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Almost Contact Metric Structures on 5-Dimensional Nilpotent Lie Algebras

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Abstract: We study almost contact metric structures on 5-dimensional nilpotent Lie algebras and investigate the class of left invariant almost contact metric structures on corresponding Lie groups. We determine certain classes that a five-dimensional nilpotent Lie group can not be equipped with.

Keywords: 5-dimensional nilpotent Lie algebra; almost contact metric structure; left invariant almost contact metric structure

1. Introduction

It is well-known that every connected odd dimensional Lie group is equipped with a left invariant almost contact metric structure. These structures give rise to almost contact metric structures on corresponding Lie algebras [1]. In literature, some certain classes of such structures are studied. In [2], some general results on 5-dimensional Sasakian Lie algebras were stated, and it was proved that an odd dimensional nilpotent Lie group with a left invariant Sasakian structure is isomorphic to the real Heisenberg group. In addition, a classification of five-dimensional Sasakian Lie algebras were obtained. Then, in [3], left invariant K-contact structures on five-dimensional Lie groups were investigated. Three-dimensional homogeneous almost contact metric structures were considered in [4]. In [5], cosymplectic and α-cosymplectic Lie algebras were investigated in terms of corresponding symplectic Lie algebras and suitable derivations on them.

Our aim in this manuscript is to determine almost contact metric structures on five-dimensional nilpotent Lie algebras by direct calculation. We use the classification of five-dimensional nilpotent Lie algebras given in [6]. We consider some certain classes of almost contact metric structures, and, by this approach, we get some general results on left invariant almost contact metric structures on five-dimensional nilpotent Lie groups.

2. Preliminaries

Let M^{2n+1} be a differentiable manifold of dimension $2n+1$. If there is a $(1,1)$ tensor field ϕ, a vector field ξ and a one-form η on M satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

then, M is said to have an almost contact structure (ϕ, ξ, η). A manifold with an almost contact structure is called an almost contact manifold. If, in addition to an almost contact structure (ϕ, ξ, η), M also admits a Riemannian metric g such that

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y, then M is an almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g). The Riemannian metric g is called a compatible metric. The one-form defined by

$$\Phi(X, Y) = g(X, \phi(Y)),$$

for all $X, Y \in \mathfrak{X}(M)$, is called the fundamental two-form of the almost contact metric manifold (M, ϕ, ξ, η, g). In [7], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental two-form. A space having the same symmetries as the covariant derivative of the fundamental two-form was written, and, then, this space was decomposed into twelve $U(n) \times 1$ irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$. There are 2^{12} invariant subspaces, each corresponding to a class of almost contact metric manifolds. For example, the trivial class for which $\nabla\Phi = 0$ [8], corresponds to the class of cosymplectic (called co-Kähler by some authors) manifolds, $\mathcal{C}_1$ is the class of nearly-K-cosymplectic manifolds, etc. [7]. For classification of almost contact metric structures (see also [9]). In this work, we focus on cosymplectic, nearly cosymplectic, α-Sasakian, β-Kenmotsu and almost cosymplectic structures.

Let (ϕ, ξ, η, g) be an almost contact metric structure on M with the fundamental two-form Φ. (ϕ, ξ, η, g) is called:

- nearly cosymplectic if $\nabla_X\Phi(X, Y) = 0$,
- α-Sasakian ($\mathcal{C}_6$) if $\nabla_X\phi(Y) = \alpha(g(X, Y)\xi - \eta(Y)X)$ for a constant α,
- β-Kenmotsu ($\mathcal{C}_5$) if $\nabla_X\phi(Y) = \beta(\Phi(X, Y)\eta(Z) - \Phi(X, Z)\eta(Y))$ for a constant β,
- semi cosymplectic ($\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$) if $\delta\Phi = 0$ and $\delta\eta = 0$, where δ denotes the coderivative of a differential form,
- almost cosymplectic ($\mathcal{C}_2 \oplus \mathcal{C}_9$) if $d\Phi = 0$ and $d\eta = 0$, where d denotes the exterior derivative of a differential form,

for all vector fields X, Y, Z on M.

In literature, there are different but related definitions of cosymplectic structures. Here, we remind them and relate to the classes we use. In [5,10], an almost cosymplectic manifold is defined as a smooth manifold with a one-form η and a two-form Φ such that $\eta \wedge \Phi^n$ is a volume form. If both η and Φ are closed, then the manifold is said to be cosymplectic. In the same context, if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \phi$ for a constant α, then the manifold is called α-cosymplectic. An almost contact metric manifold (M, ϕ, ξ, η, g), where (η, Φ) is a α cosymplectic structure is called an almost co-Kählbaser manifold. In addition, if this manifold is normal, then it is said to be co-Kähler. An almost contact metric manifold (M, ϕ, ξ, η, g) such that (η, Φ) is an α-cosymplectic structure is called an almost α co-Kähler manifold. A normal almost α co-Kähler manifold is said to be α co-Kähler. Refer to [5,10] and references therein. "Almost cosymplectic", "cosymlectic" and "α-Kenmotsu" structures in our paper correspond to "almost co-Kähler", "co-Kähler" and "α co-Kähler" in [5], respectively. Throughout the paper, the definitions in and [7,8] will be followed.

The existence of metric connections on five-dimensional almost contact metric manifolds compatible with the almost contact structure was investigated in [11]. The space of torsion tensors of a metric connection splits into ten $U(2)$-irreducible subspaces $\mathcal{W}_1$, $\mathcal{W}_2$, ..., $\mathcal{W}_{10}$. Thus, there are 2^{10} classes of almost contact metric structures in five-dimensions according to components of torsion tensor [11].

An almost contact metric structure (ϕ, ξ, η, g) on a connected Lie group G is said to be left invariant if g is left invariant and if the left multiplication map $L_a : G \longrightarrow G$, $L_a(x) = a.x$ has properties

$$\phi \circ L_a = L_a \circ \phi, \quad L_a(\xi) = \xi,$$

for all $a \in G$.

Let g be an odd dimensional Lie algebra. An almost contact metric structure on g is a quadruple (ϕ, ξ, η, g), where η is a one-form, ϕ is an endomorphism of g, $\xi \in g$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y and g is a positive definite compatible inner product on g. It is also convenient to use defining relations for the structures on Lie algebras. For instance, an almost contact metric structure (ϕ, ξ, η, g) on a Lie algebra g is said to be nearly cosymplectic if $\nabla_X \Phi(X, Y) = 0$ for any X, Y in g, etc.

Let G be a connected Lie group endowed with a left invariant almost contact metric structure (ϕ, ξ, η, g) and $g \cong T_e G$ be the corresponding Lie algebra of G. Then, this structure uniquely yields an almost contact metric structure (ϕ, ξ, η, g) on g.

