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Generalized Null 2-Type Surfaces in Minkowski 3-Space

Dae Won Yoon [1], Dong-Soo Kim [2], Young Ho Kim [3] and Jae Won Lee [1,*]

1 Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 52828, Korea; dwyoon@gnu.ac.kr
2 Department of Mathematics, Chonnam National University, Gwangju 61186, Korea; dosokim@chonnam.ac.kr
3 Department of Mathematics, Kyungpook National University, Daegu 41566, Korea; yhkim@knu.ac.kr
* Correspondence: leejaew@gnu.ac.kr; Tel.: +82-55-772-2251

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Abstract: For the mean curvature vector field $\mathbf{H}$ and the Laplace operator Δ of a submanifold in the Minkowski space, a submanifold satisfying the condition $\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C}$ is known as a generalized null 2-type, where f and g are smooth functions, and $\mathbf{C}$ is a constant vector. The notion of generalized null 2-type submanifolds is a generalization of null 2-type submanifolds defined by B.-Y. Chen. In this paper, we study flat surfaces in the Minkowski 3-space $\mathbb{L}^3$ and classify generalized null 2-type flat surfaces. In addition, we show that the only generalized null 2-type null scroll in $\mathbb{L}^3$ is a B-scroll.

Keywords: flat surface; generalized null 2-type surface; mean curvature vector; B-scroll

1. Introduction

Let $x : M \longrightarrow \mathbb{E}^m$ be an isometric immersion of an n-dimensional connected submanifold M in an m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ, respectively, the mean curvature vector field and the Laplacian operator with respect to the induced metric on M induced from that of $\mathbb{E}^m$. Then, it is well known as

$$\Delta x = -n\mathbf{H}. \tag{1}$$

By using (1), Takahashi [1] proved that minimal submanifolds of a hypersphere of $\mathbb{E}^m$ are constructed from eigenfunctions of Δ with one eigenvalue λ ($\neq 0$). In [2,3], Chen initiated the study of submanifolds in $\mathbb{E}^m$ that are constructed from harmonic functions and eigenfunctions of Δ with a nonzero eigenvalue. The position vector x of such a submanifold admits the following simple spectral decomposition:

$$x = x_0 + x_q, \quad \Delta x_0 = 0, \quad \Delta x_q = \lambda x_q \tag{2}$$

for some non-constant maps x_0 and x_q, where λ is a nonzero constant. A submanifold satisfying (2) is said to be of null 2-type [3]. From the definition of null 2-type submanifolds and (1), it follows that the mean curvature vector field $\mathbf{H}$ satisfies the following condition:

$$\Delta\mathbf{H} = \lambda\mathbf{H}. \tag{3}$$

A result from [4] states that a surface in the Euclidean space $\mathbb{E}^3$ satisfying (3) is either a minimal surface or an open part of an ordinary sphere or a circular cylinder. Ferrández and Lucas [5] extended it to the Lorentzian case. They proved that the surface satisfying (3) is either a minimal surface or an open part of a Lorentz circular cylinder, a hyperbolic cylinder, a Lorentz hyperbolic cylinder, a hyperbolic space, a de Sitter space or a B-scroll. Afterwards, several authors studied null 2-type submanifolds in the (pseudo-)Euclidean space [6–21].

Now, we will give a generalization of null 2-type submanifolds in the Minkowski space. It is well known that a Lorentz circular cylinder $\mathbb{S}^1(r) \times \mathbb{R}^1_1$ is a null 2-type surface in the Minkowski 3-space $\mathbb{L}^3$ satisfying $\Delta\mathbf{H} = \frac{1}{r^2}\mathbf{H}$, where $\mathbb{S}^1(r)$ is a circle with radius r and $\mathbb{R}^1_1$ is a Lorentz straight line. However, the following surface has another property as follows: a parametrization

$$x(s,t) = \left(\frac{1}{4}s^2 - \frac{1}{2}\ln s, \frac{1}{4}s^2 + \frac{1}{2}\ln s, t\right)$$

is a cylindrical surface in $\mathbb{L}^3$. On the other hand, the mean curvature vector field $\mathbf{H}$ of the surface is given by

$$\mathbf{H} = \left(-\frac{1}{4} - \frac{1}{4s^2}, -\frac{1}{4} + \frac{1}{4s^2}, 0\right)$$

and the surface satisfies

$$\Delta\mathbf{H} = -\frac{6}{s^2}\left(\mathbf{H} + (\frac{1}{4}, \frac{1}{4}, 0)\right).$$

Next, we consider another surface with a parametrization

$$x(s,t) = \left(\frac{1}{2}s^2 t + t, st, \frac{1}{2}s^2 t\right).$$

The surface is a conical surface in $\mathbb{L}^3$, and it satisfies the following equation for the mean curvature vector $\mathbf{H}$

$$\Delta\mathbf{H} = \frac{1}{t^2}\mathbf{H} + \frac{1}{2t^3}(1,0,1).$$

Thus, based on the above examples, we give the definition:

Definition 1. *A submanifold M of the Minkowski space is said to be of generalized null 2-type if it satisfies the condition*

$$\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C} \tag{4}$$

for some smooth functions f, g and a constant vector $\mathbf{C}$. In particular, if the functions f and g are equal to each other in (4), then the submanifold M is called of generalized null 2-type of the first kind and of the second kind otherwise.

In [22], the authors recently classified generalized null 2-type flat surfaces in the Euclidean 3-space. Conical surfaces, cylindrical surfaces or tangent developable surfaces are developable surfaces (or flat surfaces) as ruled surfaces in the Minkowski 3-space $\mathbb{L}^3$. In this paper, we study developable surfaces in $\mathbb{L}^3$ and completely classify generalized null 2-type developable surfaces, and give some examples. In addition, we investigate null scrolls in the Minkwoski 3-space $\mathbb{L}^3$ satisfying the condition (4).

2. Preliminaries

The Minkowski 3-space $\mathbb{L}^3$ is a real space $\mathbb{R}^3$ with the standard flat metric given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{L}^3$ is said to be *space-like* if $\langle\mathbf{x},\mathbf{x}\rangle > 0$ or $\mathbf{x} = 0$, *time-like* if $\langle\mathbf{x},\mathbf{x}\rangle < 0$ and *null* if $\langle\mathbf{x},\mathbf{x}\rangle = 0$ and $\mathbf{x} \neq 0$. A time-like or null vector in $\mathbb{L}^3$ is said to be *causal*. Similarly, an arbitrary curve $\gamma = \gamma(s)$ is *space-like*, *time-like* or *null* if all of its tangent vectors $\gamma'(s)$ are space-like, time-like or null, respectively. From now on, the "prime" means the partial derivative with respect to the parameter s unless mentioned otherwise.

