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Bounded Solutions to Nonhomogeneous Linear Second-Order Difference Equations

Stevo Stević [1,2]

1 Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia; sstevic@ptt.rs

2 Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract: By using some solvability methods and the contraction mapping principle are investigated bounded, as well as periodic solutions to some classes of nonhomogeneous linear second-order difference equations on domains $\mathbb{N}_0$, $\mathbb{Z} \setminus \mathbb{N}_2$ and $\mathbb{Z}$. The case when the coefficients of the equation are constant and the zeros of the characteristic polynomial associated to the corresponding homogeneous equation do not belong to the unit circle is described in detail.

Keywords: linear second-order difference equation; bounded solution; contraction mapping principle; integer domain

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1. Introduction

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{N}_k := \{n \in \mathbb{Z} : n \geq k\}$, $k \in \mathbb{Z}$, and $\mathbb{N} = \mathbb{N}_1$. Investigations of difference equations and systems of difference equations have been conducted for a long time (see, for example, [1–37] and the references therein). The solvability of the equations and systems is one of the oldest topics in the area. It is always nice to have formulas for solutions to the equations and systems for themselves, but also since they can frequently help in studying of the long-term behavior of the solutions, as is the case in [7,33,35]. Many classical results, including the ones on solvability of the equations and systems, can be found in the following classical books: [8,14–16,19,20].

Many of solvable difference equations and systems essentially use the solvability of the linear first-order difference equation, that is, of the following difference equation

$$x_{n+1} = q_n x_n + f_n, \quad n \in \mathbb{N}_0, \tag{1}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are given sequences. We will mention here just a few recent examples; the interested reader can find many other examples in the list of the references of the mentioned papers. In [33] was studied the following difference equation

$$x_n = \frac{a_n x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{2}$$

which is transformed to an equation of the form in (1) by using the change of variables $y_n = 1/(x_n x_{n-1} \cdots x_{n-k+1})$, while [32] studies the following difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a + b x_n x_{n-k})}, \quad n \in \mathbb{N}_0,$$

which is transformed to an equation of the form in (1) by using the change of variables $y_n = 1/(x_{n+1}x_{n-k+1})$ (for an extension of the equation see [31]). Essentially the same ideas and methods were used in [23,35], while [1] studies a special case of Equation (2) in another way. In fact, all the papers use our ideas and methods from a 2004 note. Paper [27] presents several related methods and can be considered as a representative one, where a comprehensive list of relevant references on solvability is given. It is also worthy to mention that some nonlinear systems of difference equations were solved by reducing them, by using some suitable changes of variables, to solvable linear ones (see, for example, [7,34], as well as the related references therein). The solvability of some product-type equations ([27]) and systems ([28,36,37]) has been also shown by using some solvable linear ones, although in a more complex way. In fact, some of the results in papers [28,36,37] use special cases of the following equation

$$z_n = b_n z_{n-1}^{a_n}, \quad n \in \mathbb{N}_0,$$

which is a product-type analog of Equation (1). All above mentioned examples show the importance of Equation (1). Here, we will frequently use various things connected to Equation (1). Some recent applications of this and related solvable equations can be found in [5,6]. Let us also mention that beside showing the solvability of difference equations and systems by finding closed-form formulas for their solutions, in the cases when it is not possible to find them, one can try to find some of their invariants which can be also useful in studying of the long-term behavior of their solutions ([21,22]).

Motivated by the recent studies of the solvability, quite recently in [30], we have studied, among other problems, the existence of bounded solutions to the difference equation

$$x_{n+2} - q_n x_n = f_n, \quad n \in \mathbb{N}_0, \tag{3}$$

in two different ways. Applying classical method of variation of constants it is easily shown that in the case $q_n = q \in \mathbb{C} \setminus \{0\}$, Equation (3) has the general solution in the following form:

$$x_n = (\sqrt{q})^n \left(c_0 + \sum_{k=0}^{n-1} \frac{f_k}{2(\sqrt{q})^{k+2}} \right) + (-\sqrt{q})^n \left(d_0 + \sum_{k=0}^{n-1} \frac{(-1)^k f_k}{2(\sqrt{q})^{k+2}} \right), \quad n \in \mathbb{N}_0, \tag{4}$$

where c_0 and d_0 are arbitrary complex numbers, and $\sqrt{q}$ is one of two possible roots of q. By using Formula (4), as well as another method, we have shown in [30], among other results, that the equation in the case $q_n = q$, $n \in \mathbb{N}_0$, has a unique bounded solution in the case when $|q| > 1$, and used the obtained formula for the bounded solution as a motivation for introducing an operator which along with the contraction mapping principle ([4]) helps in showing the existence of a unique bounded solution to Equation (3) under some conditions posed on the sequence $(q_n)_{n\in\mathbb{N}_0}$. It is a natural problem to try to use the same ideas and methods in investigating of bounded solutions to some other classes of linear and nonlinear difference equations.

One of the aims of the paper is to present some related results to those in [30] for the case of the difference equation

$$x_{n+2} + p_n x_{n+1} + q_n x_n = f_n, \quad n \in \mathbb{N}_0. \tag{5}$$

This equation is one of the most important and widely studied difference ones, since it models many real-life quantities and processes, for example, the amplitude of oscillation of the weights on a discretely weighted vibrating string [3] (pp. 15–17). For some classical results and methods for studying Equation (5), see, for example, [16,26], as well as the references therein. Note also that by using the differences $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n$ the equation can be written in the following form

$$\Delta^2 x_n + (p_n + 2)\Delta x_n + (p_n + q_n + 1)x_n = f_n, \quad n \in \mathbb{N}_0,$$

from which it immediately follows that the equation is a discrete variant of a linear second-order differential equation.

We first investigate the case when

$$p_n = p, \quad q_n = q \neq 0, \quad n \in \mathbb{N}_0, \tag{6}$$

and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence, and after that the case when $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are nonconstant sequences. The case when (6) holds can be regarded as a folklore one, but it is difficult to find many of the information provided here in the literature, especially at one place. The case when the zeros of the characteristic polynomial associated to the corresponding homogeneous equation do not belong to the unit circle is described in detail. It should be pointed out that when one of the zeros belongs to the circle then, as usual, very different situations appears. Recall that if in Equation (1) $q_n = 1$, $n \in \mathbb{N}_0$, then, the bounded sequence $(f_n)_{n\in\mathbb{N}_0}$ highly influences on the behavior of the solutions to the equation. Namely, a solution to the equation can converge, diverge to infinity, the limit set can be even a whole interval (see, for example, [19,24], as well as [2] for the case of metric spaces), or it can be even a more complicated set.

As in [30], we first use some solvability methods and then the contraction mapping principle in showing the existence of a unique bounded solution to Equation (5) under some conditions posed on the coefficients of the equation. For some other applications of fixed-point theorems in studying difference equations and systems, see, for example, [10–12,17,18,25] and the related references therein. Note that beside the contraction mapping principle, very frequent situation is application of a variant of the Schauder fixed-point theorem ([10–12,17,18]), and since recently the Darbo fixed-point theorem ([38]) which uses the notion of measure of non-compactness ([25]). It should be noted that many of these papers essentially use a similar idea, that is, a combination of a solvability method, which is frequently hidden by some summations, and a fixed point theorem. The existence of periodic solutions in the case when $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are constant sequences, while $(f_n)_{n\in\mathbb{N}_0}$ is a periodic sequence is also studied, as well as the relationship between the periodic and non-periodic ones, which is a natural continuation of the investigations in [29].

In our recent paper [29] we have also studied bounded solutions to Equation (1), but on the set of all integers $\mathbb{Z}$, which has motivated us to conduct a similar investigation for the case of Equation (5). Hence, beside studying bounded solutions on domain $\mathbb{N}_0$, it will be also done on domains $\mathbb{Z} \setminus \mathbb{N}_2$ and $\mathbb{Z}$. One of the reasons, why instead of the domain $\mathbb{Z} \setminus \mathbb{N}$ is chosen $\mathbb{Z} \setminus \mathbb{N}_2$ is found in the fact that the initial, that is, end values for the sets $\mathbb{N}_0$ and $\mathbb{Z} \setminus \mathbb{N}_2$ are the same, so that the domains patch each other well. Let us mention that a part of our investigations in [29,30] are motivated by a problem from [9].

Let $S \subseteq \mathbb{Z}$ be an unbounded set. Then the space of bounded sequences $f = (f_n)_{n\in S}$ on S with the supremum norm

$$\|f\|_{\infty,S} = \sup_{n\in S} |f_n|, \tag{7}$$

is Banach's, and is usually denoted by $l^\infty(S)$. Throughout the paper we will simply use the notations $\|f\|_\infty$ and l^∞, no matter which set S is used, since at each point it will be clear what the set is. We will also use the standard convention $\sum_{j=m}^{l} a_j = 0$, when $m, l \in \mathbb{Z}$ are such that $l < m$.