In this work, we study almost contact metric structures on five-dimensional nilpotent Lie algebras. The classification of nilpotent Lie algebras of dimension ≤ 5 was obtained in [6] (see also [12,13]). Indeed, g_i are five-dimensional nilpotent algebras with the corresponding basis $\{e_1, \ldots, e_5\}$ and non-zero brackets as follows:

$$\begin{aligned}
g_1 &: [e_1, e_2] = e_5, [e_3, e_4] = e_5, \\
g_2 &: [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5, \\
g_3 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5, \\
g_4 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, \\
g_5 &: [e_1, e_2] = e_4, [e_1, e_3] = e_5, \\
g_6 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5.
\end{aligned}$$

The rest of the classes g_7, g_8, g_9 are abelian.

3. Almost Contact Metric Structures on g_i

Let G be a connected Lie group and (ϕ, ξ, η, g) a left invariant a.c.m.s. (almost contact metric structure) on G. Denote the corresponding a.c.m.s. on g by the same symbols. Choose the basis $\{e_1, \ldots, e_5\}$ such that basis elements are g-orthonormal.

First, we investigate the existence of some classes of almost contact metric structures on each g_i.

The algebra g_1: By Kozsul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{llll}
\nabla_{e_1} e_2 = \frac{1}{2} e_5, & \nabla_{e_1} e_5 = -\frac{1}{2} e_2, & \nabla_{e_2} e_1 = -\frac{1}{2} e_5, & \nabla_{e_2} e_5 = \frac{1}{2} e_1, \\
\nabla_{e_3} e_4 = \frac{1}{2} e_5, & \nabla_{e_3} e_5 = -\frac{1}{2} e_4, & \nabla_{e_4} e_3 = -\frac{1}{2} e_5, & \nabla_{e_4} e_5 = \frac{1}{2} e_3, \\
\nabla_{e_5} e_1 = -\frac{1}{2} e_2, & \nabla_{e_5} e_2 = \frac{1}{2} e_1, & \nabla_{e_5} e_3 = -\frac{1}{2} e_4, & \nabla_{e_5} e_4 = \frac{1}{2} e_3.
\end{array}$$

- There exists no cosymplectic structure on g_1.
  To see this, assume $\Phi = \sum b_{ij} e^{ij}$ is a two-form on g_1 such that $\nabla \Phi = 0$. Then, for any elements e_i, e_j, e_k of the basis:

$$\begin{aligned}
(\nabla_{e_i} \Phi)(e_j, e_k) &= e_i[\Phi(e_j, e_k)] - \Phi(\nabla_{e_i} e_j, e_k) - \Phi(e_j, \nabla_{e_i} e_k) \\
&= -\Phi(\nabla_{e_i} e_j, e_k) - \Phi(e_j, \nabla_{e_i} e_k) = 0.
\end{aligned} \tag{1}$$

  It is easy to see that $\nabla \Phi = 0$ if and only if $b_{ij} = 0$ for any i, j. Thus, (ϕ, ξ, η, g) is not cosymplectic.

- There is no nearly cosymplectic structure (i.e., $(\nabla_X \Phi)(X, Y) = 0$). Let $\Phi = \sum b_{ij} e^{ij}$, by direct calculation, we obtain,

$$(\nabla_{e_i} \Phi)(e_i, e_j) = 0 \iff b_{13}b_{23} + b_{14}b_{24} = b_{13}b_{14} + b_{23}b_{24} = 0,$$

where $b_{14}^2 = b_{23}^2$ and $b_{13}^2 = b_{24}^2$ and the remaining coefficents are zero. Thus $\Phi = b_{13}e^{13} + b_{14}e^{14} + b_{23}e^{23} + b_{24}e^{24}$. By polarizing the equation $(\nabla_X \Phi)(X, Y) = 0$, we get

$$\nabla_X \Phi(Y, Z) + \nabla_Y \Phi(X, Z) = 0. \tag{2}$$

Then for $X = e_2$, $Y = e_3$ and $Z = e_5$ in the equation (2), we obtain $b_{13} = -b_{24}$. In addition, replacing e_3, e_5 and e_2 for X, Y, Z respectively in the equation (2), we get $b_{13} = 2b_{24}$. Thus, $b_{13} = b_{24} = 0$. On the other hand, we get $b_{14} = b_{23}$ and $2b_{23} = -b_{14}$ for $X = e_3$, $Y = e_1$ and $Z = e_5$ and $X = e_1$, $Y = e_5$ and $Z = e_3$ respectively in the equation (2), which implies $b_{14} = b_{23} = 0$.
- There is no non-zero parallel vector field on g_1. Let $\xi = \sum a_i e_i$ be a parallel vector field on g_1 (i.e., $\nabla \xi = 0$). Then, by the Kozsul formula, we have $a_i = 0$ for $i = 1, \ldots, 5$.
Note that since $(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y)$ for all vector fields X and Y, we have $\nabla \eta \neq 0$ for any almost contact metric structure (ϕ, ξ, η, g) on g_1. In particular, (ϕ, ξ, η, g) is neither $\mathcal{C}_1$ (nearly-K-cosymplectic), nor $\mathcal{C}_2$.
- A vector field ξ on g_1 is Killing if and only if $\xi \in \langle e_5 \rangle$.
Let $\xi = \sum a_i e_i$ be a Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i} \xi, e_j) = -g(\nabla_{e_j} \xi, e_i)$. Then,

$$g(\nabla_{e_2} \xi, e_5) = -\frac{1}{2} a_1 \text{and } g(\nabla_{e_5} \xi, e_2) = -\frac{1}{2} a_1$$