We now put a 2-dimensional space form in $\mathbb{L}^3$ as follows:

$$\mathbb{Q}^2(\varepsilon) = \begin{cases} \mathbb{S}^2_1 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 | -x_1^2 + x_2^2 + x_3^2 = 1\}, \text{ if } \varepsilon = 1; \\ \mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 | -x_1^2 + x_2^2 + x_3^2 = -1\}, \text{ if } \varepsilon = -1. \end{cases}$$

We call $\mathbb{S}^2_1$ and $\mathbb{H}^2$ the de-Sitter space and the hyperbolic space, respectively.

Let $\gamma : I \longrightarrow \mathbb{L}^3$ be a space-like or time-like curve in the Minkowski 3-space $\mathbb{L}^3$ parameterized by its arc-length s. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame field along $\gamma(s)$.

If $\gamma(s)$ is a space-like curve in $\mathbb{L}^3$, the Frenet formulae of $\gamma(s)$ are given by [23]:

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s), \\ \mathbf{n}'(s) &= -\varepsilon\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ \mathbf{b}'(s) &= \varepsilon\tau(s)\mathbf{n}(s), \end{aligned} \tag{5}$$

where $\langle \mathbf{t}, \mathbf{t} \rangle = 1, \langle \mathbf{n}, \mathbf{n} \rangle = \varepsilon(= \pm 1), \langle \mathbf{b}, \mathbf{b} \rangle = -\varepsilon$. Here, the functions $\kappa(s)$ and $\tau(s)$ are the curvature function and the torsion function of a space-like curve $\gamma(s)$, respectively.

If $\gamma(s)$ is a time-like curve in $\mathbb{L}^3$, the Frenet formulae of $\gamma(s)$ are given by [23]:

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s), \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ \mathbf{b}'(s) &= -\tau(s)\mathbf{n}(s), \end{aligned} \tag{6}$$

where $\langle \mathbf{t}, \mathbf{t} \rangle = -1, \langle \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = 1$. Here $\kappa(s)$ and $\tau(s)$ are the curvature function and the torsion function of a time-like curve $\gamma(s)$, respectively.

If $\gamma(s)$ is a space-like or time-like pseudo-spherical curve parametrized by arc-length s in $\mathbb{Q}^2(\varepsilon)$, let $\mathbf{t}(s) = \gamma'(s)$ and $\mathbf{g}(s) = \gamma(s) \times \gamma'(s)$. Then, we have a pseudo-orthonormal frame $\{\gamma(s), \mathbf{t}(s), \mathbf{g}(s)\}$ along $\gamma(s)$. It is called the *pseudo-spherical Frenet frame* of the pseudo-spherical curve $\gamma(s)$. If γ is a space-like curve, then the vector $\mathbf{g}$ is time-like when γ is on $\mathbb{S}^2_1$, and the vector $\mathbf{g}$ is space-like when γ is on $\mathbb{H}^2$. Similarly, if the curve γ is time-like, then the vector $\mathbf{g}$ is space-like. The following theorem can be easily obtained.

Theorem 1. *([24,25]) Under the above notations, we have the following pseudo-spherical Frenet formulae of γ:*
(1) If γ is a pseudo-spherical space-like curve,

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= -\varepsilon\gamma(s) - \varepsilon\kappa_g(s)\mathbf{g}(s), \\ \mathbf{g}'(s) &= -\kappa_g(s)\mathbf{t}(s). \end{aligned} \tag{7}$$

(2) If γ is a pseudo-spherical time-like curve,

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= \gamma(s) + \kappa_g(s)\mathbf{g}(s), \\ \mathbf{g}'(s) &= \kappa_g(s)\mathbf{t}(s). \end{aligned} \tag{8}$$

The function $\kappa_g(s)$ is called the *geodesic curvature* of the pseudo-spherical curve γ.

Now, we define a ruled surface M in $\mathbb{L}^3$. Let I and J be open intervals in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{L}^3$ and $\beta = \beta(s)$ a vector field along α with $\alpha'(s) \times \beta(s) \neq 0$ for every $s \in J$. Then, a ruled surface M is defined by the parametrization given as follows:

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

For such a ruled surface, α and β are called the base curve and the director curve respectively. In particular, if β is constant, the ruled surface is said to be cylindrical, and if it is not so, it is called non-cylindrical. Furthermore, we have five different ruled surfaces according to the characters of the base curve α and the director curve β as follows: if the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. In addition, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When β is time-like, β' is space-like because of the causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or null, respectively [26].

However, if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface M is called a null scroll. In particular, a null scroll with Cartan frame is said to be a B-scroll [27]. It is also a time-like surface.

A non-degenerate surface in $\mathbb{L}^3$ with zero Gaussian curvature is called a developable surface. The developable surfaces in $\mathbb{L}^3$ are the same as in the Euclidean space, and they are planes, conical surfaces, cylindrical surfaces and tangent developable surfaces [13].

3. Generalized Null 2-Type Cylindrical Surfaces

For a surface in the Minkowski 3-space $\mathbb{L}^3$, the next lemma is well known and useful.

Lemma 1. *([16]) Let M be an oriented surface of $\mathbb{L}^3$. Then, the Laplacian of the mean curvature vector field* $\mathbf{H}$ *of M is given by*

$$\Delta\mathbf{H} = 2A(\nabla H) + \varepsilon\nabla H^2 + (\Delta H + \varepsilon H|A|^2)N, \tag{9}$$

where ε is the sign of the unit normal vector N of the surface M and ∇H, A are the gradient of the mean curvature H and the shape operator of M, respectively.

Theorem 2. *All cylindrical surfaces in $\mathbb{L}^3$ are of generalized null 2-type.*

Proof. Let M be a cylindrical ruled surface in the Minkowski 3-space $\mathbb{L}^3$ of type M_+^1, M_-^1 or M_+^3. Then, M is parameterized by

$$x(s,t) = \alpha(s) + t\beta,$$

where the base curve $\alpha(s)$, which is a space-like or time-like curve with the arc-length parameter s, lies in a plane with a space-like or time-like unit normal vector β that is the director of M, that is, $\langle\beta,\beta\rangle = \varepsilon_1(=\pm 1)$ and $\langle\alpha'(s),\alpha'(s)\rangle = \varepsilon_2(\pm 1)$.