It is said that a sequence $(x_n)_{n\in\mathbb{N}_k}$ converges geometrically (exponentially) to a sequence $(\tilde{x}_n)_{n\in\mathbb{N}_k}$ if there are $M > 0$ and $q \in [0,1)$, such that

$$|x_n - \tilde{x}_n| \le Mq^n, \quad \text{for} \quad n \in \mathbb{N}_k,$$

while a sequence $(x_n)_{n\in\mathbb{Z}\setminus\mathbb{N}_k}$ converges geometrically (exponentially) to a sequence $(\tilde{x}_n)_{n\in\mathbb{Z}\setminus\mathbb{N}_k}$ if there are $M > 0$ and $q \in [0, 1)$, such that

$$|x_{-n} - \tilde{x}_{-n}| \le Mq^n, \quad \text{for} \quad n \ge -k+1.$$

2. Bounded Solutions to Equation (5) on $\mathbb{N}_0$

First, we prove an auxiliary result in a standard way, for the case when $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are constant sequences. The result can be obtained from a formula for the general solution to Equation (5) when the fundamental set of solutions to the corresponding homogeneous equation is known. We will give a proof of it for the completeness, and to avoid frequent troubles with indices which lead to some minor inaccuracies related to the formula as it is the case in [20]. Some consequences of the lemma, which should be folklore, are given.

Lemma 1. *Consider the equation*

$$x_{n+2} + px_{n+1} + qx_n = f_n, \quad n \in \mathbb{N}_0, \tag{8}$$

where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, and $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of complex numbers. Then the following statements are true.

(a) If $p^2 \ne 4q$, then the general solution to Equation (8) is given by the following formula

$$x_n = \lambda_1^n\left(c_0 - \sum_{k=0}^{n-1}\frac{f_k}{\lambda_1^{k+1}(\lambda_2-\lambda_1)}\right) + \lambda_2^n\left(d_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda_2^{k+1}(\lambda_2-\lambda_1)}\right), \tag{9}$$

for $n \in \mathbb{N}_0$, where c_0 and d_0 are arbitrary complex numbers, and

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2-4q}}{2}. \tag{10}$$

(b) If $p^2 = 4q$, then the general solution to Equation (8) is given by the following formula

$$x_n = \lambda^n\left(c_0 - \sum_{k=0}^{n-1}\frac{(k+1)f_k}{\lambda^{k+2}}\right) + n\lambda^n\left(d_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda^{k+2}}\right), \tag{11}$$

for $n \in \mathbb{N}_0$, where c_0 and d_0 are arbitrary complex numbers, and $\lambda = -p/2$.

Proof. (a) We solve the equation by the method of variation of constants ([8,20]). As it is well-known, the corresponding homogeneous equation, in this case, has the general solution in the following form:

$$x_n = c\lambda_1^n + d\lambda_2^n, \quad n \in \mathbb{N}_0,$$

where $\lambda_{1,2}$ are the zeros of the characteristic polynomial

$$P_2(\lambda) = \lambda^2 + p\lambda + q, \tag{12}$$

associated to the homogeneous equation, from which (10) follows.

Hence, the general solution to (8) is searched for in the following form:

$$x_n = c_n\lambda_1^n + d_n\lambda_2^n, \quad n \in \mathbb{N}_0, \tag{13}$$

where $(c_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ are two (undetermined) sequences.

The following condition is posed

$$x_{n+1} = c_{n+1}\lambda_1^{n+1} + d_{n+1}\lambda_2^{n+1} = c_n\lambda_1^{n+1} + d_n\lambda_2^{n+1}, \tag{14}$$

for $n \in \mathbb{N}_0$, that is,

$$(c_{n+1} - c_n)\lambda_1^{n+1} + (d_{n+1} - d_n)\lambda_2^{n+1} = 0, \tag{15}$$

for $n \in \mathbb{N}_0$.

Employing (13), (14), as well as (14) where n is replaced by $n+1$ in (8), and using that $p = -(\lambda_1 + \lambda_2)$ and $q = \lambda_1\lambda_2$, we get

$$c_{n+1}\lambda_1^{n+2} + d_{n+1}\lambda_2^{n+2} - (\lambda_1 + \lambda_2)(c_n\lambda_1^{n+1} + d_n\lambda_2^{n+1}) + \lambda_1\lambda_2(c_n\lambda_1^n + d_n\lambda_2^n) = f_n,$$

that is,

$$(c_{n+1} - c_n)\lambda_1^{n+2} + (d_{n+1} - d_n)\lambda_2^{n+2} = f_n, \tag{16}$$

for $n \in \mathbb{N}_0$.

For each fixed $n \in \mathbb{N}_0$, (15) and (16) jointly can be regarded as a two-dimensional linear system in variables $c_{n+1} - c_n$ and $d_{n+1} - d_n$.

By solving the system it is easily obtained

$$c_{n+1} - c_n = -\frac{f_n}{\lambda_1^{n+1}(\lambda_2 - \lambda_1)} \quad \text{and} \quad d_{n+1} - d_n = \frac{f_n}{\lambda_2^{n+1}(\lambda_2 - \lambda_1)}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$c_n = c_0 - \sum_{k=0}^{n-1} \frac{f_k}{\lambda_1^{k+1}(\lambda_2 - \lambda_1)} \quad \text{and} \quad d_n = d_0 + \sum_{k=0}^{n-1} \frac{f_k}{\lambda_2^{k+1}(\lambda_2 - \lambda_1)}, \tag{17}$$

for $n \in \mathbb{N}_0$.

Using (17) into (13) we get (9). That (9) represents the general solution to (8), follows from the fact that the sequence

$$\widehat{x}_n := \sum_{k=0}^{n-1} \frac{\lambda_2^{n-k-1} - \lambda_1^{n-k-1}}{\lambda_2 - \lambda_1} f_k$$

is a particular solution to Equation (8), which is easily verified, while

$$x_n^h = c_0\lambda_1^n + d_0\lambda_2^n,$$

is the general solution to the corresponding homogeneous equation ([8,20]).

(b) The corresponding homogeneous equation, in this case, has the general solution in the following form:

$$x_n = c\lambda^n + dn\lambda^n, \quad n \in \mathbb{N}_0,$$

where λ is the (double) zero of polynomial (12), that is, $\lambda = -p/2$.

So, the general solution to (8) is looked for in the following form:

$$x_n = c_n\lambda^n + d_n n\lambda^n, \quad n \in \mathbb{N}_0, \tag{18}$$

where $(c_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ are two (undetermined) sequences.

The following condition is posed

$$x_{n+1} = c_{n+1}\lambda^{n+1} + d_{n+1}(n+1)\lambda^{n+1} = c_n\lambda^{n+1} + d_n(n+1)\lambda^{n+1}, \tag{19}$$

for $n \in \mathbb{N}_0$, that is,

$$(c_{n+1} - c_n)\lambda^{n+1} + (d_{n+1} - d_n)(n+1)\lambda^{n+1} = 0, \quad n \in \mathbb{N}_0. \tag{20}$$

Employing (18), (19), as well as (19) where n is replaced by $n+1$ in (8), and using that $p = -2\lambda$ and $q = \lambda^2$, we get

$$c_{n+1}\lambda^{n+2} + d_{n+1}(n+2)\lambda^{n+2} - 2\lambda(c_n\lambda^{n+1} + d_n(n+1)\lambda^{n+1}) + \lambda^2(c_n\lambda^n + d_n n\lambda^n) = f_n,$$

that is,

$$(c_{n+1} - c_n)\lambda^{n+2} + (d_{n+1} - d_n)(n+2)\lambda^{n+2} = f_n, \quad n \in \mathbb{N}_0. \tag{21}$$

For each fixed $n \in \mathbb{N}_0$, equalities (20) and (21) can be regarded as a two-dimensional linear system in variables $c_{n+1} - c_n$ and $d_{n+1} - d_n$.

By solving the system it is easily obtained

$$c_{n+1} - c_n = -\frac{(n+1)f_n}{\lambda^{n+2}} \quad \text{and} \quad d_{n+1} - d_n = \frac{f_n}{\lambda^{n+2}}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$c_n = c_0 - \sum_{k=0}^{n-1} \frac{(k+1)f_k}{\lambda^{k+2}} \quad \text{and} \quad d_n = d_0 + \sum_{k=0}^{n-1} \frac{f_k}{\lambda^{k+2}}, \quad n \in \mathbb{N}_0. \tag{22}$$

Using (22) into (18) we get (11). That (11) represents the general solution to (8), follows from the facts that the sequence

$$\widehat{x}_n := \sum_{k=0}^{n-1} \frac{n-k-1}{\lambda^{k+2-n}} f_k, \quad n \in \mathbb{N}_0,$$

is a particular solution to difference Equation (8), which is easily verified, while the sequence

$$x_n^h = c_0\lambda^n + d_0 n\lambda^n, \quad n \in \mathbb{N}_0,$$

is the general solution to the corresponding homogeneous difference equation, as desired. □

Remark 1. If $q = 0$, then Equation (8) is essentially reduced to Equation (1), when $p \neq 0$, or to a very simple equation if $p = 0$, which is the reason why the condition $q \neq 0$ is posed in Lemma 1.

Corollary 1. *Consider Equation (8) where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, x_0 and x_1 are complex numbers, and $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of complex numbers. Then the following statements are true.*

(a) If $p^2 \neq 4q$, then the solution to Equation (8) with the initial values x_0 and x_1 is given by the following formula

$$x_n = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_1^n\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{n-1} \frac{f_k}{\lambda_1^{k+1}}\right) + \lambda_2^n\left(x_1 - \lambda_1 x_0 + \sum_{k=0}^{n-1} \frac{f_k}{\lambda_2^{k+1}}\right)\right), \tag{23}$$

for $n \in \mathbb{N}_0$, where $\lambda_{1,2}$ are given by (10).