yields $a_1 = 0$. Similarly, since $g(\nabla_{e_1} \xi, e_5) = -g(\nabla_{e_5} \xi, e_1)$, we have $a_2 = 0$ and $g(\nabla_{e_4} \xi, e_5) = -g(\nabla_{e_5} \xi, e_4)$ gives $a_3 = 0$. In addition, $g(\nabla_{e_3} \xi, e_5) = -g(\nabla_{e_5} \xi, e_3)$ implies $a_4 = 0$. As a result, $\xi = a_5 e_5$.
- There exists 1/2-Sasakian structure on g_1, where the fundamental two-form is $\Phi = -e^{12} - e^{34}$ and $\xi = e_5$. Note that this structure is given in [2] as a Sasakian structure because of the coefficient 2 in the defining relation of a Sasakian structure.
- There is no β-Kenmotsu structure.
Assume (ϕ, ξ, η, g) is a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij} e^{ij}$. Then, $g(\nabla_{e_i} \xi, e_j) = g(\nabla_{e_j} \xi, e_i)$ for any basis elements e_i, e_j, which implies that $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$ and $\eta = b_1 e^1 + b_2 e^2 + b_3 e^3 + b_4 e^4$. On the other hand for $X = e_1$, $Y = e_1$, $Z = e_2$ and for $X = e_1$, $Y = e_1$, $Z = e_5$ in the defining relation of a β-Kenmotsu structure, we obtain $b_{15} = 2\beta b_1 b_{12}$ and $b_{12} = -2\beta b_1 b_{15}$, respectively. Thus, $b_{12} = b_{15} = 0$. Similar arguments work if X, Y, Z are replaced by other basis elements. We get $b_{ij} = 0$ for all i, j. As a result the structure is not β-Kenmotsu.
- There is no almost cosymplectic structure.
Let $\eta = \sum b_i e^i$ and $\Phi = \sum b_{ij} e^{ij}$. Then, since $de^5 = -e^{12} - e^{34}$ and $de^i = 0$ for $i = 1, 2, 3, 4$, we get $d\eta = -b_5(e^{12} + e^{34})$. This yields $d\eta = 0$ iff $b_5 = 0$. On the other hand, we have $d\Phi = b_{15}e^{134} + b_{25}e^{234} + b_{35}e^{123} + b_{45}e^{124}$, which is zero iff $b_{15} = b_{25} = b_{45} = 0$. In this case, $\Phi \wedge \Phi = 2(b_{12}b_{34} + b_{14}b_{23} - b_{13}b_{24})e^{1234}$ and $\eta \wedge \Phi^2 = 0$, which contradicts with the assumption that (ϕ, ξ, η, g) is an almost contact metric structure.
- There are semi cosymplectic structures on g_1.
For any vector $X = \sum x_i e_i$ on g_1, we have $\delta\Phi(X) = x_5(b_{12} + b_{34})$. Thus, $\delta\Phi = 0$ for all X iff $b_{12} = -b_{34}$. In addition, $\delta\eta = 0$ for any one-form η. Choose, for example, the a.c.m.s. (ϕ, ξ, η, g), where $\xi = e_5$, $\eta = e^5$ and $\Phi = e^{12} - e^{34}$. This structure is semi cosymplectic.
- Consider the a.c.m.s. (ϕ, ξ, η, g), where $\phi(e_1) = -e_4$, $\phi(e_2) = -e_3$, $\phi(e_3) = e_2$, $\phi(e_4) = e_1$ and $\xi = e_5$, $\eta = e^5$ on g_1. We show that there is a metric connection ∇^c compatible with this structure.

Assume that ∇^c is a metric connection of g. Then, $\nabla^c = \nabla + A$, where A is a skew-symmetric $(2,1)$ tensor field. Since ∇^c is compatible with $\xi = e_5$, we have $\nabla^c_{e_i} e_5 = 0$ for all basis elements e_i. We obtain

$$A(e_1, e_5) = \frac{1}{2}e_2, \;\; A(e_2, e_5) = -\frac{1}{2}e_1, \;\; A(e_3, e_5) = \frac{1}{2}e_4, \;\; A(e_4, e_5) = -\frac{1}{2}e_3.$$

Metric compatibility of ∇^c yields

$$0 = e_1[g(e_1, e_2)] = g(\nabla^c_{e_1} e_1, e_2) + g(e_1, \nabla^c_{e_1} e_2),$$

and thus $g(e_1, \nabla^c_{e_1} e_2) = 0$. Note that $\nabla^c_{e_1} e_1 = \nabla_{e_1} e_1 + A(e_1, e_1) = 0$. Similarly, $g(e_2, \nabla^c_{e_1} e_2) = g(e_5, \nabla^c_{e_1} e_2) = 0$. Hence, $\nabla^c_{e_1} e_2 = a_3 e_3 + a_4 e_4$ for some constants a_3, a_4 and $A(e_1, e_2) = a_3 e_3 + a_4 e_4 - \frac{1}{2} e_5$.

Since ∇^c is also compatible with ϕ, that is, $\nabla^c \phi = 0$, we have

$$0 = (\nabla^c_{e_1} \phi)(e_2) = \nabla^c_{e_1}(\phi(e_2)) - \phi(\nabla^c_{e_1} e_2) = -\nabla^c_{e_1} e_3 - \phi(a_3 e_3 + a_4 e_4).$$

Thus,

$$\nabla^c_{e_1} e_3 = -a_3 e_2 - a_4 e_1 = A(e_1, e_3).$$

In addition, $(\nabla^c_{e_1} \phi)(e_4) = 0$ implies $\phi(\nabla^c_{e_1} e_4) = 0$. By the identity $\phi^2 = -I + \eta \otimes \xi$, we get

$$0 = \phi^2(\nabla^c_{e_1} e_4) = -\nabla^c_{e_1} e_4 + g(\nabla^c_{e_1} e_4, e_5) e_5,$$

which gives $\nabla^c_{e_1} e_4 = A(e_1, e_4) = 0$. Note that $g(\nabla^c_{e_1} e_4, e_5) = 0$ since ∇^c is a metric connection. Similarly, $\nabla^c_{e_2} e_3 = A(e_2, e_3) = 0$. By direct calculation, we get

$$\nabla^c_{e_2} e_4 = A(e_2, e_4) = a_4 e_1 + a_3 e_2,$$

$$\nabla^c_{e_3} e_4 = a_3 e_3 + a_4 e_4, \;\; A(e_3, e_4) = a_3 e_3 + a_4 e_4 - \frac{1}{2} e_5.$$

To sum up,

$$\begin{aligned} A \;\; = \;\; & e_1 \otimes \{a_4(-e^{13} + e^{24}) - \tfrac{1}{2} e^{25}\} + e_2 \otimes \{a_3(-e^{13} + e^{24}) + \tfrac{1}{2} e^{15}\} \\ & + e_3 \otimes \{a_3 e^{12} - \tfrac{1}{2} e^{45}\} + e_4 \otimes \{a_4 e^{12} + \tfrac{1}{2} e^{35}\} + e_5 \otimes \{-\tfrac{1}{2} e^{12} - \tfrac{1}{2} e^{34}\}. \end{aligned}$$

Since (ϕ, ξ, η, g) has a totally skew-symmetric metric connection, by Proposition 4.1 in [11], we conclude that (ϕ, ξ, η, g) is in the class $\mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \oplus \mathcal{W}_6$ with respect to the classification of Puhle in [11].

Similar observations can be made for existing structures on each g_i.