Now, we take a local pseudo-orthonormal frame $\{e_1, e_2, e_3\}$ on $\mathbb{L}^3$ such that $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial s}$ are tangent to M, and e_3 normal to M. It follows that the Levi–Civita connection $\widetilde{\nabla}$ of $\mathbb{L}^3$ is expressed as

$$\begin{aligned} &\widetilde{\nabla}_{e_1}e_1 = \widetilde{\nabla}_{e_1}e_2 = \widetilde{\nabla}_{e_2}e_1 = 0, \quad \widetilde{\nabla}_{e_2}e_2 = \varepsilon_3\kappa(s)e_3, \\ &\widetilde{\nabla}_{e_1}e_3 = 0, \quad \widetilde{\nabla}_{e_2}e_3 = -\varepsilon_2\kappa(s)e_2, \end{aligned} \tag{10}$$

where $\kappa(s)$ is the curvature function of $\alpha(s)$ and $\varepsilon_3(=\pm 1)$ is the sign of e_3. From this, the mean curvature vector field $\mathbf{H}$ of M is given by

$$\mathbf{H} = \frac{\varepsilon_2\kappa(s)}{2}e_3 \tag{11}$$

and the Laplacian $\Delta\mathbf{H}$ of $\mathbf{H}$ is expressed as

$$\Delta\mathbf{H} = \frac{3}{2}\varepsilon_1\varepsilon_2\kappa(s)\kappa'(s)e_2 - \frac{1}{2}(\kappa^3(s) + \varepsilon_1\kappa''(s))e_3. \tag{12}$$

Suppose that M is of generalized null 2-type. With the help of (4) and (12), we obtain the following equations:

$$gC_1 = 0, \tag{13}$$

$$\frac{3}{2}\varepsilon_1\varepsilon_2\kappa(s)\kappa'(s) = gC_2, \tag{14}$$

$$\frac{1}{2}\varepsilon_1\varepsilon_2\kappa^3(s) + \frac{1}{2}\varepsilon_2\kappa''(s) = -\frac{1}{2}\varepsilon_1\kappa(s)f + gC_3, \tag{15}$$

where $\mathbf{C} = \varepsilon_1 C_1 e_1 + \varepsilon_2 C_2 e_2 - \varepsilon_1\varepsilon_2 C_3 e_3$ with $C_1 = \langle\mathbf{C}, e_1\rangle$, $C_2 = \langle\mathbf{C}, e_2\rangle$ and $C_3 = \langle\mathbf{C}, e_3\rangle$. In this case, C_1 is a constant, and C_2, C_3 are functions of the variable s.

If g is identically zero, then, from (14), the curvature $\kappa(s)$ is constant, and from (15), the function f is constant, say λ. Thus, M satisfies $\Delta\mathbf{H} = \lambda\mathbf{H}$, that is, it is either a Euclidean plane, a Minkowski plane, a Lorentz circular cylinder $\mathbb{S}^2 \times \mathbb{R}^1_1$, a hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{R}$ or a Lorentz hyperbolic cylinder $\mathbb{S}^1_1 \times \mathbb{R}$ according to [16].

We now assume that $g \neq 0$. It follows from (13) that $C_1 = 0$. By using (10), we can show that the component functions of $\mathbf{C}$ satisfy the following equations:

$$\begin{aligned} C_2'(s) + \varepsilon_1\varepsilon_2\kappa(s)C_3(s) = 0, \\ C_3'(s) + \varepsilon_2\kappa(s)C_2(s) = 0, \end{aligned} \tag{16}$$

which yield $\varepsilon_2 C_2^2(s) - \varepsilon_1\varepsilon_2 C_3^2(s) = \eta d_0^2$ for some nonzero constant d_0, where $\eta = \langle\mathbf{C}, \mathbf{C}\rangle$.

Case 1: If M is of type M^3_+, then $\varepsilon_1 = -1$, $\varepsilon_2 = 1$ and $\eta = 1$. We may put from (16)

$$C_2(s) = d_0 \sin\theta(s), \quad C_3(s) = d_0\cos\theta(s), \tag{17}$$

where $\theta(s) = \kappa_0 + \int \kappa(s)ds$ for some constant κ_0. Therefore, the constant vector $\mathbf{C}$ becomes

$$\mathbf{C} = d_0\sin\theta(s)e_2 + d_0\cos\theta(s)e_3. \tag{18}$$

Combining (14), (15) and (17), one also gets

$$g = -\frac{3\kappa(s)\kappa'(s)}{2d_0}\csc\theta(s), \quad f = \frac{\kappa''(s)}{\kappa(s)} - \kappa^2(s) + 3\kappa'(s)\cot\theta(s). \tag{19}$$

Thus, the mean curvature vector field $\mathbf{H}$ of the cylindrical surface M_3^+ satisfies

$$\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C},$$

where f, g and $\mathbf{C}$ are given in (18) and (19), respectively.

Case 2: Let M be of type M^1_+. In this case, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$ and the constant vector $\mathbf{C}$ is space-like, time-like or null.

First of all, we consider the constant vector $\mathbf{C}$ is non-null. Then, from (16), we may put

$$\begin{cases} C_2(s) = d_0\cosh\theta(s), & C_3 = d_0\sinh\theta(s) \qquad \text{if } \eta = 1, \\ C_2(s) = d_0\sinh\theta(s), & C_3 = d_0\cosh\theta(s) \qquad \text{if } \eta = -1, \end{cases} \tag{20}$$

where $\theta(s) = -\int\kappa(s)ds + \kappa_0$ with a constant κ_0.

By using (14), (15) and (17), the functions $f(s)$ and $g(s)$ are determined by

$$\begin{cases} f(s) = -\frac{\kappa''(s)}{\kappa(s)} + \kappa^2(s) + 3\kappa'(s)\tanh\theta(s), & g(s) = \frac{3\kappa(s)\kappa'(s)}{2d_0\cosh\theta(s)} \quad \text{if } \eta = 1, \\ f(s) = -\frac{\kappa''(s)}{\kappa(s)} - \kappa^2(s) + 3\kappa'(s)\coth\theta(s), & g(s) = \frac{3\kappa(s)\kappa'(s)}{2d_0\sinh\theta(s)} \quad \text{if } \eta = -1. \end{cases} \tag{21}$$

Thus, for the non-null constant vector $\mathbf{C}$, the cylindrical surface M_1^+ is of generalized null 2-type, that is, it satisfies

$$\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C},$$

where f, g and $\mathbf{C}$ are given by (20) and (21), respectively.

Next, let the constant vector $\mathbf{C}$ be null, that is, $\eta = 0$. Then, we get

$$C_2(s) = \pm C_3(s).$$

We will consider the case $C_2(s) = C_3(s)$. It follows from (16) $C_2(s) = e^{\theta(s)}$, where $\theta(s) = -\int \kappa(s)ds + \kappa_0$ for some constant κ_0. In this case, we have

$$f(s) = -\frac{\kappa''(s)}{\kappa(s)} - \kappa^2(s) + 3\kappa'(s), \quad g(s) = \frac{3}{2}e^{-\theta(s)}\kappa(s)\kappa'(s) \tag{22}$$

and, for the null constant vector $\mathbf{C}$, the surface satisfies the condition $\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C}$.

Case 3: Let M be of type M_-^1, that is, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. In this case, the constant vector $\mathbf{C}$ is space-like, time-like or null.

