is conveniently achieved by means of the Kustaanheimo-Stiefel transformation [15], which we write as follows [16,22]

$$\begin{aligned} u_1 &= \sqrt{r}\cos(\theta/2)\cos(\varphi - \Phi); \quad & u_2 &= \sqrt{r}\cos(\theta/2)\sin(\varphi - \Phi); \\ u_3 &= \sqrt{r}\sin(\theta/2)\cos(\Phi); & u_4 &= \sqrt{r}\sin(\theta/2)\sin(\Phi), \end{aligned} \tag{72}$$

where $r$, $\theta$, $\varphi$ are three-dimensional polar coordinates with $r > 0$, $0 < \theta < \pi$, $0 \leq \varphi < 2\pi$, and $0 \leq \Phi < 2\pi$ generates the extension to the fourth dimension. The vector $\mathbf{u} = \{u_1, u_2, u_3, u_3\}$ covers the $\mathbf{R}^4$ and the volume elements are related as [16]

$$\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 = (1/8)r\sin(\theta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\Phi. \tag{73}$$

The stationary Schrödinger equation $H\psi = E\psi$ for the Hamiltonian $H = p^2/(2m) - \lambda/r$ is transformed into the following form of a four-dimensional harmonic oscillator [14]:

$$H_u\Psi(\mathbf{u}) = \lambda\Psi(\mathbf{u}), \quad H_u = -\hbar^2/(8m)\Delta_{\mathbf{u}} - E\,\mathbf{u}\cdot\mathbf{u}, \quad \Delta_{\mathbf{u}} = \partial_{u_1}^2 + ...\partial_{u_4}^2 \tag{74}$$

with the constraint

$$\partial_\Phi\Psi(\mathbf{u}) = 0. \tag{75}$$

It should be noticed that, by (72), the components $u_i^2$ have the dimension of a length rather than length square. As a consequence, in the evolution equation $\mathrm{i}\hbar\partial_\sigma\Psi = H_u\Psi$, the parameter $\sigma$, which has the dimension time/length, is not the time parameter of the original problem. For negative energies with $E < 0$, four-dimensional coherent oscillator states (of type-I) were used in [14] to show that elliptic orbits emerge in the classical limit whereby $\sigma$ turns out being proportional to the eccentric anomaly.

In the spectrum of positive energies (ionized states of the hydrogen atom) with $E > 0$, coherent states of the RO were constructed in [16] and gave rise to hyperbolic orbits in the classical limit; by analytic continuation, also the elliptic orbits were derived from the RO states in the classical limit [17]. In addition, Kepler's equation was obtained by the assumption that time-dependence enters through the curve parameter $\sigma$ only. Recently [18], based on the coherent RO states, the first order quantum correction to Kepler's equation could be established for the smallness parameter $\epsilon = \hbar/L$ where $L$ denotes the orbital angular momentum.

## 7. Conclusions

Besides the standard coherent states of the harmonic oscillator (H0), a further solution family of the time-dependent Schrödinger equation was derived with the following properties: (i) The functions are normalizable of Gaussian type and contain a disposable width parameter. The latter allows us, for instance, to use arbitrarily concentrated one-particle states independently of the parameters of a harmonic trap; (ii) The functions are complete and isomorphic to the standard coherent states at time $t = 0$; (iii) The states minimize the position-momentum uncertainty product at the discrete times $T_n = n\,\pi/(2\omega)$, $n = 0, 1, \ldots$; (iv) The width of the wave packets "breathes" periodically with period $T/2 = \pi/\omega$. (v) There is no diffusion, $T = 2\pi/\omega$ is the recurrence time of the states.

In the case of the reversed harmonic oscillator (RO), there exists only one family of time-dependent solutions. They share the properties (i) and (ii) of the type-II HO states, and (iii) is fulfilled at time $t = 0$, only. There is no recurrence, instead there is diffusion with a broadening which increases exponentially with time. The application to the Kepler-Coulomb problem was briefly discussed. The HO coherent states of type-I and the RO coherent states served as basis to derive, in the classical limit, the elliptic Kepler orbits [14] and the hyperbolic ones [16,17], respectively.

**Acknowledgments:** The author expresses his gratitude to Jürgen Parisi for his constant encouragement and support. He also profited from his critical reading of the manuscript.

**Conflicts of Interest:** The author declares no conflict of interest.

## Appendix A. Probability Density for Type-II States

We have to decompose the functions $\beta(\tau)$ and $c(\tau)$, as given by (21)and (22), into their real and imaginary parts. To this end, we set $C_2 = A_2 + \mathrm{i}\,B_2$ with real constants $A_2$ and $B_2$ and $\beta = \beta_R + \mathrm{i}\,\beta_I$. Using the definitions of $N_1$ and $C_1$ in terms of $\gamma_0$, we obtain

$$\begin{aligned}
\beta_R &= \frac{1+\gamma_0}{2}\,\frac{A_2\cos(\tau) + B_2\gamma_0\sin(\tau)}{1+(\gamma_0^2-1)\sin^2(\tau)},\\
\beta_I &= \frac{1+\gamma_0}{2}\,\frac{B_2\cos(\tau) - A_2\gamma_0\sin(\tau)}{1+(\gamma_0^2-1)\sin^2(\tau)}.
\end{aligned} \tag{A1}$$