The algebra g_2: By Kozsul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{llll} \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_3 = -\frac{1}{2} e_2 + \frac{1}{2} e_5, & \nabla_{e_1} e_5 = -\frac{1}{2} e_3, & \nabla_{e_2} e_1 = -\frac{1}{2} e_3, \\ \nabla_{e_2} e_3 = \frac{1}{2} e_1, & \nabla_{e_2} e_4 = \frac{1}{2} e_5, & \nabla_{e_2} e_5 = -\frac{1}{2} e_4, & \nabla_{e_3} e_1 = -\frac{1}{2} e_2 - \frac{1}{2} e_5, \\ \nabla_{e_3} e_2 = \frac{1}{2} e_1, & \nabla_{e_3} e_5 = \frac{1}{2} e_1, & \nabla_{e_4} e_2 = -\frac{1}{2} e_5 \,, & \nabla_{e_4} e_5 = \frac{1}{2} e_2, \\ \nabla_{e_5} e_1 = -\frac{1}{2} e_3, & \nabla_{e_5} e_2 = -\frac{1}{2} e_4, & \nabla_{e_5} e_3 = \frac{1}{2} e_1, & \nabla_{e_5} e_4 = \frac{1}{2} e_2, \end{array}$$

- There exists no cosymplectic structure.
  The proof is similar to that of g_1.
- There exists no nearly cosymplectic structure.
  Assume that there exists a nearly cosymplectic structure (ϕ, ξ, η, g) with the fundamental two-form $\Phi = \sum b_{ij}e^{ij}$. Then, for any basis elements e_i, e_j, we have $(\nabla_{e_i}\Phi)(e_i, e_j) = -\Phi(e_i, \nabla_{e_i}e_j) = 0$. Thus, we get:

  $(\nabla_{e_1}\Phi)(e_1, e_2) = 0 \Rightarrow b_{13} = 0$, $(\nabla_{e_1}\Phi)(e_1, e_3) = 0 \Rightarrow b_{12} = b_{15}$, $(\nabla_{e_2}\Phi)(e_2, e_1) = 0 \Rightarrow b_{23} = 0$,
  $(\nabla_{e_2}\Phi)(e_2, e_3) = 0 \Rightarrow b_{12} = 0$,
  $(\nabla_{e_2}\Phi)(e_2, e_4) = 0 \Rightarrow b_{25} = 0$, $(\nabla_{e_2}\Phi)(e_2, e_5) = 0 \Rightarrow b_{24} = 0$,
  $(\nabla_{e_3}\Phi)(e_3, e_1) = 0 \Rightarrow b_{35} = 0$ and $(\nabla_{e_4}\Phi)(e_4, e_2) = 0 \Rightarrow b_{45} = 0$.

  Thus, the fundamental two-form is of type $\Phi = b_{14}e^{14} + b_{34}e^{34}$. From the equation $\Phi(X, Y) = g(X, \phi(Y))$, the endomorphism ϕ is defined by $\phi(e_1) = -b_{14}e_4$, $\phi(e_2) = 0$, $\phi(e_3) = -b_{34}e_4$, $\phi(e_4) = b_{14}e_1 + b_{34}e_3$, $\phi(e_5) = 0$. Let $\xi = \sum_{i=1}^{5} a_i e_i$ and $\eta = \sum_{i=1}^{5} b_i e^i$. Then,
  $\phi^2(e_2) = 0 = -e_2 + \eta(e_2)\xi \Rightarrow b_2a_2 = 1, b_2a_5 = 0 \Rightarrow a_5 = 0$.
  On the other hand,
  $\phi^2(e_5) = 0 = -e_5 + \eta(e_5) \Rightarrow b_5a_5 = 1 \Rightarrow a_5 \neq 0$. Therefore, the condition $\phi^2 = -I + \eta \otimes \xi$ does not hold. Thus, the structure is not nearly cosymplectic.
- There is no non-zero parallel vector field on g_2.
  If a non-zero vector field $\xi = \sum a_i e_i$ is parallel ($\nabla\xi = 0$), by calculating $g(\nabla_{e_i}\xi, e_j)$ for basis elements, we get $a_i = 0$, for $i = 1, \cdots, 5$. It also shows that $\nabla\eta \neq 0$ for any almost contact metric structure (ϕ, ξ, η, g) on g_2. In particular, (ϕ, ξ, η, g) is neither $\mathcal{C}_1$ (nearly-K-cosymplectic), nor $\mathcal{C}_2$.
- A vector field ξ on g_2 is Killing if and only if $\xi \in \langle e_5 \rangle$.
  Assume $\xi = \sum a_i e_i$ is a Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$. Thus,
  $$g(\nabla_{e_2}\xi, e_3) = -\frac{1}{2}a_1,\ g(\nabla_{e_3}\xi, e_2) = -\frac{1}{2}a_1 \Rightarrow a_1 = 0,$$
  and similarly,
  $$g(\nabla_{e_4}\xi, e_5) = -g(\nabla_{e_5}\xi, e_4) \Rightarrow a_2 = 0,$$
  $$g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1) \Rightarrow a_3 = 0,$$
  $$g(\nabla_{e_2}\xi, e_5) = -g(\nabla_{e_5}\xi, e_2) \Rightarrow a_4 = 0.$$
- There is no α-Sasakian structure. Assume that a structure (ϕ, ξ, η, g) on g_2 is α-Sasakian. Then, $\xi \in \langle e_5 \rangle$, since it is a Killing vector field. On the other hand, by considering the relation $\nabla_X\xi = -\alpha\phi(X)$, we get the endomorphism:
  $$\phi(e_1) = \frac{a_5}{2\alpha}e_3,\ \phi(e_2) = \frac{a_5}{2\alpha}e_4,\ \phi(e_3) = -\frac{a_5}{2\alpha}e_1,\ \phi(e_4) = -\frac{a_5}{2\alpha}e_2.$$
  In addition, the structure must satisfy the defining relation of the class of α-Sasakian structures:
  $$(\nabla_X\phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X).$$
  However, it is easy to see that this relation is not satisfied for $X = Y = e_1$. Hence, the structure is not α-Sasakian.
  Let (ϕ, ξ, η, g) is a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij}e^{ij}$. Then, $g(\nabla_{e_i}\xi, e_j) = g(\nabla_{e_j}\xi, e_i)$ for any basis elements e_i, e_j. Since $g(\nabla_{e_1}\xi, e_2) = -\frac{a_3}{2}$, $g(\nabla_{e_2}\xi, e_1) = \frac{a_3}{2} \Rightarrow a_3 = 0$ and $g(\nabla_{e_2}\xi, e_4) = -\frac{a_5}{2}$, $g(\nabla_{e_4}\xi, e_2) = \frac{a_5}{2} \Rightarrow a_4 = 0$, we have $\xi = a_1e_1 + a_2e_2 + a_4e_4$ and $\eta = b_1e^1 + b_2e^2 + b_4e^4$. On the other hand, for $X = e_1$, $Y = e_3$, $Z = e_5$ and for $X = e_3$, $Y = e_3$, $Z = e_5$ in the defining relation of a β-Kenmotsu structure, we obtain $b_{25} = 0$ and $b_{13} = 0$,

respectively. Similar arguments work if X, Y, Z are replaced by other basis elements. Thus, we get $b_{ij} = 0$ for all i, j. As a result, the structure is not β-Kenmotsu.