(b) *If $p^2 = 4q$, then the solution to Equation (8) with the initial values x_0 and x_1 is given by the following formula*

$$x_n = \lambda^n \left(x_0 - \sum_{k=0}^{n-1} \frac{(k+1)f_k}{\lambda^{k+2}} \right) + n\lambda^{n-1} \left(x_1 - \lambda x_0 + \sum_{k=0}^{n-1} \frac{f_k}{\lambda^{k+1}} \right), \tag{24}$$

for $n \in \mathbb{N}_0$, where $\lambda = -p/2$.

Proof. (a) Using Formula (9) with $n = 0, 1$, and by some calculations, we see that it must be

$$c_0 + d_0 = x_0, \quad \lambda_1 c_0 + \lambda_2 d_0 = x_1.$$

By solving the two-dimensional linear system, we obtain

$$c_0 = \frac{\lambda_2 x_0 - x_1}{\lambda_2 - \lambda_1}, \quad d_0 = \frac{x_1 - \lambda_1 x_0}{\lambda_2 - \lambda_1}. \tag{25}$$

Using (25) in (9) is obtained (23).
(b) From (11) with $n = 0, 1$, and some calculations, we see that it must be

$$c_0 = x_0, \quad c_0 + d_0 = \frac{x_1}{\lambda},$$

from which it follows that

$$c_0 = x_0, \quad d_0 = \frac{x_1 - \lambda x_0}{\lambda}. \tag{26}$$

Using (26) in (11) is obtained (24). □

Remark 2. Corollary 1, which essentially includes Lemma 1, can be also obtained by another standard method, the method of decomposition. We would like to point out that the method produces a slightly different formula. Namely, Equation (8) can be written in the following form:

$$x_{n+2} - \lambda_1 x_{n+1} = \lambda_2 (x_{n+1} - \lambda_1 x_n) + f_n, \quad n \in \mathbb{N}_0. \tag{27}$$

By using the change of variables

$$y_n = x_{n+1} - \lambda_1 x_n, \quad n \in \mathbb{N}_0,$$

Equation (27) becomes

$$y_{n+1} = \lambda_2 y_n + f_n, \quad n \in \mathbb{N}_0, \tag{28}$$

which is a special case of Equation (1), so it is solvable in closed-form, and its solution is

$$y_n = \lambda_2^n y_0 + \sum_{j=0}^{n-1} f_j \lambda_2^{n-1-j}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_n = \lambda_1 x_{n-1} + \lambda_2^{n-1}(x_1 - \lambda_1 x_0) + \sum_{j=0}^{n-2} f_j \lambda_2^{n-2-j}, \quad n \in \mathbb{N}. \tag{29}$$

By solving Equation (29) and after some calculation, in the case $\lambda_1 \neq \lambda_2$, we obtain

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n - (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_1 - \lambda_2} + \sum_{j=0}^{n-2} f_j \frac{\lambda_1^{n-j-1} - \lambda_2^{n-j-1}}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}, \tag{30}$$

which, on the first site, seems a bit different from the formula in (23). However, since

$$\sum_{j=0}^{n-1} f_j \frac{\lambda_1^{n-j-1} - \lambda_2^{n-j-1}}{\lambda_1 - \lambda_2} = \sum_{j=0}^{n-2} f_j \frac{\lambda_1^{n-j-1} - \lambda_2^{n-j-1}}{\lambda_1 - \lambda_2} + f_{n-1} \frac{1-1}{\lambda_1 - \lambda_2},$$

we see that Formulas (23) and (30) are the same.

The same situation appears in the case $\lambda_1 = \lambda_2$. We leave the verification of the fact as an exercise. In fact, a similar situation appears at several points, and at some different contexts, in the paper. We will not mention them, and suggest the reader to have the remark on his mind.

The following folklore result is another consequence of Lemma 1.

Corollary 2. *Consider Equation (8), where the zeros $\lambda_{1,2}$ of polynomial (12) satisfy the condition*

$$M := \max\{|\lambda_1|, |\lambda_2|\} < 1, \tag{31}$$

and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then every solution to the equation is bounded.

Proof. According to Lemma 1 we know that in the case $p^2 \neq 4q$, the general solution to difference Equation (8) is given by Formula (9), while if $p^2 = 4q$, the general solution is given by (11).

Assume first that $p^2 \neq 4q$. Then, by using (9), we have

$$\begin{aligned} |x_n| \le & |\lambda_1|^n \left(|c_0| + \sum_{k=0}^{n-1} \frac{|f_k|}{|\lambda_1|^{k+1}|\lambda_2 - \lambda_1|} \right) + |\lambda_2|^n \left(|d_0| + \sum_{k=0}^{n-1} \frac{|f_k|}{|\lambda_2|^{k+1}|\lambda_2 - \lambda_1|} \right) \\ & \le M^n(|c_0| + |d_0|) + \frac{\|f\|_\infty}{|\lambda_2 - \lambda_1|} \sum_{k=0}^{n-1} (|\lambda_1|^{n-k-1} + |\lambda_2|^{n-k-1}) \\ & \le |c_0| + |d_0| + \frac{2\|f\|_\infty}{|\lambda_2 - \lambda_1|(1-M)}, \end{aligned}$$

for every $n \in \mathbb{N}_0$, from which the result follows, in this case.

Now assume that $p^2 = 4q$. Since in this case $\lambda_1 = \lambda_2 = \lambda = -p/2$ and $|\lambda| = M$, by using (11), we have

$$\begin{aligned} |x_n| \le & |\lambda|^n \left(|c_0| + \sum_{k=0}^{n-1} \frac{(n-k-1)|f_k|}{|\lambda|^{k+2}} \right) + n|\lambda|^n |d_0| \\ & \le n|\lambda|^n(|c_0| + |d_0|) + \|f\|_\infty \sum_{s=0}^{n-1} s|\lambda|^{s-1} \\ & \le (|c_0| + |d_0|) \sup_{n\in\mathbb{N}} nM^n + \frac{\|f\|_\infty}{(1-M)^2}, \end{aligned}$$

for every $n \in \mathbb{N}_0$, from which along with the boundedness of the sequence $(nM^n)_{n\in\mathbb{N}}$, the result follows, in this case. □

If the sequence $(f_n)_{n\in\mathbb{N}_0}$ is T-periodic, that is,

$$f_n = f_{n+T}, \quad n \in \mathbb{N}_0, \tag{32}$$

for some $T \in \mathbb{N}$ (for $T = 1$ is said that f_n is eventually constant [13]), a natural question is if Equation (8) in this case has periodic solutions, and if so what is the relation between the periodic ones and the other solutions ([29]). The following result gives an answer to the question.

Theorem 1. *Consider Equation (8), where the zeros $\lambda_{1,2}$ of polynomial (12) satisfy condition (31) and $(f_n)_{n\in\mathbb{N}_0}$ is a T-periodic sequence. Then the following statements hold.*

(a) There is a unique T-periodic solution to Equation (8).
(b) All the solutions to Equation (8) converge geometrically to the periodic one.

Proof. (a) If $(x_n)_{n\in\mathbb{N}_0}$ is a T-periodic solution to Equation (8), then specially we have

$$x_0 = x_T \quad \text{and} \quad x_1 = x_{T+1}. \tag{33}$$

On the other hand, if (33) holds, then from (8) and (32), we have

$$x_{T+2} = -px_{T+1} - qx_T + f_T = -px_1 - qx_0 + f_0 = x_2.$$

A simple inductive argument along with a use of (8) shows that

$$x_{mT+l} = x_l,$$

for every $m \in \mathbb{N}$ and $l \in \{0, 1, \dots, T-1\}$, that is, such a solution to Equation (8) is T-periodic.

Case $p^2 \neq 4q$. From this and (23), we see that it is enough to show that the linear system

$$\begin{aligned} x_0 =& \frac{1}{\lambda_2-\lambda_1}\left(\lambda_1^T\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{T-1}\frac{f_k}{\lambda_1^{k+1}}\right) + \lambda_2^T\left(x_1 - \lambda_1 x_0 + \sum_{k=0}^{T-1}\frac{f_k}{\lambda_2^{k+1}}\right)\right) \\ x_1 =& \frac{1}{\lambda_2-\lambda_1}\left(\lambda_1^{T+1}\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{T}\frac{f_k}{\lambda_1^{k+1}}\right) + \lambda_2^{T+1}\left(x_1 - \lambda_1 x_0 + \sum_{k=0}^{T}\frac{f_k}{\lambda_2^{k+1}}\right)\right), \end{aligned} \tag{34}$$

has a unique solution in variables x_0 and x_1.