In view of the function $c(\tau)$, we make use of the following auxiliary relations

$$\begin{aligned}
F_c &\equiv -C_2^2\,[4\,(\exp(2\mathrm{i}\,\tau) + C_1)]^{-1} = F_R + \mathrm{i}\,F_I,\\
F_R &= (1/(4N_1))\left[(B_2^2 - A_2^2)\cos(2\tau) - 2A_2B_2\sin(2\tau) + (B_2^2 - A_2^2)C_1\right],\\
F_I &= (1/(4N_1))\left[(A_2^2 - B_2^2)\sin(2\tau) - 2A_2B_2\cos(2\tau) - 2A_2B_2C_1\right],
\end{aligned} \tag{A2}$$

$$\exp\left[c(\tau) + c^*(\tau)\right] = (1/\sqrt{N_1})\exp\left[2C_3 + 2F_R\right], \tag{A3}$$

where the integration constant $C_3$ is assumed being real and the star suffix means complex conjugation.

The probability density $P$ results from the wave function (10) in the form

$$P(\xi,\tau) = \frac{C_0^2}{\sqrt{N_1}}\exp\left[2C_3 + 2F_R + 2\beta_R\xi - \gamma_R\xi^2\right], \tag{A4}$$

where $C_0$ is defined through the normalization integral

$$1 = \int_{-\infty}^{\infty}\mathrm{d}\xi\,P(\xi,\tau) = \frac{C_0^2\sqrt{\pi}}{\sqrt{N_1\gamma_R}}\exp(G), \quad G = 2C_3 + 2F_R + \beta_R^2/\gamma_R. \tag{A5}$$

From the expression of $G$, it is not obvious that $C_0$ is independent of $\tau$ which was assumed in (10). Clearly, since $\Phi := \psi/C_0$ obeys the Schrödinger equation and $H$ is hermitian, one has the property

$$\partial_\tau\langle\Phi|\Phi\rangle = 0. \tag{A6}$$

As a matter of fact, it is straightforward to show that

$$2F_R + \beta_R^2/\gamma_R = \left[B_2^2(C_1-1) - A_2^2(1+C_1)\right]\left[2(C_1^2-1)\right]^{-1} \tag{A7}$$

does not depend on $\tau$. We now dispose of the integration constant $C_3$ such that the exponent $G$ vanishes:

$$C_3 = -\left[B_2^2(C_1-1) - A_2^2(1+C_1)\right]\left[4(C_1^2-1)\right]^{-1}. \tag{A8}$$

In view of $G=0$, we replace $2C_3+2F_R$ by $-\beta_R^2/\gamma_R$, so that

$$P(\xi,\tau) = \frac{C_0^2}{\sqrt{N_1}}\exp\left[-\gamma_R\left(\xi-\beta_R/\gamma_R\right)^2\right], \tag{A9}$$

which is the result (24). The normalization condition comes out immediately in the form

$$1 = \frac{C_0^2\sqrt{\pi}}{\sqrt{N_1\gamma_R}} = \frac{C_0^2\sqrt{\pi}}{\sqrt{1-C_1^2}} = \frac{C_0^2\sqrt{\pi}(1+\gamma_0)}{2\sqrt{\gamma_0}}. \tag{A10}$$

## Appendix B. Proof of Completeness

In order to prove the completeness of the functions (5), *i.e.*, for the type-I HO case, we take advantage of the following generating function of the Hermite polynomials [23]:

$$\exp\left[2XZ-Z^2\right] = \sum_{n=0}^{\infty}\frac{Z^n}{n!}H_n(X). \tag{A11}$$

In the function (5), we replace $z$ by $\sqrt{2}\,Z$ to obtain

$$\psi_z(\zeta) = \pi^{-1/4}\exp\left[-ZZ^*-(1/2)\zeta^2\right]\exp\left[-Z^2+2\zeta Z\right]. \tag{A12}$$

With the aid of (A11), one can write

$$\psi_z(\zeta) = \exp\left[-(1/2)zz^*\right]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\varphi_n(\zeta), \tag{A13}$$

where

$$\varphi_n(\zeta) = \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}H_n(\zeta)\exp\left[-(1/2)\zeta^2\right]. \tag{A14}$$

By means of (A13) and setting $z = u\exp[\mathrm{i}\,\varphi]$, we obtain

$$\langle\zeta_2|z\rangle\langle z|\zeta_1\rangle = \exp\left[-u^2\right]\sum_{m,n=0}^{\infty}\frac{u^{n+m}\exp\left[\mathrm{i}\,(m-n)\varphi\right]}{\sqrt{m!n!}}\varphi_m(\zeta_2)\varphi_n(\zeta_1). \tag{A15}$$