- There exists a semi cosymplectic structure.
  By checking covariant derivatives, it can be seen that $\delta\eta = 0$ for any one-form η. In addition, for a two-form $\Phi = \sum b_{ij}e^{ij}$, by assuming $\delta\Phi = 0$, we get $b_{12} = 0$ and $b_{13} = -b_{24}$. If we choose $\Phi = e^{13} - e^{24}$, then the endomorphism ϕ is $\phi(e_1) = -e_3$, $\phi(e_2) = e_4$, $\phi(e_3) = e_1$, $\phi(e_4) = -e_2$, $\phi(e_5) = 0$. For $\xi = e_5$ and $\eta = e^5$, (ϕ, ξ, η, g) is a semi cosymplectic structure on g_2.
- There exists an almost cosymplectic structure.
  The almost contact metric structure (ϕ, ξ, η, g), where $\xi = e_2$, $\eta = e^2$ and $\Phi = e^{15} + e^{34}$ is almost cosymplectic, that is $d\Phi = d\eta = 0$.

The algebra g_3: By Kozsul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{array}{llll}
\nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_4, & \nabla_{e_1}e_4 = -\frac{1}{2}e_3 + \frac{1}{2}e_5, & \nabla_{e_1}e_5 = -\frac{1}{2}e_4, \\
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_5, & \nabla_{e_2}e_5 = -\frac{1}{2}e_3, & \\
\nabla_{e_3}e_1 = -\frac{1}{2}e_2 - \frac{1}{2}e_4, & \nabla_{e_3}e_2 = \frac{1}{2}e_1 - \frac{1}{2}e_5, & \nabla_{e_3}e_4 = \frac{1}{2}e_1, & \nabla_{e_3}e_5 = \frac{1}{2}e_2, \\
\nabla_{e_4}e_1 = -\frac{1}{2}e_3 - \frac{1}{2}e_5, & \nabla_{e_4}e_3 = \frac{1}{2}e_1, & \nabla_{e_4}e_5 = \frac{1}{2}e_1, & \\
\nabla_{e_5}e_1 = -\frac{1}{2}e_4, & \nabla_{e_5}e_2 = -\frac{1}{2}e_3, & \nabla_{e_5}e_3 = \frac{1}{2}e_2, & \nabla_{e_5}e_4 = \frac{1}{2}e_1.
\end{array}
$$

- There exists no cosymplectic structure.
  The proof is similar to that of g_1.
- There exists no nearly cosymplectic structure.
  Let (ϕ, ξ, η, g) be a nearly cosymplectic structure with fundamental two-form $\Phi = \sum b_{ij}e^{ij}$. Then, for any basis elements e_i, e_j, we have $(\nabla_{e_i}\Phi)(e_i, e_j) = -\Phi(e_i, \nabla_{e_i}e_j) = 0$. After some calculations, we see that this equation holds if and only if $\Phi = b_{24}e^{24}$. However, the condition $\eta \wedge \Phi \wedge \Phi \neq 0$ is not satisfied since $\Phi \wedge \Phi = 0$.
- There is no non-zero parallel vector field on g_3.
  The proof is similar to these of g_1 and g_2.
- A vector field ξ on g_3 is Killing if and only if $\xi \in \langle e_5 \rangle$.
  Let $\xi = \sum a_i e_i$ be a Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$.
  Now, $g(\nabla_{e_1}\xi, e_3) = -g(\nabla_{e_3}\xi, e_1) \Rightarrow a_2 = 0$,
  $g(\nabla_{e_1}\xi, e_4) = -g(\nabla_{e_4}\xi, e_1) \Rightarrow a_3 = 0$,
  $g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1) \Rightarrow a_4 = 0$,
  $g(\nabla_{e_2}\xi, e_3) = -g(\nabla_{e_3}\xi, e_2) \Rightarrow a_1 = 0$. In other words, ξ is Killing if and only if $\xi = a_5 e_5$
- There is no α-Sasakian structure.
  Let (ϕ, ξ, η, g) be an α-Sasakian structure on g_3. Then, $\xi \in \langle e_5 \rangle$, since it is a Killing vector field. On the other hand, by considering the relation $\nabla_X \xi = -\alpha\phi(X)$, we get the endomorphism as:

  $$\phi(e_1) = \frac{a_5}{2\alpha}e_4,\ \phi(e_2) = \frac{a_5}{2\alpha}e_3,\ \phi(e_3) = -\frac{a_5}{2\alpha}e_2,\ \phi(e_4) = -\frac{a_5}{2\alpha}e_1.$$

  However, for $X = Y = e_1$, this structure does not satisfy the the defining relation $(\nabla_X\phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X)$.
- There is no β-Kenmotsu structure.
  Let (ϕ, ξ, η, g) be a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij}e^{ij}$, $\xi = \sum a_i e_i$, $\eta = \sum b_i e^i$. Then, $g(\nabla_{e_i}\xi, e_j) = g(\nabla_{e_j}\xi, e_i)$ for any basis elements e_i, e_j, which implies that $\xi = a_1e_1 + a_2e_2$ and $\eta = b_1e^1 + b_2e^2$. However, replacing basis elements for vector fields in the

defining relation of a β-Kenmotsu structure, we get $b_{ij} = 0$, for any i, j. Thus, there does not exist a β-Kenmotsu structure.

- There exists a semi cosymplectic structure.
  The almost contact metric structure (ϕ, ξ, η, g) for which $\xi = e_5$, $\eta = e^5$ and $\Phi = e^{14} - e^{23}$ is semi cosymplectic, that is, $\delta\Phi = \delta\eta = 0$.
- There exists an almost cosymplectic structure.
  The almost contact metric structure (ϕ, ξ, η, g), such that $\xi = e_1$, $\eta = e^1$ and $\Phi = e^{25} - e^{34}$ is almost cosymplectic.