System (34) can be written in the following form:

$$\begin{aligned} &(\lambda_1\lambda_2(\lambda_1^{T-1} - \lambda_2^{T-1}) + \lambda_1 - \lambda_2)x_0 + (\lambda_2^T - \lambda_1^T)x_1 = \lambda_1^T S_1 - \lambda_2^T S_2 \\ &\lambda_1\lambda_2(\lambda_1^T - \lambda_2^T)x_0 + (\lambda_2^{T+1} - \lambda_1^{T+1} + \lambda_1 - \lambda_2)x_1 = \lambda_1^{T+1} S_1 - \lambda_2^{T+1} S_2, \end{aligned} \tag{35}$$

where

$$S_1 := \sum_{j=0}^{T-1}\frac{f_j}{\lambda_1^{j+1}} \quad \text{and} \quad S_2 := \sum_{j=0}^{T-1}\frac{f_j}{\lambda_2^{j+1}}.$$

After some standard but interesting calculation it is shown that the determinant of system (35) is:

$$\begin{aligned} \Delta =& \begin{vmatrix} \lambda_1\lambda_2(\lambda_1^{T-1} - \lambda_2^{T-1}) + \lambda_1 - \lambda_2 & \lambda_2^T - \lambda_1^T \\ \lambda_1\lambda_2(\lambda_1^T - \lambda_2^T) & \lambda_2^{T+1} - \lambda_1^{T+1} + \lambda_1 - \lambda_2 \end{vmatrix} \\ =& (\lambda_1 - \lambda_2)^2(\lambda_1^T - 1)(\lambda_2^T - 1) \neq 0, \end{aligned} \tag{36}$$

due to (31) and $\lambda_1 \neq \lambda_2$.

Also, we have

$$\begin{aligned} \Delta_1 =& \begin{vmatrix} \lambda_1^T S_1 - \lambda_2^T S_2 & \lambda_2^T - \lambda_1^T \\ \lambda_1^{T+1} S_1 - \lambda_2^{T+1} S_2 & \lambda_2^{T+1} - \lambda_1^{T+1} + \lambda_1 - \lambda_2 \end{vmatrix} \\ =& (\lambda_2 - \lambda_1)(\lambda_1^T(\lambda_2^T - 1)S_1 - \lambda_2^T(\lambda_1^T - 1)S_2), \end{aligned} \tag{37}$$

and

$$\Delta_2 = \begin{vmatrix} \lambda_1\lambda_2(\lambda_1^{T-1} - \lambda_2^{T-1}) + \lambda_1 - \lambda_2 & \lambda_1^T S_1 - \lambda_2^T S_2 \\ \lambda_1\lambda_2(\lambda_1^T - \lambda_2^T) & \lambda_1^{T+1} S_1 - \lambda_2^{T+1} S_2 \end{vmatrix}$$
$$= (\lambda_2 - \lambda_1)(\lambda_1^{T+1}(\lambda_2^T - 1)S_1 - \lambda_2^{T+1}(\lambda_1^T - 1)S_2). \tag{38}$$

From (36) to (38), it follows that

$$x_0 = \frac{\lambda_1^T(\lambda_2^T - 1)S_1 - \lambda_2^T(\lambda_1^T - 1)S_2}{(\lambda_2 - \lambda_1)(\lambda_1^T - 1)(\lambda_2^T - 1)} \tag{39}$$

and

$$x_1 = \frac{\lambda_1^{T+1}(\lambda_2^T - 1)S_1 - \lambda_2^{T+1}(\lambda_1^T - 1)S_2}{(\lambda_2 - \lambda_1)(\lambda_1^T - 1)(\lambda_2^T - 1)}, \tag{40}$$

are the initial values for which is obtained the T-periodic solution to Equation (8) in this case.

Case $p^2 = 4q$. Using (24), we see that (33) becomes the linear system

$$\begin{aligned} x_0 =& \lambda^T\left(x_0 - \sum_{k=0}^{T-1}\frac{(k+1)f_k}{\lambda^{k+2}}\right) + T\lambda^{T-1}\left(x_1 - \lambda x_0 + \sum_{k=0}^{T-1}\frac{f_k}{\lambda^{k+1}}\right) \\ x_1 =& \lambda^{T+1}\left(x_0 - \sum_{k=0}^{T}\frac{(k+1)f_k}{\lambda^{k+2}}\right) + (T+1)\lambda^T\left(x_1 - \lambda x_0 + \sum_{k=0}^{T}\frac{f_k}{\lambda^{k+1}}\right). \end{aligned} \tag{41}$$

System (41) can be rewritten in the following form:

$$\begin{aligned} &(1 + (T-1)\lambda^T)x_0 - T\lambda^{T-1}x_1 = T\lambda^{T-1}\widehat{S}_1 - \lambda^T\widehat{S}_2 \\ &T\lambda^{T+1}x_0 + (1 - (T+1)\lambda^T)x_1 = (T+1)\lambda^T\widehat{S}_1 - \lambda^{T+1}\widehat{S}_2, \end{aligned} \tag{42}$$

where

$$\widehat{S}_1 := \sum_{j=0}^{T-1}\frac{f_j}{\lambda^{j+1}} \quad \text{and} \quad \widehat{S}_2 := \sum_{j=0}^{T-1}\frac{(j+1)f_j}{\lambda^{j+2}}.$$

After some calculation it is shown that the determinant of system (42) is:

$$\Delta = \begin{vmatrix} 1 + (T-1)\lambda^T & -T\lambda^{T-1} \\ T\lambda^{T+1} & 1 - (T+1)\lambda^T \end{vmatrix} = (\lambda^T - 1)^2 \neq 0, \tag{43}$$

due to (31).

Also, we have

$$\Delta_1 = \begin{vmatrix} T\lambda^{T-1}\widehat{S}_1 - \lambda^T\widehat{S}_2 & -T\lambda^{T-1} \\ (T+1)\lambda^T\widehat{S}_1 - \lambda^{T+1}\widehat{S}_2 & 1 - (T+1)\lambda^T \end{vmatrix}$$
$$= T\lambda^{T-1}\widehat{S}_1 + \lambda^T(\lambda^T - 1)\widehat{S}_2, \tag{44}$$

and

$$\Delta_2 = \begin{vmatrix} 1 + (T-1)\lambda^T & T\lambda^{T-1}\widehat{S}_1 - \lambda^T\widehat{S}_2 \\ T\lambda^{T+1} & (T+1)\lambda^T\widehat{S}_1 - \lambda^{T+1}\widehat{S}_2 \end{vmatrix}$$
$$= \lambda^T(1 + T - \lambda^T)\widehat{S}_1 + \lambda^{T+1}(\lambda^T - 1)\widehat{S}_2. \tag{45}$$

From (43)–(45), it follows that

$$x_0 = \frac{T\lambda^{T-1}\widehat{S}_1 + \lambda^T(\lambda^T - 1)\widehat{S}_2}{(\lambda^T - 1)^2}$$

and

$$x_1 = \frac{\lambda^T(1+T-\lambda^T)\widehat{S}_1 + \lambda^{T+1}(\lambda^T-1)\widehat{S}_2}{(\lambda^T-1)^2},$$

are the initial values for which is obtained the T-periodic solution to Equation (8) in this case.

(b) If $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ is the T-periodic solution to Equation (8) and $(x_n)_{n\in\mathbb{N}_0}$ is any solution to the equation, then if $p^2 \neq 4q$, from (23) we have

$$\begin{aligned}|\tilde{x}_n - x_n| = \frac{1}{|\lambda_2-\lambda_1|}\Bigg|&\Bigg(\lambda_1^n\Bigg(\lambda_2\tilde{x}_0 - \tilde{x}_1 - \sum_{k=0}^{n-1}\frac{f_k}{\lambda_1^{k+1}}\Bigg) + \lambda_2^n\Bigg(\tilde{x}_1 - \lambda_1\tilde{x}_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda_2^{k+1}}\Bigg)\Bigg)\\ &- \Bigg(\lambda_1^n\Bigg(\lambda_2 x_0 - x_1 - \sum_{k=0}^{n-1}\frac{f_k}{\lambda_1^{k+1}}\Bigg) + \lambda_2^n\Bigg(x_1 - \lambda_1 x_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda_2^{k+1}}\Bigg)\Bigg)\Bigg|\\ \le&\frac{(|\lambda_2||\tilde{x}_0-x_0| + |\tilde{x}_1-x_1|)|\lambda_1|^n + (|\lambda_1||\tilde{x}_0-x_0| + |\tilde{x}_1-x_1|)|\lambda_2|^n}{|\lambda_2-\lambda_1|},\end{aligned}$$

from which along with (31) the statement follows in this case.

If $p^2 = 4q$, then from (24) we have

$$\begin{aligned}|\tilde{x}_n - x_n| =&\Bigg|\lambda^n\Bigg(\tilde{x}_0 - \sum_{k=0}^{n-1}\frac{(k+1)f_k}{\lambda^{k+2}}\Bigg) + n\lambda^{n-1}\Bigg(\tilde{x}_1 - \lambda\tilde{x}_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda^{k+1}}\Bigg)\\ &- \lambda^n\Bigg(x_0 - \sum_{k=0}^{n-1}\frac{(k+1)f_k}{\lambda^{k+2}}\Bigg) - n\lambda^{n-1}\Bigg(x_1 - \lambda x_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda^{k+1}}\Bigg)\Bigg|\\ \le&|\tilde{x}_0 - x_0||\lambda|^n + (|\tilde{x}_1 - x_1| + |\lambda||\tilde{x}_0 - x_0|)n|\lambda|^{n-1}\\ \le&M\left(\frac{1+|\lambda|}{2}\right)^n,\end{aligned}$$

for some $M = M(x_0, x_1, \tilde{x}_0, \tilde{x}_1, \lambda)$, from which the statement follows in this case. $\square$

The following result solves the problem of existence of a unique bounded solution to Equation (5) for the case $p_n = p$, $q_n = q$, $n \in \mathbb{N}_0$, when

$$m := \min\{|\lambda_1|, |\lambda_2|\} > 1. \tag{46}$$

Theorem 2. *Consider Equation (8), where the zeros $\lambda_{1,2}$ of the polynomial (12) satisfy condition (46), and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then, there is a unique bounded solution to the equation.*

Proof. According to Lemma 1 (a) we know that in the case $p^2 \neq 4q$, the general solution to (8) is given by (9), while if $p^2 = 4q$, the general solution is given by (11).