Applying the same method as in Case 2, the functions $f(s)$ and $g(s)$ are determined by

$$\begin{cases} f(s) = \frac{\kappa''(s)}{\kappa(s)} + \kappa^2(s) + 3\kappa'(s)\coth\theta(s), & g(s) = \frac{3\kappa(s)\kappa'(s)}{2d_0\sinh\theta(s)} \quad \text{if } \eta = 1, \\ f(s) = \frac{\kappa''(s)}{\kappa(s)} + \kappa^2(s) + 3\kappa'(s)\tanh\theta(s), & g(s) = \frac{3\kappa(s)\kappa'(s)}{2d_0\cosh\theta(s)} \quad \text{if } \eta = -1, \\ f(s) = \frac{\kappa''(s)}{\kappa(s)} + \kappa^2(s) + 3\kappa'(s), & g(s) = \frac{3}{2}e^{-\theta(s)}\kappa(s)\kappa'(s) \quad \text{if } \eta = 0, \end{cases} \tag{23}$$

and the component functions of $\mathbf{C}$ are given by

$$\begin{cases} C_2(s) = d_0\sinh\theta(s), & C_3(s) = d_0\cosh\theta(s), \quad \text{if } \eta = 1, \\ C_2(s) = d_0\cosh\theta(s), & C_3(s) = d_0\sinh\theta(s), \quad \text{if } \eta = -1, \\ C_2(s) = \pm C_3(s), & \quad \text{if } \eta = 0, \end{cases} \tag{24}$$

where $\theta(s) = \int \kappa(s)ds + \kappa_0$ for some constant κ_0.

Thus, from Cases 1, 2 and 3, Theorem 2 is proved. $\square$

Example 1. *We consider a surface defined by*

$$x(s,t) = \left(\frac{1}{4}s^2 - \frac{1}{2}\ln s, \frac{1}{4}s^2 + \frac{1}{2}\ln s, t\right). \tag{25}$$

This parametrization is a cylindrical ruled surface of type M_+^1. In this case, the mean curvature vector field $\mathbf{H}$ *of the surface is given by*

$$\mathbf{H} = \left(-\frac{1}{4} - \frac{1}{4s^2}, -\frac{1}{4} + \frac{1}{4s^2}, 0\right).$$

By a direct computation, the Laplacian $\Delta\mathbf{H}$ of the mean curvature vector field $\mathbf{H}$ becomes

$$\Delta\mathbf{H} = \left(\frac{3}{2s^4}, -\frac{3}{2s^4}, 0\right),$$

and it can be rewritten in terms of the mean curvature vector field **H** *and a constant vector* **C** *as follows:*

$$\Delta\mathbf{H} = -\frac{6}{s^2}(\mathbf{H}+\mathbf{C}),$$

where $\mathbf{C} = (\frac{1}{4}, \frac{1}{4}, 0)$ *is a null vector. Thus, the cylindrical ruled surface defined by (25) is a generalized null 2-type surface of the first kind.*

Remark 1. *A cylindrical surface in* $\mathbb{L}^3$ *generated by the base curve* $\alpha(s)$ *with the curvature* $\kappa(s) = \frac{1}{s}$ *and a constant director* β *is a generalized null 2-type surface of the first kind if the constant vector* **C** *is null.*

4. Generalized Null 2-Type Non-Cylindrical Flat Surfaces

In this section, we classify non-cylindrical flat surfaces satisfying

$$\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C}. \tag{26}$$

It is well-known that a non-cylindrical flat surface in the Minkowski 3-space $\mathbb{L}^3$ is an open part of a conical surface or a tangent developable surface.

First of all, we consider a conical surface M in $\mathbb{L}^3$. Then, we may give the parametrization of M by

$$x(s,t) = \alpha_0 + t\beta(s), \quad s \in I, \quad t > 0,$$

such that $\langle \beta'(s), \beta'(s)\rangle = \varepsilon_1$ and $\langle \beta(s), \beta(s)\rangle = \varepsilon_2$, where α_0 is a constant vector. We take the orthonormal tangent frame $\{e_1, e_2\}$ on M such that $e_1 = \frac{1}{t}\frac{\partial}{\partial s}$ and $e_2 = \frac{\partial}{\partial t}$. The unit normal vector of M is given by $e_3 = e_1 \times e_2$. By the Gauss and Weingarten formulas, we have

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = -\frac{\varepsilon_1\varepsilon_2}{t}e_2 + \frac{\varepsilon_1\varepsilon_2\kappa_g(s)}{t}e_3, \quad \tilde{\nabla}_{e_1} e_2 = \frac{1}{t}e_1, \quad \tilde{\nabla}_{e_2} e_1 = \tilde{\nabla}_{e_2} e_2 = 0, \\ &\tilde{\nabla}_{e_1} e_3 = \frac{\varepsilon_1\kappa_g(s)}{t}e_1, \quad \tilde{\nabla}_{e_2} e_3 = 0, \end{aligned} \tag{27}$$

where $\kappa_g(s) = \langle \beta(s), \beta'(s) \times \beta''(s)\rangle$, which is the geodesic curvature of the pseudo-spherical curve $\beta(s)$ in $\mathbb{Q}^2(\varepsilon)$. From (27), the mean curvature vector field **H** of M is given by

$$\mathbf{H} = -\frac{\varepsilon_1\kappa_g(s)}{2t}e_3, \tag{28}$$

and the Laplacian $\Delta\mathbf{H}$ of the mean curvature vector field **H** is expressed as

$$\Delta\mathbf{H} = \frac{3\varepsilon_1}{2t^3}\kappa_g(s)\kappa_g'(s)e_1 - \frac{\varepsilon_2}{2t^3}\kappa_g^2(s)e_2 + \left(\frac{1}{2t^3}\kappa_g''(s) + \frac{\varepsilon_2}{2t^3}\kappa_g^3(s) + \frac{\varepsilon_1\varepsilon_2}{2t^3}\kappa_g(s)\right)e_3. \tag{29}$$

Suppose that κ_g is constant. If $\kappa_g = 0$, by a rigid motion, the pseudo-spherical curve $\beta(s)$ in $\mathbb{Q}^2(\varepsilon)$ lies on yz-plane or xz-plane. Thus M is an open part of a Euclidean plane or a Minkowski plane. If κ_g is a non-zero constant, from (27), we can obtain by a straightforward computation

$$\beta'''(s) = \varepsilon_2(\kappa_g^2(s) - \varepsilon_1)\beta'(s). \tag{30}$$

Case 1: $\varepsilon_2(\kappa_g^2(s) - \varepsilon_1) = k^2$ for some real number k.