In (A15), the $\varphi$ integration projects out the terms $n=m$ with the result

$$\frac{1}{\pi}\int_0^{\infty}u\mathrm{d}u\int_0^{2\pi}\mathrm{d}\varphi\,\langle\zeta_2|z\rangle\langle z|\zeta_1\rangle = 2\int_0^{\infty}u\mathrm{d}u\,\exp[-u^2]\sum_{n=0}^{\infty}\frac{u^{2n}}{n!}\,\varphi_n(\zeta_2)\varphi_n(\zeta_1). \tag{A16}$$

After changing the integration variable $u\to v$ with $v=u^2$ with $u\mathrm{d}u = \mathrm{d}v/2$, one uses

$$\int_0^{\infty}\mathrm{d}v\,\frac{v^n}{n!}\exp[-v] = 1, \quad n = 0,1,\ldots \tag{A17}$$

and, in view of the completeness of the Hermite polynomials, arrives at

$$\frac{1}{\pi}\int_0^{\infty}u\mathrm{d}u\int_0^{2\pi}\mathrm{d}\varphi\,\langle\zeta_2|z\rangle\langle z|\zeta_1\rangle = \sum_{n=0}^{\infty}\varphi_n(\zeta_2)\varphi_n(\zeta_1) = \delta\left(\zeta_2-\zeta_1\right). \tag{A18}$$

In the type-II HO and the RO cases, there appear additional purely imaginary phases in the wave function, which do not depend on $\zeta_1$, $\zeta_2$, and drop out at the step (A15) of the completeness proof above.

## References


1. Schrödinger, E. Der stetige Übergang von der Mikro-zur Makromechanik. *Naturwissenschaften* **1926**, *14*, 664–666.
2. Glauber, R.J. Coherent and incoherent states of the radiation field. *Phys. Rev.* **1963**, *131*, 2766.
3. Glauber, R.J. Photon Correlations. *Phys. Rev. Lett.* **1963**, *10*, 84.
4. Antoniou, I.E.; Progogine, I. Intrinsic irreversibility and integrability of dynamics. *Phys. A Stat. Mech. Appl.* **1993**, *192*, 443–464.
5. Gentilini, S.; Braidotti, M.C.; Marcucci, G.; DelRe, E.; Conti, C. Physical realization of the Glauber quantum oscillator. *Sci. Rep.* **2015**, *5*, 15816.
6. Glauber, R.J. Amplifiers, attenuators, and schrödinger's cat. *Ann. N. Y. Acad. Sci.* **1986**, *480*, 336–372.
7. Barton, G. Quantum mechanis of the inverted oscillator potential. *Ann. Phys.* **1986**, *166*, 322–363.
8. Bhaduri, R.K.; Khare, A.; Reimann, S.M.; Tomisiek, E.L. The riemann zeta function and the inverted harmonic oscillator. *Ann. Phys.* **1997**, *264*, 25–40.
9. Guo, G.-J.; Ren, Z.-Z.; Ju, G.-X.; Guo, X.-Y. Quantum tunneling effect of a time-dependent inverted harmonic oscillator. *J. Phys. A Math. Theor.* **2011**, *44*, 185301.
10. Galindo, A.; Pascual, P. *Quantum Mechanics I*; Springer: Berlin, Germany, 1990.
11. Fock, V.A. Zur Theorie des Wassenstoffatoms. *Z. Phys.* **1935**, *98*, 145–154.
12. Chen, A.C. Hydrogen atom as a four-dimensional oscillator. *Phys. Rev. A* **1980**, *22*, 333–335.
13. Gracia-Bondia, J.M. Hydrogen atom in the phase-space formulation of quantum mechanics. *Phys. Rev. A* **1984**, *30*, 691–697.
14. Gerry, C.C. Coherent states and the Kepler-Coulomb problem. *Phys. Rev. A* **1986**, *33*, 6–11.
15. Kustaanheimo, P.; Stiefel, E. Perturbation theory of Kepler motion based on spinor regularization. *J. Reine Angew. Math.* **1965**, *218*, 204–219.
16. Rauh, A.; Parisi, J. Quantum mechanics of hyperbolic orbits in the Kepler problem. *Phys. Rev. A* **2011**, *83*, 042101.
17. Rauh, A.; Parisi, J. Quantum mechanics of Kepler orbits. *Adv. Stud. Theor. Phys.* **2014**, *8*, 889–938.
18. Rauh, A.; Parisi, J. Quantum mechanical correction to Kepler's equation. *Adv. Stud. Theor. Phys.* **2016**, *10*, 1–22.
19. Baletto, F.; Riccardo, F. Structural properties of nanoclusters: Energetic, thermodynamic, and kinetic effects. *Rev. Mod. Phys.* **2015**, *77*, 371–423.
20. Bauch, S.; Balzer, K.; Bonitz, M. Quantum breathing mode of trapped bosons and fermions at arbitrary coupling. *Phys. Rev. B* **2009**, *80*, 054515.
21. Wolfram Research, Inc. *Mathematica*; Version 10.1.0.0; Wolfram Research, Inc.: Champaign, IL, USA, 2015.
22. Chen, C.; Kibler, M. Connection between the hydrogen atom and the four-dimensional oscillator. *Phys. Rev. A* **1985**, *31*, 3960–3963.
23. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Academic Press: New York, NY, USA, 1965.