Assume first that $p^2 \neq 4q$. If in the case there is a bounded solution to Equation (8), then it must be

$$c_0 = \sum_{k=0}^{\infty}\frac{f_k}{\lambda_1^{k+1}(\lambda_2-\lambda_1)} =: \widetilde{S}_1 \quad \text{and} \quad d_0 = -\sum_{k=0}^{\infty}\frac{f_k}{\lambda_2^{k+1}(\lambda_2-\lambda_1)} =: \widetilde{S}_2. \tag{47}$$

Note that sums $\widetilde{S}_1$ and $\widetilde{S}_2$ are finite since due to condition (46) and the boundedness of $(f_n)_{n\in\mathbb{N}_0}$, we have

$$\left|\sum_{k=0}^{\infty}\frac{f_k}{\lambda_j^{k+1}(\lambda_2-\lambda_1)}\right| \le \frac{\|f\|_\infty}{|\lambda_2-\lambda_1|(|\lambda_j|-1)} < +\infty, \quad j = 1, 2.$$

Indeed, if $c_0 \neq \widetilde{S}_1$ and $d_0 \neq \widetilde{S}_2$, then from (9), we easily get

$$|x_n| \asymp (\max\{|\lambda_1|, |\lambda_2|\})^n. \tag{48}$$

If $c_0 \neq \widetilde{S}_1$ and $d_0 = \widetilde{S}_2$, then

$$x_n = \lambda_1^n \left(c_0 - \sum_{k=0}^{n-1} \frac{f_k}{\lambda_1^{k+1}(\lambda_2 - \lambda_1)} \right) - \lambda_2^n \sum_{k=n}^{\infty} \frac{f_k}{\lambda_2^{k+1}(\lambda_2 - \lambda_1)}.$$

From this and since

$$\left| \lambda_2^n \sum_{k=n}^{\infty} \frac{f_k}{\lambda_2^{k+1}(\lambda_2 - \lambda_1)} \right| \leq \frac{\|f\|_\infty}{|\lambda_2 - \lambda_1|(|\lambda_2| - 1)}$$

we get

$$x_n \asymp \lambda_1^n. \tag{49}$$

If $c_0 = \widetilde{S}_1$ and $d_0 \neq \widetilde{S}_2$, then

$$x_n = \lambda_1^n \sum_{k=n}^{\infty} \frac{f_k}{\lambda_1^{k+1}(\lambda_2 - \lambda_1)} + \lambda_2^n \left(d_0 + \sum_{k=0}^{n-1} \frac{f_k}{\lambda_2^{k+1}(\lambda_2 - \lambda_1)} \right).$$

From this and since

$$\left| \lambda_1^n \sum_{k=n}^{\infty} \frac{f_k}{\lambda_1^{k+1}(\lambda_2 - \lambda_1)} \right| \leq \frac{\|f\|_\infty}{|\lambda_2 - \lambda_1|(|\lambda_1| - 1)}$$

we get

$$x_n \asymp \lambda_2^n. \tag{50}$$

Hence, in these three cases from (48)–(50) it would follow that the solutions would be unbounded, a contradiction.

By using (47) in (9), we get

$$x_n = \sum_{k=n}^{\infty} \frac{\lambda_1^{n-k-1} - \lambda_2^{n-k-1}}{\lambda_2 - \lambda_1} f_k, \quad n \in \mathbb{N}_0. \tag{51}$$

A direct calculation shows that sequence $(x_n)_{n\in\mathbb{N}_0}$ defined by (51) is a solution to Equation (8). On the other hand, by using the assumptions of the theorem we easily get

$$|x_n| \leq \frac{2\|f\|_\infty}{|\lambda_2 - \lambda_1|} \sum_{k=n}^{\infty} m^{n-k-1} = \frac{2\|f\|_\infty}{|\lambda_2 - \lambda_1|(m-1)} < \infty, \quad n \in \mathbb{N}_0,$$

from which the boundedness of $(x_n)_{n\in\mathbb{N}_0}$, follows. From this and since by (47), (c_0, d_0) is uniquely determined it follows that (51) is a unique bounded solution to Equation (8), in this case.

Now, assume that $p^2 = 4q$. If in the case there is a bounded solution to (8), then it must be

$$c_0 = \sum_{k=0}^{\infty} \frac{(k+1)f_k}{\lambda^{k+2}} =: S_3 \quad \text{and} \quad d_0 = -\sum_{k=0}^{\infty} \frac{f_k}{\lambda^{k+2}} =: S_4. \tag{52}$$

Note that sums S_3 and S_4 are also finite since due to condition (46) and the boundedness of $(f_n)_{n\in\mathbb{N}_0}$ we have

$$\left| \sum_{k=0}^{\infty} \frac{f_k}{\lambda^{k+2}} \right| \leq \frac{\|f\|_\infty}{|\lambda|(|\lambda| - 1)} < +\infty$$

and
$$\left|\sum_{k=0}^{\infty}\frac{(k+1)f_k}{\lambda^{k+2}}\right| \le \frac{\|f\|_\infty}{(|\lambda|-1)^2} < +\infty.$$

Indeed, if $c_0 \neq S_3$ and $d_0 \neq S_4$, then from (11), we easily get
$$|x_n| \asymp n|\lambda|^n. \tag{53}$$

If $c_0 \neq S_3$ and $d_0 = S_4$, then
$$x_n = \lambda^n\left(c_0 - \sum_{k=0}^{n-1}\frac{(k+1)f_k}{\lambda^{k+2}}\right) - n\lambda^n\sum_{k=n}^{\infty}\frac{f_k}{\lambda^{k+2}}.$$

From this, since
$$\left|n\lambda^n\sum_{k=n}^{\infty}\frac{f_k}{\lambda^{k+2}}\right| \le \frac{n\|f\|_\infty}{|\lambda|(|\lambda|-1)}$$
and $n = o(\lambda^n)$, we get
$$x_n \asymp \lambda^n. \tag{54}$$

If $c_0 = S_3$ and $d_0 \neq S_4$, then
$$x_n = \lambda^n\sum_{k=n}^{\infty}\frac{(k+1)f_k}{\lambda^{k+2}} + n\lambda^n\left(d_0 + \sum_{k=0}^{n-1}\frac{f_k}{\lambda^{k+2}}\right).$$

From this, since
$$\left|\lambda^n\sum_{k=n}^{\infty}\frac{(k+1)f_k}{\lambda^{k+2}}\right| \le \frac{\|f\|_\infty}{|\lambda|^2}\sum_{s=0}^{\infty}\frac{s+n+1}{|\lambda|^s} = \frac{\|f\|_\infty}{|\lambda|^2}\left(\frac{|\lambda|n}{|\lambda|-1} + \frac{|\lambda|^2}{(|\lambda|-1)^2}\right),$$
and $n = o(n\lambda^n)$, we get
$$x_n \asymp n\lambda^n. \tag{55}$$

Hence, in these three cases from (53)–(55) it would follow that the solutions are unbounded, a contradiction.

By using (52) in (11), we get
$$x_n = \sum_{k=n}^{\infty}\frac{(k+1-n)}{\lambda^{k+2-n}}f_k, \quad n \in \mathbb{N}_0. \tag{56}$$

A direct calculation shows that sequence $(x_n)_{n\in\mathbb{N}_0}$ defined by (56) is a solution to Equation (8). On the other hand, by using the assumptions of the theorem we easily get
$$|x_n| \le \|f\|_\infty\sum_{k=n}^{\infty}(k+1-n)|\lambda|^{n-k-2} = \frac{\|f\|_\infty}{(|\lambda|-1)^2} < \infty, \quad n \in \mathbb{N}_0,$$
from which the boundedness of $(x_n)_{n\in\mathbb{N}_0}$ follows. From this and since by (52), (c_0, d_0) is uniquely determined it follows that (56) is a unique bounded solution to Equation (8), in this case. □

Theorem 3. *Consider Equation (8), where $p \in \mathbb{C}$, $q \in \mathbb{C}\setminus\{0\}$, the zeros $\lambda_{1,2}$ of the polynomial (12) satisfy condition (46) and $(f_n)_{n\in\mathbb{N}_0}$ is a T-periodic sequence. Then, the unique bounded solution to Equation (8) is T-periodic.*

Proof. If $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ is the bounded solution to Equation (8), then by Theorem 2 we see that it is given by (51) when $p^2 \neq 4q$ and (56) if $p^2 = 4q$.