The algebra g_4: By Kozsul's formula, the covariant derivatives of the basis elements are as follows:

$$\nabla_{e_1} e_2 = \frac{1}{2} e_3, \ \nabla_{e_1} e_3 = -\frac{1}{2} e_2 + \frac{1}{2} e_4, \ \nabla_{e_1} e_4 = -\frac{1}{2} e_3 + \frac{1}{2} e_5, \ \nabla_{e_1} e_5 = -\frac{1}{2} e_4,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2} e_3, \ \nabla_{e_2} e_3 = \frac{1}{2} e_1,$$

$$\nabla_{e_3} e_1 = -\frac{1}{2} e_2 - \frac{1}{2} e_4, \ \nabla_{e_3} e_2 = \frac{1}{2} e_1, \ \nabla_{e_3} e_4 = \frac{1}{2} e_1,$$

$$\nabla_{e_4} e_1 = -\frac{1}{2} e_3 - \frac{1}{2} e_5, \ \nabla_{e_4} e_3 = \frac{1}{2} e_1, \ \nabla_{e_4} e_5 = \frac{1}{2} e_1,$$

$$\nabla_{e_5} e_1 = -\frac{1}{2} e_4, \ \nabla_{e_5} e_4 = \frac{1}{2} e_1.$$

- There exists no cosymplectic structure.
  Assume that the two-form $\Phi = \sum b_{ij} e^{ij}$ is parallel. Then, $(\nabla_{e_i} \Phi)(e_j, e_k) = 0$ for any basis elements e_i, e_j, e_k. This gives $b_{ij} = 0$. Thus, there is no non-zero parallel two-form on g_4.
- There is no nearly cosymplectic structure on g_4.
  Let $\Phi = \sum b_{ij} e^{ij}$ be the two-form of a nearly cosymplectic a.c.m.s. Replacing X and Y by basis elements, we have $(\nabla_{e_1} \Phi)(e_1, e_2) = -\frac{1}{2} b_{13} = 0$ and similarly $b_{ij} = 0$, except for b_{24}, b_{25} and b_{35}. Thus, $\Phi = b_{24} e^{24} + b_{25} e^{25} + b_{35} e^{35}$. We get $b_{24} = 0$, $b_{25} = 0$ and $b_{35} = 0$ for $X = e_1$, $Y = e_2$, $Z = e_3$; $X = e_1$, $Y = e_2$, $Z = e_4$ and $X = e_1$, $Y = e_4$, $Z = e_5$ respectively from the equation (2).
- There is no non-zero parallel vector field on g_4.
  If a non-zero vector field $\xi = \sum a_i e_i$ is parallel ($\nabla \xi = 0$), by calculating $g(\nabla_{e_i} \xi, e_j)$ for basis elements, we get $a_i = 0$, for $i = 1, \cdots, 5$. It also shows that $\nabla \eta \neq 0$ for any almost contact metric structure (ϕ, ξ, η, g) on g_4. In particular, (ϕ, ξ, η, g) is neither $\mathcal{C}_1$ (nearly-K-cosymplectic), nor $\mathcal{C}_2$.
- A vector field ξ on g_4 is Killing if and only if $\xi \in \langle e_5 \rangle$.
  Let $\xi = \sum a_i e_i$ be a non-zero Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i} \xi, e_j) = -g(\nabla_{e_j} \xi, e_i)$. Thus,

$$g(\nabla_{e_1} \xi, e_3) = -g(\nabla_{e_3} \xi, e_1) \Rightarrow a_2 = 0,$$

$$g(\nabla_{e_1} \xi, e_4) = -g(\nabla_{e_4} \xi, e_1) \Rightarrow a_3 = 0,$$

$$g(\nabla_{e_1} \xi, e_5) = -g(\nabla_{e_5} \xi, e_1) \Rightarrow a_4 = 0,$$

$$g(\nabla_{e_2} \xi, e_3) = -g(\nabla_{e_3} \xi, e_2) \Rightarrow a_1 = 0.$$

  No condition is obtained for a_5. In other words, ξ is Killing if and only if $\xi = a_5 e_5$.
- There is no α-Sasakian structure.
  Let (ϕ, ξ, η, g) be an α-Sasakian structure on g_4. Then, $\xi = e_5$, since it is a unit Killing vector field. On the other hand, by considering the relation $\nabla_X \xi = -\alpha \phi(X)$, we get $\phi(e_2) = -\frac{1}{2\alpha} \nabla_{e_2} e_5 = 0$. However, in this case, $g(\phi(e_2), \phi(e_2)) \neq g(e_2, e_2) - \eta(e_2)\eta(e_2)$.
- There is no β-Kenmotsu structure.
  Let (ϕ, ξ, η, g) be a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij} e^{ij}$, $\xi = \sum a_i e_i$,

$\eta = \sum b_i e^i$. Then, $g(\nabla_{e_i}\xi, e_j) = g(\nabla_{e_j}\xi, e_i)$ for any basis elements e_i, e_j, which implies that $\xi = a_1e_1 + a_2e_2$ and $\eta = b_1e^1 + b_2e^2$. However, after an easy calculation on the defining relation of a β-Kenmotsu structure, we get $b_{ij} = 0$, for any i, j.

- There exists a semi cosymplectic structure.
  For any two-form $\Phi = \sum b_{ij}e^{ij}$ any $X = \sum x_i e_i \in g_4$,
$$\delta\Phi(X) = -\sum(\nabla_{e_i}\Phi)(e_i, X) = -\{x_3b_{12} + x_4b_{13} + x_5b_{14}\}.$$
  Thus $\delta\Phi(X) = 0$ for any X iff $b_{12} = b_{13} = b_{14} = 0$. In addition, for any one-form $\eta = \sum b_ie^i$, we have
$$\delta\eta = -\sum(\nabla_{e_i}\eta)(e_i) = -\sum g(\nabla_{e_i}\xi, e_i) = 0.$$
  Thus for example, the a.c.m.s. for which $\xi = e_1$, $\eta = e^1$ and $\Phi = e^{23} + e^{45}$ is semi cosymplectic.
- There exists no almost cosymplectic structure.
  Since $d\eta(X,Y) = \frac{1}{2}\{(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)\}$, $d\eta(X,Y) = 0$ iff $(\nabla_X\eta)(Y) = (\nabla_Y\eta)(X)$, or equivalently, $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$ for all X, Y in g_4. Substituting basis elements for X and Y implies that $d\eta = 0$ iff $\xi = a_1e_1 + a_2e_2$. Any almost cosymplectic structure is almost-K-contact, thus for the fundamental form $\Phi = \sum b_{ij}e^{ij}$ of an almost cosymplectic structure, we have $\nabla_\xi\Phi = 0$, where $\xi = a_1e_1 + a_2e_2$. $\nabla_\xi\Phi(e_i, e_j) = 0$ yields $\Phi = 0$.