Let $\varepsilon_1 = 1$. Without loss of generality, we may assume $\beta'(0) = (0,1,0)$. Thus, $\beta'''(s) = k^2\beta'(s)$ implies

$$\beta'(s) = (B_1 \sinh ks, \cosh ks + B_2 \sinh ks, B_3 \sinh ks)$$

for some constants B_1, B_2 and B_3. Since $\varepsilon_1 = 1$, we have $B_1^2 - B_3^2 = 1$ and $B_2 = 0$. From this, we can obtain

$$\beta(s) = \left(\frac{B_1}{k}\cosh ks + D_1, \frac{1}{k}\sinh ks, \frac{B_3}{k}\cosh ks + D_3\right) \tag{31}$$

for some constants D_1, D_3 satisfying $D_3^2 - D_1^2 = \frac{1}{k^2} + \varepsilon_2$, $B_1 D_1 = B_3 D_3$ and $B_1^2 - B_3^2 = 1$. We now change the coordinates by $\bar{x}, \bar{y}, \bar{z}$ such that $\bar{x} = B_1 x - B_3 z$, $\bar{y} = y$, $\bar{z} = -B_3 x + B_1 z$, that is,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} B_1 & 0 & -B_3 \\ 0 & 1 & 0 \\ -B_3 & 0 & B_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With respect to the coordinates $(\bar{x}, \bar{y}, \bar{z})$, $\beta(s)$ turns into

$$\beta(s) = \left(\frac{1}{k}\cosh ks, \frac{1}{k}\sinh ks, D\right) \tag{32}$$

for a constant $D = B_1 D_3 - B_3 D_1$ with $D^2 = \frac{1}{k^2} + \varepsilon_2$. Thus, up to a rigid motion M has the parametrization of the form

$$x(s,t) = \alpha_0 + t\left(\frac{1}{k}\cosh ks, \frac{1}{k}\sinh ks, D\right).$$

We call such a surface a *hyperbolic conical surface of the first kind*, and it satisfies

$$\Delta \mathbf{H} = \left(\frac{\varepsilon_2(1 - D^2 - k^2)}{k^2 t^2}\right)\mathbf{H} + \left(\frac{\varepsilon_2 D(1 - D^2 k^2)}{2k^4 t^3}\right)(0,0,1).$$

Next, let $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. We now consider a initial condition $\beta'(0) = (1,0,0)$ of the ordinary differential equation (ODE) (30). Quite similarly as we did, we obtain

$$\beta(s) = \left(\frac{1}{k}\sinh ks, \frac{B_2}{k}\cosh ks + D_2, \frac{B_3}{k}\cosh ks + D_3\right),$$

satisfying $B_2^2 + B_3^2 = 1$, $B_2 D_2 + B_3 D_3 = 0$ and $D_2^2 + D_3^2 = 1 - \frac{1}{k^2}$.

If we adopt the coordinates' transformation,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B_2 & B_3 \\ 0 & -B_3 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With respect to the new coordinates $(\bar{x}, \bar{y}, \bar{z})$, the vector $\beta(s)$ becomes

$$\beta(s) = \left(\frac{1}{k}\sinh ks, \frac{1}{k}\cosh ks, D\right), \tag{33}$$

where $D = B_2 D_3 - B_3 D_2$ with $D^2 = 1 - \frac{1}{k^2}$. We call such a surface generated by (33) a *hyperbolic conical surface of the second kind* and it satisfies

$$\Delta \mathbf{H} = \left(\frac{1 + D^2 + k^2}{k^2 t^2}\right)\mathbf{H} + \left(\frac{D(1 + k^2 D^2)}{2k^4 t^3}\right)(0,0,1).$$

Case 2: $\varepsilon_2(\kappa_g^2(s) - \varepsilon_1) = -k^2$ for some real number k.

Let $\varepsilon_1 = 1$. We may give the initial condition by $\beta'(0) = (0,1,0)$ for the differential equation $\beta'''(s) + k^2\beta'(s) = 0$. Under such an initial condition, a vector field $\beta(s)$ is given by

$$\beta(s) = \left(-\frac{B_1}{k}\cos ks + D_1, \frac{1}{k}\sin ks, -\frac{B_3}{k}\cos ks + D_3\right),$$

where B_1, B_3, D_1 and D_3 are some constants satisfying $B_3^2 - B_1^2 = 1$, $B_1D_1 = B_3D_3$ and $D_1^2 - D_3^2 = \frac{1}{k^2} - \varepsilon_2$. If we take another coordinate system $(\bar{x}, \bar{y}, \bar{z})$ such that

$$\bar{x} = -B_3x + B_1z, \quad \bar{y} = y, \quad \bar{z} = B_1x - B_3z,$$

then a vector $\beta(s)$ takes the form

$$\beta(s) = \left(D, \frac{1}{k}\sin ks, \frac{1}{k}\cos ks\right), \tag{34}$$

where $D = B_1D_3 - B_3D_1$ satisfying $D^2 = \frac{1}{k^2} - \varepsilon_2$. We call such a surface generated by (34) an *elliptic conical surface* and it satisfies

$$\Delta\mathbf{H} = \left(\frac{\varepsilon_1 - \varepsilon_1D^2 - k^2}{k^2t^2}\right)\mathbf{H} - \left(\frac{D(k^2D^2-1)}{2k^4t^3}\right)(1,0,0).$$

Case of $\varepsilon_1 = -1$ gives $\varepsilon_2 = -1$. It is impossible by the causal character of Lorentz geometry.

Case 3: $\kappa_g^2(s) - \varepsilon_1 = 0$.

In this case, $\kappa_g^2(s) = 1$, in other words, $\varepsilon_1 = 1$, which implies by using (27) $\langle\beta''(s), \beta''(s)\rangle = 0$. Since $\beta''(s)$ is a constant vector by (30), we may put $\beta''(s) = (d_1, d_2, d_3)$ for some constants d_1, d_2, d_3 satisfying $-d_1^2 + d_2^2 + d_3^2 = 0$ and so $\beta'(s) = (d_1s + k_1, d_2s + k_2, d_3s + k_3)$ for some constants k_1, k_2 and k_3. Since $\langle\beta'(s), \beta'(s)\rangle = \varepsilon_1 = 1$, we may set $(k_1, k_2, k_3) = (0,1,0)$ up to an isometry and hence $\beta(s) = (\frac{d_1}{2}s^2 + c_1, \frac{d_2}{2}s^2 + s + c_2, \frac{d_3}{2}s^2 + c_3)$ for some constants c_1, c_2 and c_3. However, $\langle\beta(s), \beta(s)\rangle = \varepsilon_2$ implies $d_2 = c_2 = 0$ and $d_1^2 = d_3^2$, $-c_1^2 + c_3^2 = \varepsilon_2$, $-d_1c_1 + d_3c_3 + 1 = 0$. Thus, $\beta(s)$ takes the form

$$\beta(s) = \left(\frac{d_1}{2}s^2 + c_1, s, \frac{d_3}{2}s^2 + c_3\right). \tag{35}$$

We call such a surface generated by (35) a *quadric conical surface*.