Hence, if $p^2 \neq 4q$, then we have

$$\begin{aligned}\tilde{x}_{n+T} &= \sum_{k=n+T}^{\infty} \frac{\lambda_1^{n+T-k-1} - \lambda_2^{n+T-k-1}}{\lambda_2 - \lambda_1} f_k \\ &= \sum_{j=n}^{\infty} \frac{\lambda_1^{n-j-1} - \lambda_2^{n-j-1}}{\lambda_2 - \lambda_1} f_{j+T} \\ &= \sum_{j=n}^{\infty} \frac{\lambda_1^{n-j-1} - \lambda_2^{n-j-1}}{\lambda_2 - \lambda_1} f_j = \tilde{x}_n,\end{aligned} \tag{57}$$

for $n \in \mathbb{N}_0$, while if $p^2 = 4q$, we have

$$\begin{aligned}\tilde{x}_{n+T} &= \sum_{k=n+T}^{\infty} \frac{(k+1-n-T)}{\lambda^{k+2-n-T}} f_k \\ &= \sum_{j=n}^{\infty} \frac{(j+1-n)}{\lambda^{j+2-n}} f_{j+T} \\ &= \sum_{j=n}^{\infty} \frac{(j+1-n)}{\lambda^{j+2-n}} f_j = \tilde{x}_n,\end{aligned} \tag{58}$$

for $n \in \mathbb{N}_0$. From (57) and (58) the result follows. $\square$

Theorem 4. *Consider Equation (8), where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, the zeros $\lambda_{1,2}$ of the polynomial (12) satisfy the following condition*

$$\min\{|\lambda_1|, |\lambda_2|\} < 1 < \max\{|\lambda_1|, |\lambda_2|\}, \tag{59}$$

and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then, the following statements are true.

(a) If $|\lambda_1| < 1 < |\lambda_2|$, then a solution to Equation (8) is bounded if and only if

$$\lambda_1 x_0 - x_1 = \sum_{j=0}^{\infty} \frac{f_j}{\lambda_2^{j+1}}. \tag{60}$$

(b) If $|\lambda_2| < 1 < |\lambda_1|$, then a solution to Equation (8) is bounded if and only if

$$\lambda_2 x_0 - x_1 = \sum_{j=0}^{\infty} \frac{f_j}{\lambda_1^{j+1}}. \tag{61}$$

Proof. (a) Since $|\lambda_1| < 1$, we have that

$$\begin{aligned}\left|\lambda_1^n\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{n-1} \frac{f_k}{\lambda_1^{k+1}}\right)\right| &\le (|\lambda_2||x_0| + |x_1|)|\lambda_1|^n + \|f\|_\infty \sum_{k=0}^{n-1} |\lambda_1|^{n-k-1} \\ &\le |\lambda_2||x_0| + |x_1| + \frac{\|f\|_\infty}{1-|\lambda_1|} < \infty.\end{aligned} \tag{62}$$

From (23), (62), and since $|\lambda_2| > 1$, it follows that the boundedness of a solution x_n to Equation (8) implies (60). Indeed, since $|\lambda_2| > 1$, we have

$$\left|\sum_{k=0}^{\infty} \frac{f_k}{\lambda_2^{k+1}}\right| \le \sum_{k=0}^{\infty} \frac{|f_k|}{|\lambda_2|^{k+1}} \le \frac{\|f\|_\infty}{|\lambda_2|-1} < +\infty,$$

that is, the last series is absolutely convergent. So, if (60) were not hold, then for the solution would be

$$x_n \asymp \lambda_2^n,$$

which would contradict with its boundedness.

Now assume that (60) holds. Then from (23) and (60) it follows that the solution in the case must be

$$x_n = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_1^n\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{n-1} \frac{f_k}{\lambda_1^{k+1}}\right) - \lambda_2^n\left(\sum_{k=n}^{\infty} \frac{f_k}{\lambda_2^{k+1}}\right)\right). \tag{63}$$

Since $|\lambda_2| > 1$, we have

$$\left|\lambda_2^n\left(\sum_{k=n}^{\infty} \frac{f_k}{\lambda_2^{k+1}}\right)\right| \le \|f\|_\infty \sum_{k=n}^{\infty} |\lambda_2|^{n-k-1} = \frac{\|f\|_\infty}{|\lambda_2|-1} < \infty. \tag{64}$$

Using (62) and (64) in (63), we have

$$|x_n| \le \frac{1}{|\lambda_2 - \lambda_1|}\left(|\lambda_2||x_0| + |x_1| + \frac{\|f\|_\infty}{1-|\lambda_1|} + \frac{\|f\|_\infty}{|\lambda_2|-1}\right),$$

from which it follows that the solution to Equation (8) is bounded.

(b) The proof of the statement is similar/dual to the one in (a). Hence, it is omitted. □

Theorem 5. *Consider Equation (8), where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, the zeros $\lambda_{1,2}$ of the polynomial (12) satisfy condition (59) and $(f_n)_{n\in\mathbb{N}_0}$ is a T-periodic sequence. Then, the following statements are true.*

(a) There is a unique T-periodic solution to Equation (8).
(b) All bounded solutions to Equation (8) converge geometrically to the periodic one.

Proof. (a) We may assume that the condition holds $|\lambda_1| < 1 < |\lambda_2|$, since the other case is essentially the same and is obtained by changing some letters only. By Theorem 4, we see that a solution to Equation (8) is bounded if and only if (60) holds, and that bounded solutions to Equation (8) have the form in (63). If a solution to the equation is T-periodic, then it must be $x_0 = x_T$, that is,

$$x_0 = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_1^T\left(\lambda_2 x_0 - x_1 - \sum_{k=0}^{T-1} \frac{f_k}{\lambda_1^{k+1}}\right) - \lambda_2^T\left(\sum_{k=T}^{\infty} \frac{f_k}{\lambda_2^{k+1}}\right)\right), \tag{65}$$

from which, along with (63) and by some calculation is obtained

$$x_0 = \frac{\lambda_2^T \sum_{j=T}^{\infty} \frac{f_j}{\lambda_2^{j+1}} - \lambda_1^T \sum_{j=0}^{\infty} \frac{f_j}{\lambda_2^{j+1}} + \lambda_1^T \sum_{j=0}^{T-1} \frac{f_j}{\lambda_1^{j+1}}}{(\lambda_2 - \lambda_1)(\lambda_1^T - 1)}. \tag{66}$$

By using equalities (60) and (66) in (63) and after some calculation it is shown that for such chosen x_0 is obtained a T-periodic solution to Equation (8). Since initial value x_0 is uniquely defined by (66), and consequently by (60) initial value x_1 is also uniquely defined, the T-periodic solution is unique too, as claimed.

(b) If $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ is the T-periodic solution to Equation (8) and $(x_n)_{n\in\mathbb{N}_0}$ is any bounded solution to the equation, then from (60) and (63) we have

$$\begin{aligned}|\tilde{x}_n - x_n| = \frac{1}{|\lambda_2-\lambda_1|}\Bigg|&\lambda_1^n\Bigg((\lambda_2-\lambda_1)\tilde{x}_0 + \sum_{k=0}^{\infty}\frac{f_k}{\lambda_2^{k+1}} - \sum_{k=0}^{n-1}\frac{f_k}{\lambda_1^{k+1}}\Bigg) - \lambda_2^n\Bigg(\sum_{k=n}^{\infty}\frac{f_k}{\lambda_2^{k+1}}\Bigg)\\ &-\lambda_1^n\Bigg((\lambda_2-\lambda_1)x_0 + \sum_{k=0}^{\infty}\frac{f_k}{\lambda_2^{k+1}} - \sum_{k=0}^{n-1}\frac{f_k}{\lambda_1^{k+1}}\Bigg) - \lambda_2^n\Bigg(\sum_{k=n}^{\infty}\frac{f_k}{\lambda_2^{k+1}}\Bigg)\Bigg|\\ \le&|\tilde{x}_0 - x_0||\lambda_1|^n,\end{aligned}$$

from which the statement follows. □

Remark 3. Since the sequence $(f_n)_{n\in\mathbb{N}_0}$ in Theorem 5 is T-periodic, then the expression for x_0 in (66) can be written in a somewhat nicer way. Namely, since the series $\sum_{j=0}^{\infty}\frac{f_j}{\lambda_2^{j+1}}$ is absolutely convergent, we have

$$\begin{aligned}\sum_{j=mT}^{\infty}\frac{f_j}{\lambda_2^{j+1}} &= \sum_{k=m}^{\infty}\sum_{j=kT}^{(k+1)T-1}\frac{f_j}{\lambda_2^{j+1}} = \sum_{k=m}^{\infty}\sum_{i=0}^{T-1}\frac{f_{kT+i}}{\lambda_2^{kT+i+1}} = \sum_{k=m}^{\infty}\frac{1}{\lambda_2^{kT}}\sum_{i=0}^{T-1}\frac{f_i}{\lambda_2^{i+1}}\\ &= \frac{\lambda_2^{(1-m)T}}{\lambda_2^T-1}\sum_{i=0}^{T-1}\frac{f_i}{\lambda_2^{i+1}},\end{aligned} \tag{67}$$

for every $m \in \mathbb{N}_0$.