The algebra g_5: By Kozsul's formula; the covariant derivatives of the basis elements are as follows:
$$\nabla_{e_1}e_2 = \tfrac{1}{2}e_4, \quad \nabla_{e_1}e_3 = \tfrac{1}{2}e_5, \quad \nabla_{e_1}e_4 = -\tfrac{1}{2}e_2, \quad \nabla_{e_1}e_5 = -\tfrac{1}{2}e_3,$$
$$\nabla_{e_2}e_1 = -\tfrac{1}{2}e_4, \quad \nabla_{e_2}e_4 = \tfrac{1}{2}e_1, \quad \nabla_{e_3}e_1 = -\tfrac{1}{2}e_5, \quad \nabla_{e_3}e_5 = \tfrac{1}{2}e_1,$$
$$\nabla_{e_4}e_1 = -\tfrac{1}{2}e_2 \quad \nabla_{e_4}e_2 = \tfrac{1}{2}e_1, \quad \nabla_{e_5}e_1 = -\tfrac{1}{2}e_3, \quad \nabla_{e_5}e_3 = \tfrac{1}{2}e_1$$

- There exists no cosymplectic structure.
  The proof is similar to that in other algebras.
- There is no nearly cosymplectic structure on g_5.
  Let $\Phi = \sum b_{ij}e^{ij}$ be the two-form of a nearly cosymplectic a.c.m.s. Replacing X and Y by basis elements, we have $(\nabla_{e_i}\Phi)(e_i, e_j) = 0$, which shows that $b_{ij} = 0$, except for b_{23}, b_{25}, b_{34} and b_{45}. Then $\Phi = b_{23}e^{23} + b_{25}e^{25} + b_{34}e^{34} + b_{45}e^{45}$. We obtain $b_{23} = b_{25} = b_{34} = b_{45} = 0$ for $X = e_1$, $Y = e_2$, $Z = e_5$; $X = e_1$, $Y = e_2$, $Z = e_3$; $X = e_3$, $Y = e_1$, $Z = e_2$ and $X = e_4$, $Y = e_1$, $Z = e_3$ respectively in the equation (2).
- There is no non-zero parallel vector field on g_5.
  The proof is similar to other cases. In particular, (ϕ, ξ, η, g) is neither $\mathcal{C}_1$ (nearly-K-cosymplectic), nor $\mathcal{C}_2$.
- A vector field ξ on g_5 is Killing if and only if $\xi \in \langle e_4, e_5\rangle$.
  Let $\xi = \sum a_ie_i$ be a non-zero Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$. Thus,
$$g(\nabla_{e_1}\xi, e_4) = -g(\nabla_{e_4}\xi, e_1) \Rightarrow a_2 = 0,$$
$$g(\nabla_{e_1}\xi, e_5) = -g(\nabla_{e_5}\xi, e_1) \Rightarrow a_3 = 0,$$
$$g(\nabla_{e_2}\xi, e_4) = -g(\nabla_{e_4}\xi, e_2) \Rightarrow a_1 = 0.$$
  No condition is obtained for a_4 and a_5. Thus, ξ is Killing if and only if $\xi = a_4e_4 + a_5e_5$.
- There is no α-Sasakian structure.
  Let (ϕ, ξ, η, g) be an α-Sasakian structure on g_5. Then, $\xi = a_4e_4 + a_5e_5$, where $a_4^2 + a_5^2 = 1$ and $\eta = b_4e_4 + b_5e_5$. By the relation $\nabla_X\xi = -\alpha\phi(X)$, we get $\phi(e_2) = -\frac{a_4}{2\alpha}e_1$ and $\phi(e_3) = -\frac{a_5}{2\alpha}e_1$.

Since $g(\phi(e_2), \phi(e_3)) = g(e_2, e_3) - \eta(e_2)\eta(e_3)$, we have $a_4.a_5 = 0$. This implies $\phi(e_2) = 0$, or $\phi(e_3) = 0$. Assume without loss of generality that $\phi(e_2) = 0$. Then, $g(\phi(e_2), \phi(e_2)) \neq g(e_2, e_2) - \eta(e_2)\eta(e_2)$.

- There is no β-Kenmotsu structure.
  Let (ϕ, ξ, η, g) be a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij}e^{ij}$, $\xi = \sum a_i e_i$, $\eta = \sum b_i e^i$. Then, $g(\nabla_{e_i}\xi, e_j) = g(\nabla_{e_j}\xi, e_i)$ for any basis elements e_i, e_j, which implies that $\xi = a_1e_1 + a_2e_2 + a_3e_3$ and $\eta = b_1e^1 + b_2e^2 + b_3e^3$. Replacing basis elements for X, Y, Z in the defining relation of β-Kenmotsu structures results in $\Phi = 0$. Thus, there does not exist a β-Kenmotsu structure.
- There exists a semi cosymplectic structure.
  For any two-form $\Phi = \sum b_{ij}e^{ij}$ any $X = \sum x_i e_i \in g_5$,

$$\delta\Phi(X) = -\sum(\nabla_{e_i}\Phi)(e_i, X) = -\{x_4b_{12} + x_5b_{13}\}$$

  Thus, $\delta\Phi(X) = 0$ for any X iff $b_{12} = b_{13} = 0$. In addition, for any one-form $\eta = \sum b_i e^i$, we have

$$\delta\eta = -\sum(\nabla_{e_i}\eta)(e_i) = -\sum g(\nabla_{e_i}\xi, e_i) = 0.$$

  Thus, for example, the a.c.m.s. for which $\xi = e_5$, $\eta = e^5$ and $\Phi = e^{14} + e^{23}$ is semi cosymplectic.
- There exists an almost cosymplectic structure.
  Consider, for instance, the a.c.m.s. given by $\xi = e_1$, $\eta = e^1$ and $\Phi = e^{25} + e^{34}$.

The algebra g_6: By Kozsul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{array}{llll}
\nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}e_2 + \frac{1}{2}e_4, & \nabla_{e_1}e_4 = -\frac{1}{2}e_3, & \\
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_5, & \nabla_{e_2}e_5 = -\frac{1}{2}e_3, & \\
\nabla_{e_3}e_1 = -\frac{1}{2}e_2 - \frac{1}{2}e_4, & \nabla_{e_3}e_2 = \frac{1}{2}e_1 - \frac{1}{2}e_5, & \nabla_{e_3}e_4 = \frac{1}{2}e_1, & \nabla_{e_3}e_5 = \frac{1}{2}e_2, \\
\nabla_{e_4}e_1 = -\frac{1}{2}e_3, & \nabla_{e_4}e_3 = \frac{1}{2}e_1, & \nabla_{e_5}e_2 = -\frac{1}{2}e_3, & \nabla_{e_5}e_3 = \frac{1}{2}e_2.
\end{array}$$