As shown in the Introduction, a quadric conical surface is of generalized null 2-type of the first kind. Let us suppose that κ_g is a non-constant, i.e., $\kappa_g' \neq 0$ on an open interval. Suppose that M is of generalized null 2-type, that is, M satisfies the condition (4). Then, we have the following equations:

$$\frac{3\kappa_g(s)\kappa_g'(s)}{2t^3} = gC_1, \tag{36}$$

$$-\frac{\kappa_g^2(s)}{2t^3} = gC_2, \tag{37}$$

$$-\frac{1}{2t^3}\left(\varepsilon_1\varepsilon_2\kappa_g''(s) + \varepsilon_1\kappa_g^3(s) + \kappa_g(s)\right) = \frac{\varepsilon_2\kappa_g(s)}{2t}f + gC_3, \tag{38}$$

where $\mathbf{C} = \varepsilon_1C_1e_1 + \varepsilon_2C_2e_2 - \varepsilon_1\varepsilon_2C_3e_3$ with $C_1 = \langle\mathbf{C}, e_1\rangle$, $C_2 = \langle\mathbf{C}, e_2\rangle$ and $C_3 = \langle\mathbf{C}, e_3\rangle$. Since $e_1 = \beta'(s)$, $e_2 = \beta(s)$ and $e_3 = \beta'(s) \times \beta(s)$, the component functions C_i $(i = 1,2,3)$ of $\mathbf{C}$ depend only on variable s. Let us differentiate C_1, C_2 and C_3 covariantly with respect to e_1. Then, from (27), we have the following equations:

$$C_1'(s) + \varepsilon_1\varepsilon_2C_2(s) - \varepsilon_1\varepsilon_2\kappa_g(s)C_3(s) = 0, \tag{39}$$

$$C_2'(s) - C_1(s) = 0, \tag{40}$$

$$C_3'(s) - \varepsilon_1 \kappa_g(s) C_1(s) = 0. \tag{41}$$

Combining (36) and (37), and using (40), we have

$$C_1 = -\frac{3c\kappa_g'}{\kappa_g^4} \quad \text{and} \quad C_2 = \frac{c}{\kappa_g^3}, \tag{42}$$

where c is a constant of integration.

Together with (37) and (42), we can find

$$g = -\frac{\kappa_g^5}{2ct^3}. \tag{43}$$

Substituting (42) into (39), we get

$$C_3 = \frac{c(\kappa_g^2 - 3\varepsilon_1\varepsilon_2\kappa_g\kappa_g'' + 12\varepsilon_1\varepsilon_2{\kappa_g'}^2)}{\kappa_g^6}. \tag{44}$$

Then, (38) and (44) lead to

$$f = -\frac{\varepsilon_2}{t^2\kappa_g^2}(4\varepsilon_1\varepsilon_2\kappa_g\kappa_g'' - 12\varepsilon_1\varepsilon_2{\kappa_g'}^2 + \varepsilon_1\kappa_g^4). \tag{45}$$

Furthermore, it follows from (41) and (42) that

$$C_3' = -\frac{3\varepsilon_1 c\kappa_g'}{\kappa_g^3}$$

and its solution is given by

$$C_3 = \frac{3\varepsilon_1 c}{2\kappa_g^2} + a_1 \tag{46}$$

for some constant a_1.

Combining (44) and (46), the geodesic curvature κ_g satisfies the following equation:

$$\kappa_g'' - \frac{4}{\kappa_g}{\kappa_g'}^2 - \frac{1}{3}\varepsilon_1\varepsilon_2\kappa_g + \frac{1}{2}\varepsilon_2\kappa_g^3 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2\kappa_g^5 = 0. \tag{47}$$

To solve the ODE, we put $p = \kappa_g'$. Then, (47) can be written of the form

$$\frac{dp}{d\kappa_g} - \frac{4}{\kappa_g}p = \frac{1}{p}\left(\frac{1}{3}\varepsilon_1\varepsilon_2\kappa_g - \frac{1}{2}\varepsilon_2\kappa_g^3 - \frac{a_1}{3c}\varepsilon_1\varepsilon_2\kappa_g^5\right), \tag{48}$$

and it is a Bernoulli differential equation. Thus, the solution is given by

$$p = \pm\kappa_g\left(a_2\kappa_g^6 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2\kappa_g^4 + \frac{1}{4}\varepsilon_2\kappa_g^2 - \frac{1}{9}\varepsilon_1\varepsilon_2\right)^{\frac{1}{2}},$$

which is equivalent to

$$\kappa_g^{-1}\left(a_2\kappa_g^6 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2\kappa_g^4 + \frac{1}{4}\varepsilon_2\kappa_g^2 - \frac{1}{9}\varepsilon_1\varepsilon_2\right)^{-\frac{1}{2}} d\kappa_g = \pm ds$$

for some constant a_2. If we put

$$F(v) = \int \psi(v)dv,$$

where

$$\psi(v) = v^{-1}\left(a_2 v^6 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2 v^4 + \frac{1}{4}\varepsilon_2 v^2 - \frac{1}{9}\varepsilon_1\varepsilon_2\right)^{-\frac{1}{2}},$$

and then we have

$$F(\kappa_g) = \pm s + a_3 \tag{49}$$

for some constant a_3. Thus, the geodesic curvature κ_g is given by

$$\kappa_g(s) = F^{-1}(\pm s + a_3). \tag{50}$$

Furthermore, the constant vector $\mathbf{C}$ can be expressed as

$$\mathbf{C} = -\frac{3c\kappa_g'}{\kappa_g^4}e_1 + \frac{c}{\kappa_g^3}e_2 + \frac{c(\kappa_g^2 - 3\varepsilon_1\varepsilon_2\kappa_g\kappa_g'' + 12\varepsilon_1\varepsilon_2\kappa_g'^2)}{\kappa_g^6}e_3. \tag{51}$$

Conversely, for some constants a_1, a_2 and c such that the function

$$\psi(v) = v^{-1}\left(a_2 v^6 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2 v^4 + \frac{1}{4}\varepsilon_2 v^2 - \frac{1}{9}\varepsilon_1\varepsilon_2\right)^{-\frac{1}{2}} \tag{52}$$

is well-defined on an open interval $J \subset (0, \infty)$, we take an indefinite integral $F(v)$ of the function $\psi(v)$. Let I be the image of the function F. We can take an open subinterval $J_1 \subset J$ such that $F : J_1 \to I$ is a strictly increasing function with $F'(v) = \psi(v)$. Let us consider the function φ defined by $\varphi(s) = F^{-1}(\pm s + a_3)$ for some constant a_3. Then, the function φ satisfies $F(\varphi) = \pm s + a_3$.