Using (67) in (66) for $m = 0$ and $m = 1$ and after some calculation it follows that

$$x_0 = \frac{\lambda_1^T(\lambda_2^T-1)\sum_{j=0}^{T-1}\frac{f_j}{\lambda_1^{j+1}} - \lambda_2^T(\lambda_1^T-1)\sum_{j=0}^{T-1}\frac{f_j}{\lambda_2^{j+1}}}{(\lambda_2-\lambda_1)(\lambda_1^T-1)(\lambda_2^T-1)}.$$

From this, (60) and some calculation we get

$$x_1 = \frac{\lambda_1^{T+1}(\lambda_2^T-1)\sum_{j=0}^{T-1}\frac{f_j}{\lambda_1^{j+1}} - \lambda_2^{T+1}(\lambda_1^T-1)\sum_{j=0}^{T-1}\frac{f_j}{\lambda_2^{j+1}}}{(\lambda_2-\lambda_1)(\lambda_1^T-1)(\lambda_2^T-1)}.$$

Note that the initial values match with the ones in (39) and (40).

Now we are in a position to formulate and prove the main results in this section. The results give some sufficient conditions for the unique existence of bounded solutions to Equation (5), that is, when the sequences $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$, in general, are not constant, and they are in the spirit of the main result in our recent paper [30].

Theorem 6. *Assume that $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are sequences of complex numbers such that*

$$\hat{q}_2 := \sup_{n\in\mathbb{N}_0}\frac{|p_n+r_1+r_2|+|q_n-r_1r_2|}{|r_1-r_2|(r_m-1)} < \frac{1}{2}, \tag{68}$$

for some distinct numbers r_1 and r_2, such that $r_m := \min\{|r_1|,|r_2|\} > 1$, and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then, Equation (5) has a unique bounded solution.

Proof. Write Equation (5) in the following form

$$x_{n+2} - (r_1+r_2)x_{n+1} + r_1r_2x_n = -(p_n+r_1+r_2)x_{n+1} + (r_1r_2-q_n)x_n + f_n, \tag{69}$$

for $n \in \mathbb{N}_0$.

Let A be the following operator defined on the class of all sequences

$$A(u) = \left(\sum_{k=n}^{\infty} \frac{r_1^{n-k-1} - r_2^{n-k-1}}{r_2 - r_1} ((r_1 r_2 - q_k) u_k - (p_k + r_1 + r_2) u_{k+1} + f_k) \right)_{n \in \mathbb{N}_0}. \tag{70}$$

If $u \in l^\infty$, then from (70), by using condition (68) and some elementary estimates, it follows that

$$\begin{aligned} \|A(u)\|_\infty &= \sup_{n \in \mathbb{N}_0} \left| \sum_{k=n}^{\infty} \frac{r_1^{n-k-1} - r_2^{n-k-1}}{r_2 - r_1} ((r_1 r_2 - q_k) u_k - (p_k + r_1 + r_2) u_{k+1} + f_k) \right| \\ &\le 2 \sup_{n \in \mathbb{N}_0} \sum_{k=n}^{\infty} \frac{\|u\|_\infty (|r_1 r_2 - q_k| + |p_k + r_1 + r_2|) + \|f\|_\infty}{|r_2 - r_1| r_m^{k-n+1}} \\ &\le \frac{\|u\|_\infty |r_1 - r_2| (r_m - 1) + 2\|f\|_\infty}{|r_2 - r_1| (r_m - 1)} < \infty, \end{aligned}$$

which means that operator A maps the Banach space l^∞ into itself.

On the other hand, for every $u, v \in l^\infty$ we have

$$\begin{aligned} &\|A(u) - A(v)\|_\infty \\ &= \sup_{n \in \mathbb{N}_0} \left| \sum_{k=n}^{\infty} \frac{r_1^{n-k-1} - r_2^{n-k-1}}{r_2 - r_1} ((r_1 r_2 - q_k)(u_k - v_k) - (p_k + r_1 + r_2)(u_{k+1} - v_{k+1})) \right| \\ &\le \|u - v\|_\infty 2 \sup_{n \in \mathbb{N}_0} \sum_{k=n}^{\infty} \frac{|r_1 r_2 - q_k| + |p_k + r_1 + r_2|}{|r_2 - r_1| r_m^{k-n+1}} \\ &\le \hat{2} q_2 \|u - v\|_\infty, \end{aligned} \tag{71}$$

from which along with condition (68) it follows that the operator $A : l^\infty \to l^\infty$ is a contraction.

By the Banach fixed point theorem ([4]) it follows that the operator has a unique fixed point, say $x^* = (x_n^*)_{n \in \mathbb{N}_0} \in l^\infty$, that is, $A(x^*) = x^*$, which can be written as follows

$$x_n^* = \sum_{k=n}^{\infty} \frac{r_1^{n-k-1} - r_2^{n-k-1}}{r_2 - r_1} ((r_1 r_2 - q_k) x_k^* - (p_k + r_1 + r_2) x_{k+1}^* + f_k), \tag{72}$$

for $n \in \mathbb{N}_0$.

A direct calculation shows that this bounded sequence satisfies difference Equation (69), that is, Equation (5) for every $n \in \mathbb{N}_0$, from which the theorem follows. □

Theorem 7. *Assume that $(p_n)_{n \in \mathbb{N}_0}$ and $(q_n)_{n \in \mathbb{N}_0}$ are sequences of complex numbers such that*

$$\hat{q} := \sup_{n \in \mathbb{N}_0} \frac{|p_n + 2r| + |q_n - r^2|}{(r - 1)^2} < 1, \tag{73}$$

for some number $r > 1$, and $(f_n)_{n \in \mathbb{N}_0}$ is a bounded sequence of complex numbers. Then the difference Equation (5) has a unique bounded solution.

Proof. Write Equation (5) in the following form

$$x_{n+2} - 2r x_{n+1} + r^2 x_n = -(p_n + 2r) x_{n+1} + (r^2 - q_n) x_n + f_n, \tag{74}$$

for $n \in \mathbb{N}_0$.

Let A be the following operator defined on the class of all sequences

$$A(u) = \left(\sum_{k=n}^{\infty} \frac{(k+1-n)}{r^{k+2-n}} ((r^2 - q_k)u_k - (p_k + 2r)u_{k+1} + f_k) \right)_{n \in \mathbb{N}_0}. \tag{75}$$

If $u \in l^{\infty}$, then from (75) it follows that

$$\begin{aligned} \|A(u)\|_{\infty} &= \sup_{n \in \mathbb{N}_0} \left| \sum_{k=n}^{\infty} \frac{(k+1-n)}{r^{k+2-n}} ((r^2 - q_k)u_k - (p_k + 2r)u_{k+1} + f_k) \right| \\ &\leq \sup_{n \in \mathbb{N}_0} \sum_{k=n}^{\infty} \frac{(k+1-n)(|r^2 - q_k||u_k| + |p_k + 2r||u_{k+1}| + |f_k|)}{r^{k+2-n}} \\ &\leq \frac{(r-1)^2 \|u\|_{\infty} + \|f\|_{\infty}}{(r-1)^2} < \infty, \end{aligned}$$

which means that operator A maps the Banach space l^{∞} into itself.

On the other hand, for every $u, v \in l^{\infty}$ we have

$$\begin{aligned} &\|A(u) - A(v)\|_{\infty} \\ &= \sup_{n \in \mathbb{N}_0} \left| \sum_{k=n}^{\infty} \frac{(k+1-n)}{r^{k+2-n}} ((r^2 - q_k)(u_k - v_k) - (p_k + 2r)(u_{k+1} - v_{k+1}) \right| \\ &\leq \|u - v\|_{\infty} \sup_{n \in \mathbb{N}_0} \sum_{k=n}^{\infty} \frac{(k+1-n)}{r^{k+2-n}} (|r^2 - q_k| + |p_k + 2r|) \\ &\leq \hat{q} \|u - v\|_{\infty}. \end{aligned} \tag{76}$$

From (76) and condition (73) it follows that the operator $A : l^{\infty} \to l^{\infty}$ is a contraction.

By the Banach fixed point theorem we get that the operator has a unique fixed point, say $x^* = (x_n^*)_{n \in \mathbb{N}_0} \in l^{\infty}$, that is, $A(x^*) = x^*$ or equivalently

$$x_n^* = \sum_{k=n}^{\infty} \frac{(k+1-n)}{r^{k+2-n}} ((r^2 - q_k)x_k^* - (p_k + 2r)x_{k+1}^* + f_k), \tag{77}$$

for $n \in \mathbb{N}_0$.

A direct calculation shows that this bounded sequence satisfies difference Equation (74), that is, Equation (5) for every $n \in \mathbb{N}_0$, from which the theorem follows. $\square$

3. Bounded Solutions to Equation (5) on the Domain $\mathbb{Z} \setminus \mathbb{N}_2$

Now we consider Equation (8) on domain $\mathbb{Z} \setminus \mathbb{N}_2$. Recall that the "initial" values on the domain are again x_0 and x_1 (the values, are, in a way, the end values). To deal with a real second-order difference equation it is natural to assume that $q \neq 0$ (the case $q = 0$, $p \neq 0$, has been recently studied in [29]). In this case, the equation can be written in the following form

$$x_n + \frac{p}{q} x_{n+1} + \frac{1}{q} x_{n+2} = \frac{f_n}{q}, \quad n \leq -1, \tag{78}$$

or equivalently as follows:

$$x_{-n} - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) x_{-(n-1)} + \frac{x_{-(n-2)}}{\lambda_1 \lambda_2} = \frac{f_{-n}}{\lambda_1 \lambda_2}, \quad n \in \mathbb{N}, \tag{79}$$

where λ_1 and λ_2 are zeros of the polynomial in (12).