- There exists no cosymplectic structure on g_6.
  It is easy to see that $\nabla\Phi = 0$ if and only if $\Phi = 0$, where Φ is a two-form.
- There is no nearly cosymplectic structure.
  Let $\Phi = \sum b_{ij}e^{ij}$ be a two-form with the property that $(\nabla_X\Phi)(X, Y) = 0$. Then, we obtain $\Phi = b_{15}e^{15} + b_{24}e^{24} + b_{45}e^{45}$. By considering Φ as the fundamental two-form of an almost contact metric structure (ϕ, ξ, η, g), from the condition $\phi^2 = -I + \eta \otimes \xi$, we get $b_{15}^2 = 1$ and $b_{45} = 0$. We get $b_{15} = b_{24} = 0$ for $X = e_5$, $Y = e_2$, $Z = e_3$ and $X = e_1$, $Y = e_3$, $Z = e_4$ respectively in the equation (2).
- There is no non-zero parallel vector field on g_6.
  The proof is the same as before.
- A vector field ξ on g_6 is Killing if and only if $\xi \in \langle e_4, e_5 \rangle$.
  Let $\xi = \sum a_i e_i$ be a Killing vector field. Then, for any e_i, e_j, we have $g(\nabla_{e_i}\xi, e_j) = -g(\nabla_{e_j}\xi, e_i)$. Thus,

$$g(\nabla_{e_2}\xi, e_3) = -g(\nabla_{e_3}\xi, e_2) \Rightarrow a_1 = 0,$$

$$g(\nabla_{e_1}\xi, e_3) = -g(\nabla_{e_3}\xi, e_1) \Rightarrow a_2 = 0,$$

$$g(\nabla_{e_1}\xi, e_4) = -g(\nabla_{e_4}\xi, e_1) \Rightarrow a_3 = 0.$$

  No conditions are obtained for a_4 and a_5.

- There exists no α-Sasakian structure.
  Let (ϕ, ξ, η, g) be an α-Sasakian structure on g_6. Then, ξ has the form $\xi = a_4 e_4 + a_5 e_5$ and satisfies the equation $\nabla_X \xi = -\alpha\phi(X)$. Thus, the endomorphism can be expressed with:
  $$\phi(e_1) = \frac{a_4}{2\alpha}e_3,\ \phi(e_2) = \frac{a_5}{2\alpha}e_3,\ \phi(e_3) = -\frac{a_5}{2\alpha}e_2,\ \phi(e_4) = 0,\ \phi(e_5) = 0.$$
  From the condition $\phi^2 = -I + \eta \otimes \xi$, we have
  $$\phi^2(e_4) = 0 = (a_4^2 - 1)e_4 + a_4 a_5 e_5 \Rightarrow a_4^2 = 1,\ a_4 a_5 = 0$$
  and
  $$\phi^2(e_5) = 0 = a_5 a_4 e_4 + (a_5^2 - 1)e_5 \Rightarrow a_5^2 = 1,\ a_4 a_5 = 0.$$
  However, since $a_5^2 = a_4^2 = 1$, the number $a_4 a_5$ is non-zero.
- There is no β-Kenmotsu structure.
  Let (ϕ, ξ, η, g) be a β-Kenmotsu structure with fundamental two-form $\Phi = \sum b_{ij} e^{ij}$, $\xi = \sum a_i e_i$. Then, $g(\nabla_{e_i}\xi, e_j) = g(\nabla_{e_j}\xi, e_i)$ for any basis elements e_i, e_j, which implies that $\xi = a_1 e_1 + a_2 e_2$. However, after calculations on the defining relation, we get $\Phi = b_{12} e^{12}$. However, in this case, $\Phi \wedge \Phi = 0$.
- There exists a semi cosymplectic structure.
  The a.c.m.s. (ϕ, ξ, η, g) with $\xi = e_3$, $\eta = e^3$ and $\Phi = e^{14} + e^{25}$ is semi cosymplectic.
- There is no almost cosymplectic structure.
  Obviously, $de^1 = 0$, $de^2 = 0$, $de^3 = -e^{12}$, $de^4 = -e^{13}$, $de^5 = -e^{23}$. Thus, for a one-form $\eta = \sum b_i e^i$, we have $d\eta = 0 \iff b_3 = b_4 = b_5 = 0$, and for a two-form $\Phi = \sum b_{ij} e^{ij}$, we get $d\Phi = 0 \iff b_{15} = b_{24}$, $b_{34} = b_{35} = b_{45} = 0$. So, if (ϕ, ξ, η, g) with the fundamental two-form $\Phi = \sum b_{ij} e^{ij}$ is an almost cosymplectic structure on g_6, then, Φ and η have the forms $\Phi = b_{12}e^{12} + b_{13}e^{13} + b_{14}e^{14} + b_{15}e^{15} + b_{23}e^{23} + b_{15}e^{24} + b_{25}e^{25}$ and $\eta = b_1 e^1 + b_2 e^2$. However, it is easy to see that $\eta \wedge \Phi \wedge \Phi$ vanishes. Thus, the structure is not almost cosymplectic.

In summary, we state the following.

Theorem 1. *An almost contact metric structure on a five-dimensional nilpotent Lie algebra g is cosymplectic if and only if g is abelian.*

The existence of cosymplectic structures on Lie groups and on their compact quotients by uniform discrete subgroups was studied in [14]. We state Theorem 1 by direct calculation.

In the sequel, we deduce

Corollary 2. *There is no cosymplectic left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.*

Theorem 3. *There is no nearly cosymplectic structure on any five-dimensional nilpotent Lie algebra.*

Corollary 4. *There is no nearly cosymplectic left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.*

Theorem 5. *There exists no non-zero parallel vector field on any five-dimensional nilpotent Lie algebra.*

There are non-zero Killing vector fields on g_i for $i \in \{1, 2, 3, 4, 5, 6\}$.

Theorem 6. *Let g be one of g_1, g_2, g_3 or g_4. A vector field ξ on g is Killing if and only if $\xi \in < e_5 >$. In addition, if g is g_5 or g_6, then ξ is Killing iff $\xi \in < e_4, e_5 >$.*

Theorem 7. *If g has an α-Sasakian structure, then g is isomorphic to g_1.*

Theorem 8. *There is no β-Kenmotsu a.c.m.s. on any five-dimensional nilpotent Lie algebra.*

We may conclude

Corollary 9. *There is no β-Kenmotsu left invariant almost contact metric structure on a five-dimensional connected Lie group whose corresponding Lie algebra is nilpotent.*

Theorem 10. *There exist semi cosymplectic a.c.m. structures on each g_i.*

Theorem 11. *An a.c.m.s. on g is almost cosymplectic iff g is isomorphic to one of g_2, g_3 or g_5.*

Let G be a simply-connected nilpotent Lie group with Lie algebra g. It is known that there exists a co-compact discrete subgroup Γ of G such that G/Γ is a compact nilmanifold [15]. Giving examples of discrete subgroups Γ for simply-connected nilpotent Lie group G_i with Lie algebra g_i is an ongoing study.

4. Conclusions

In this paper, we examined almost contact metric structures on five dimensional nilpotent Lie algebras by direct calculation and obtained some results about the relations between the classes of almost contact metric structures and five dimensional nilpotent Lie algebras. In addition, we got some general results on left invariant almost contact metric structures on five dimensional nilpotent Lie groups by studying their corresponding Lie algebras.

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