For any unit speed pseudo-spherical curve $\beta(s)$ in $\mathbb{Q}^2(\varepsilon)$ with geodesic curvature $\kappa_g(s) = \varphi(s)$, we consider the conical surface M in $\mathbb{L}^3$ parametrized by

$$x(s,t) = \alpha_0 + t\beta(s), \quad s \in I, \quad t > 0, \tag{53}$$

where α_0 is a constant vector. Given any nonzero constant c, we put f and g the functions, respectively, given by

$$f(s,t) = -\frac{\varepsilon_2}{t^2\varphi^2}(4\varepsilon_1\varepsilon_2\varphi\varphi'' - 12\varepsilon_1\varepsilon_2\varphi'^2 + \varepsilon_1\varphi^4), \quad g(s,t) = -\frac{\varphi^5}{2ct^3}. \tag{54}$$

For a nonzero constant c and the pseudo-orthonormal frame $\{e_1, e_2, e_3\}$ on $\mathbb{L}^3$ such that $e_1 = \frac{1}{t}\frac{\partial}{\partial s}$ and $e_2 = \frac{\partial}{\partial t}$ are tangent to M and e_3 normal to M, we put

$$\mathbf{C} = -\frac{3c\varphi'}{\varphi^4}e_1 + \frac{c}{\varphi^3}e_2 + \frac{c(\varphi^2 - 3\varepsilon_1\varepsilon_2\varphi\varphi'' + 12\varepsilon_1\varepsilon_2\varphi'^2)}{\varphi^6}e_3. \tag{55}$$

Note that it follows from the definition of φ that the function φ satisfies (47). Hence, using (27), it is straightforward to show that

$$\widetilde{\nabla}_{e_1}\mathbf{C} = \widetilde{\nabla}_{e_2}\mathbf{C} = 0,$$

which implies that $\mathbf{C}$ is a constant vector. Furthermore, the same argument as in the first part of this subsection yields the mean curvature vector field $\mathbf{H}$ of the conical surface M satisfies

$$\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C},$$

where f, g and $\mathbf{C}$ are given in (54) and (55), respectively. This shows that the conical surface is of generalized null 2-type.

Thus, we have the following:

Theorem 3. *Let M be a conical surface in the Minkowski 3-space $\mathbb{L}^3$. Then, M is of generalized null 2-type if and only if it is an open part of one of the following surfaces:*

(1) a Euclidean plane;
(2) a Minkowski plane;
(3) a hyperbolic conical surface of the first kind;
(4) a hyperbolic conical surface of the second kind;
(5) an elliptic conical surface;
(6) a quadric conical surface;
(7) a conical surface parameterized by

$$x(s,t) = \alpha_0 + t\beta(s),$$

where α_0 is a constant vector and $\beta(s)$ is a unit speed pseudo-spherical curve in $\mathbb{Q}^2(\varepsilon)$ with the non-constant geodesic curvature κ_g which is, for some indefinite integral $F(v)$ of the function

$$\psi(v) = v^{-1}\left(a_2v^6 + \frac{a_1}{3c}\varepsilon_1\varepsilon_2 v^4 + \frac{1}{4}\varepsilon_2 v^2 - \frac{1}{9}\varepsilon_1\varepsilon_2\right)^{-\frac{1}{2}}$$

with $a_1, a_2, c \in R$, given by

$$\kappa_g(s) = F^{-1}(\pm s + a_3),$$

where a_3 is constant.

Next, we study tangent developable surfaces in the Minkowski 3-space $\mathbb{L}^3$.

Theorem 4. *Let M be a tangent developable surface in the Minkowski 3-space $\mathbb{L}^3$. Then, M is of generalized null 2-type if and only if M is an open part of a Euclidean plane or a Minkowski plane.*

Proof. Let $\alpha(s)$ be a curve parameterized by arc-length s in $\mathbb{L}^3$ with non-zero curvature $\kappa(s)$. Then, a non-degenerate tangent developable surface M in $\mathbb{L}^3$ is defined by

$$x(s,t) = \alpha(s) + t\alpha'(s), \quad t \neq 0.$$

In the case, we can take the pseudo-orthonormal frame $\{e_1, e_2, e_3\}$ of $\mathbb{L}^3$ such that $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\varepsilon_2}{t\kappa(s)}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)$ are tangent to M and e_3 is normal to M. By a direct calculation, we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{e_1}e_1 = \widetilde{\nabla}_{e_1}e_2 = 0, \quad \widetilde{\nabla}_{e_2}e_1 = \frac{1}{t}e_2, \quad \widetilde{\nabla}_{e_2}e_2 = -\frac{\varepsilon_1\varepsilon_2}{t}e_1 - \frac{\varepsilon_1\tau(s)}{t\kappa(s)}e_3, \\
&\widetilde{\nabla}_{e_1}e_3 = 0, \quad \widetilde{\nabla}_{e_2}e_3 = \frac{\varepsilon_2\tau(s)}{t\kappa(s)}e_2,
\end{aligned} \tag{56}$$

where $\langle e_1, e_1\rangle = \varepsilon_1(= \pm 1)$, $\langle e_2, e_2\rangle = \varepsilon_2(= \pm 1)$ and $\tau(s)$ is the torsion of $\alpha(s)$. Therefore, the mean curvature vector field $\mathbf{H}$ of M is given by

$$\mathbf{H} = \frac{\tau(s)}{2t\kappa(s)}e_3. \tag{57}$$

By a long computation, the Laplacian $\Delta\mathbf{H}$ of the mean curvature vector field $\mathbf{H}$ turns out to be

$$\begin{aligned}\Delta\mathbf{H} = &-\frac{\varepsilon_1\tau^2}{2\kappa^2t^3}e_1 + \frac{1}{2\kappa^4t^4}\left(3\kappa\tau^2 - 2\kappa'\tau^2t + 3\kappa\tau\tau't\right)e_2 \\ &+ \frac{1}{2\kappa^4t^5}\Big(-\varepsilon_1\kappa^3\tau t^2 + \varepsilon_1\kappa'\tau t - 3\varepsilon_2\kappa\tau - (\varepsilon_1\kappa^2t + \varepsilon_2\kappa\kappa't^2)(\frac{\tau}{\kappa})' \\ &-\varepsilon_2\kappa^2(\frac{\tau}{\kappa})''t^2 - \varepsilon_1\tau^3\kappa t^2\Big)e_3.\end{aligned} \tag{58}$$

Suppose that M is of generalized null 2-type, that is, M satisfies $\Delta\mathbf{H} = f\mathbf{H} + g\mathbf{C}$ for some smooth functions f, g and a constant vector $\mathbf{C}$. With the help of (57) and (58), (4) can be written in the form

$$\begin{aligned} gC_1 &= -\frac{\tau^2}{2\kappa^2t^3}, \\ gC_2 &= \frac{\varepsilon_2}{2\kappa^4t^4}\left(3\kappa\tau^2 - 2\kappa'\tau^2t + 3\kappa\tau\tau't\right), \\ -\frac{\varepsilon_1\varepsilon_2\tau}{2\kappa t}f + gC_3 &= -\frac{\varepsilon_1\varepsilon_2}{2\kappa^4t^5}\Big(-\varepsilon_1\kappa^3\tau t^2 + \varepsilon_1\kappa'\tau t - 3\varepsilon_2\kappa\tau - (\varepsilon_1\kappa^2t + \varepsilon_2\kappa\kappa't^2)(\frac{\tau}{\kappa})' \\ &\quad -\varepsilon_2\kappa^2(\frac{\tau}{\kappa})''t^2 - \varepsilon_1\tau^3\kappa t^2\Big), \end{aligned} \tag{59}$$

where $\mathbf{C} = \varepsilon_1C_1e_1 + \varepsilon_2C_2e_2 - \varepsilon_1\varepsilon_2C_3e_3$ with $C_1 = \langle\mathbf{C}, e_1\rangle$, $C_2 = \langle\mathbf{C}, e_2\rangle$ and $C_3 = \langle\mathbf{C}, e_3\rangle$. In this case, the components C_i of $\mathbf{C}$ are functions of only s. It follows from (56) that we have

$$C_1' - \varepsilon_2\kappa C_2 = 0, \tag{60}$$

$$C_2' + \varepsilon_1\kappa C_1 + \varepsilon_1\varepsilon_2\tau C_3 = 0, \tag{61}$$

$$C_3' + \varepsilon_2\tau C_2 = 0. \tag{62}$$

By combining the first and second equations of (59), we get

$$3\varepsilon_2\kappa\tau^2C_1 + (3\varepsilon_2\kappa\tau\tau'C_1 - 2\varepsilon_2\kappa'\tau^2C_1 + \kappa^2\tau^2C_2)t = 0.$$

This shows that we obtain

$$\begin{aligned} 3\varepsilon_2\kappa\tau^2C_1 &= 0, \\ 3\varepsilon_2\kappa\tau\tau'C_1 - 2\varepsilon_2\kappa'\tau^2C_1 + \kappa^2\tau^2C_2 &= 0. \end{aligned} \tag{63}$$

Consider the open set $\mathcal{O} = \{p \in M | \tau(p) \neq 0\}$. Suppose that $\mathcal{O}$ is a non-empty set. (63) shows that $C_1 = 0$ and $C_2 = 0$, and it follows from (61) that $C_3 = 0$. That is, $\mathbf{C} = 0$ on $\mathcal{O}$. In addition, (59) gives $\tau = 0$, and it is a contradiction. Thus, the open set $\mathcal{O}$ is empty and τ is identically zero. Therefore, $\alpha(s)$ is a plane curve, and the surface M is an open part of a Euclidean plane or a Minkowski plane.

The converse of Theorem 4 follows a straightforward calculation. □

5. Null Scrolls

Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{L}^3$ and $\beta = \beta(s)$ a null vector field along α satisfying $\langle\alpha', \beta\rangle = -1$. Then, the null scroll M is parameterized by

$$x(s,t) = \alpha(s) + t\beta(s). \tag{64}$$

Furthermore, without loss of generality, we may choose $\alpha(s)$ as a null geodesic of M, i.e, $\langle\alpha'(s), \beta'(s)\rangle = 0$ for all s. By putting $\gamma(s) = \alpha'(s) \times \beta(s)$, then $\{\alpha'(s), \beta(s), \gamma(s)\}$ is a pseudo-orthonormal frame along $\alpha(s)$ in $\mathbb{L}^3$. We define the smooth functions k and u by

$$k(s) = \langle\alpha''(s), \gamma(s)\rangle, \quad u(s) = \langle\beta(s), \gamma'(s)\rangle.$$

On the other hand, the induced Lorentz metric on M is given by $g_{11} = u(s)^2 t^2$, $g_{12} = -1$ and $g_{22} = 0$. Since M is a non-degenerate surface, $u(s)t$ is non-vanishing everywhere.
In terms of the pseudo-orthonormal frame, we have

$$\begin{aligned} \alpha''(s) &= k(s)\gamma(s), \\ \beta'(s) &= -u(s)\gamma(s), \\ \gamma'(s) &= -u(s)\alpha'(s) + k(s)\beta(s). \end{aligned} \tag{65}$$

The mean curvature vector field $\mathbf{H}$ of M is given by

$$\mathbf{H} = -u^2 t\beta + u\gamma,$$

and its Laplacian $\Delta\mathbf{H}$ is expressed as

$$\Delta\mathbf{H} = (-4uu' - 2u^4 t)\beta + 2u^3\gamma. \tag{66}$$

Suppose that M is a generalized null 2-type surface. Then, we have

$$\begin{aligned} 4uu' + 2u^4 t &= u^2 tf - gC_2, \\ gC_1 &= 0, \\ 2u^3 &= uf + gC_3, \end{aligned} \tag{67}$$

for a constant vector $\mathbf{C} = C_1\alpha' + C_2\beta + C_3\gamma$ with $C_1 = -\langle \mathbf{C}, \beta\rangle$, $C_2 = -\langle \mathbf{C}, \alpha'\rangle$ and $C_3 = \langle \mathbf{C}, \gamma\rangle$.

Suppose that g is identically zero. By combining the first and third Equations in (67), we see that u is constant, say u_0. In this case, we have $f = 2u_0^2$. Thus, M is a B-scroll, and it satisfies $\Delta\mathbf{H} = 2u_0^2\mathbf{H}$ (see [16]).

Consider the open set $\mathcal{O} = \{p \in M | g(p) \neq 0\}$. Suppose that $\mathcal{O}$ is a non-empty set. Then, from (67), we find $C_1 = 0$ on a component $\mathcal{O}_0$ on $\mathcal{O}$. Let us differentiate C_1 with respect to s and use (65). Then, $C_3 = 0$ on $\mathcal{O}_0$. Since

$$\alpha' \times \beta' = -u\alpha', \quad \alpha'' \times \beta = k\beta,$$

by differentiating the equation $C_3 = 0$ with respect to s, we can obtain

$$kC_1 - uC_2 = 0.$$

It follows that $C_2 = 0$ on $\mathcal{O}_0$ because $C_1 = 0$ and $u \neq 0$. Since $\mathbf{C}$ is a constant vector, it is a zero vector. From the first and third Equations in (67), u is a non-zero constant, say u_0, and $f = 2u_0^2$ on M. Thus, M is of null 2-type and it is a B-scroll.

Consequently, we have

Theorem 5. *Let M be a null scroll in the Minkowski 3-space $\mathbb{L}^3$. Then, M is of generalized null 2-type if and only if M is an open piece of a B-scroll.*

We now propose an open problem.

Problem 1. *Classify all generalized null 2-type surfaces in the Euclidean space or pseudo-Euclidean space.*

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