Equation (79) can be considered by using the change of variables $y_n = x_{-n}$ which will transform the equation into an equation of the form in (8), but with shifted indices. To avoid some technical problems due to non-symmetricity of domains $\mathbb{N}_0$ and $\mathbb{Z} \setminus \mathbb{N}_2$, instead of this, we will use the method of decomposition mentioned in Remark 2 (see, for example, [15,20]).

If we write (79) in the form

$$x_{-n} - \frac{x_{-(n-1)}}{\lambda_1} = \frac{1}{\lambda_2}\left(x_{-(n-1)} - \frac{x_{-(n-2)}}{\lambda_1}\right) + \frac{f_{-n}}{\lambda_1\lambda_2}, \quad n \in \mathbb{N}, \tag{80}$$

and multiply the following equation

$$x_{-j} - \frac{x_{-(j-1)}}{\lambda_1} = \frac{1}{\lambda_2}\left(x_{-(j-1)} - \frac{x_{-(j-2)}}{\lambda_1}\right) + \frac{f_{-j}}{\lambda_1\lambda_2} \tag{81}$$

by $\lambda_2^{-(n-j)}$, $j = \overline{1,n}$, and suming up such obtained equalities, we get

$$x_{-n} = \frac{x_{-(n-1)}}{\lambda_1} + \frac{1}{\lambda_2^n}\left(x_0 - \frac{x_1}{\lambda_1}\right) + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n}\frac{f_{-j}}{\lambda_2^{n-j}}, \tag{82}$$

for $n \in \mathbb{N}$.

Multiplying the following equality

$$x_{-i} = \frac{x_{-(i-1)}}{\lambda_1} + \frac{1}{\lambda_2^i}\left(x_0 - \frac{x_1}{\lambda_1}\right) + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{i}\frac{f_{-j}}{\lambda_2^{i-j}}, \tag{83}$$

by $\lambda_1^{-(n-i)}$, $i = \overline{1,n}$, and summing up such obtained equalities, in the case $\lambda_1 \neq \lambda_2$, that is, $p^2 \neq 4q$, we get

$$\begin{aligned}
x_{-n} =& \frac{x_0}{\lambda_1^n} + \frac{1}{\lambda_2}\left(x_0 - \frac{x_1}{\lambda_1}\right)\sum_{j=0}^{n-1}\frac{1}{\lambda_1^j\lambda_2^{n-1-j}} + \frac{1}{\lambda_1\lambda_2}\sum_{i=1}^{n}\frac{1}{\lambda_1^{n-i}}\sum_{j=1}^{i}\frac{f_{-j}}{\lambda_2^{i-j}} \\
=& x_0\frac{\lambda_1^{-(n+1)} - \lambda_2^{-(n+1)}}{\lambda_1^{-1} - \lambda_2^{-1}} - \frac{x_1}{\lambda_1\lambda_2}\frac{\lambda_1^{-n} - \lambda_2^{-n}}{\lambda_1^{-1} - \lambda_2^{-1}} + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n}\frac{f_{-j}}{\lambda_1^n}\lambda_2^j\sum_{i=j}^{n}\left(\frac{\lambda_1}{\lambda_2}\right)^i \\
=& x_0\frac{\lambda_1^{-(n+1)} - \lambda_2^{-(n+1)}}{\lambda_1^{-1} - \lambda_2^{-1}} - \frac{x_1}{\lambda_1\lambda_2}\frac{\lambda_1^{-n} - \lambda_2^{-n}}{\lambda_1^{-1} - \lambda_2^{-1}} + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n}f_{-j}\frac{\lambda_1^{j-n-1} - \lambda_2^{j-n-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \\
=& \frac{\lambda_1^{-(n+1)}(x_0 - x_1\lambda_2^{-1} + (\lambda_1\lambda_2)^{-1}\sum_{j=1}^{n}f_{-j}\lambda_1^j)}{\lambda_1^{-1} - \lambda_2^{-1}} \\
&- \frac{\lambda_2^{-(n+1)}(x_0 - x_1\lambda_1^{-1} + (\lambda_1\lambda_2)^{-1}\sum_{j=1}^{n}f_{-j}\lambda_2^j)}{\lambda_1^{-1} - \lambda_2^{-1}} \\
=& \frac{\lambda_1^{-n}(\lambda_2 x_0 - x_1 + \sum_{j=1}^{n}f_{-j}\lambda_1^{j-1}) - \lambda_2^{-n}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{n}f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1},
\end{aligned} \tag{84}$$

for $n \in \mathbb{N}$. In fact, the formula also holds for $n = 0$ and $n = 1$, which is easily verified by direct calculation and by using the convention for summations mentioned in introduction.

Now assume that $p^2 = 4q$, then (84) holds with $\lambda_1 = \lambda_2 =: \lambda$, from which along with some calculation we get

$$\begin{aligned} x_{-n} =& \frac{x_0}{\lambda^n} + \left(x_0 - \frac{x_1}{\lambda}\right)\frac{n}{\lambda^n} + \frac{1}{\lambda^{n+2}}\sum_{i=1}^{n}\lambda^i\sum_{j=1}^{i}\frac{f_{-j}}{\lambda^{i-j}} \\ =& \lambda^{-(n+1)}\left(\lambda x_0 + (\lambda x_0 - x_1)n + \sum_{j=1}^{n} f_{-j}(n-j+1)\lambda^{j-1}\right), \end{aligned}$$

for $n \in \mathbb{N}$.

The last formula can be also written in the following form

$$\begin{aligned} x_{-n} =& \lambda^{-(n+1)}\left(\lambda x_0 - \sum_{j=1}^{n} f_{-j}(j-1)\lambda^{j-1}\right) \\ &+ n\lambda^{-(n+1)}\left(\lambda x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda^{j-1}\right), \quad n \in \mathbb{N}. \end{aligned} \tag{85}$$

Note that as in the previous case Formula (85) also holds for $n = 0$ and $n = 1$, which is easily verified by direct calculation and by using the convention for summations mentioned in introduction.

As a consequence of the above consideration we have that the following result holds.

Lemma 2. *Consider Equation (78) where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, x_0 and x_1 are given complex numbers, and $(f_{-n})_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then the following statements are true:*

(a) If $p^2 \neq 4q$, then the solution to Equation (78) with initial/end values x_0 and x_1 is given by

$$x_{-n} = \frac{\lambda_1^{-n}(\lambda_2 x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda_1^{j-1}) - \lambda_2^{-n}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1}, \tag{86}$$

where $\lambda_{1,2}$ are given by (10).

(b) If $p^2 = 4q$, then the solution to Equation (78) with initial/end values x_0 and x_1 is given by

$$x_{-n} = \lambda^{-(n+1)}\left(\lambda x_0 + (\lambda x_0 - x_1)n + \sum_{j=1}^{n} f_{-j}(n-j+1)\lambda^{j-1}\right), \tag{87}$$

where $\lambda = -p/2$.

Remark 4. Note that Formula (86) can be written in the form

$$x_{-n} = \hat{x}_{-n} + x^h_{-n}, \tag{88}$$

with

$$x^h_{-n} = \frac{(\lambda_2 x_0 - x_1)\lambda_1^{-n} - (\lambda_1 x_0 - x_1)\lambda_2^{-n}}{\lambda_2 - \lambda_1}, \tag{89}$$

$$\widehat{x}_{-n} = \frac{\sum_{j=1}^{n} f_{-j}(\lambda_1^{-n+j-1} - \lambda_2^{-n+j-1})}{\lambda_2 - \lambda_1}, \quad n \in \mathbb{N}, \tag{90}$$

where x^h_{-n} is the solution to the homogeneous difference equation corresponding to (79) with initial/end values x_0 and x_1, while $\widehat{x}_{-n}$ is a particular solution to (79), in the case $p^2 \neq 4q$.

Also, Formula (87) can be written in the form in (88) with

$$x^h_{-n} = (\lambda x_0 + (\lambda x_0 - x_1)n)\lambda^{-(n+1)}, \tag{91}$$

$$\widehat{x}_{-n} = \sum_{j=1}^{n} f_{-j}(n-j+1)\lambda^{-n+j-2}, \quad n \in \mathbb{N}, \tag{92}$$

where x^h_{-n} is the solution to the homogeneous difference equation corresponding to (79) with initial/end values x_0 and x_1, while $\widehat{x}_{-n}$ is a particular solution to (79), in the case $p^2 = 4q$.

From (23), (24), (89) and (91) it follows that the solution to Equation (8) with $f_n = 0$, $n \in \mathbb{Z}$, with initial values x_0 and x_1 is given by

$$x_n = \frac{(\lambda_2 x_0 - x_1)\lambda_1^n + (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_2 - \lambda_1}, \quad n \in \mathbb{Z},$$

when $p^2 \neq 4q$, that is, by

$$x_n = (\lambda x_0 + (x_1 - \lambda x_0)n)\lambda^{n-1}, \quad n \in \mathbb{Z},$$

when $p^2 = 4q$.

From (86) and (87) similar to Corollary 2 is proved the following result. Hence, we omit the details.

Corollary 3. *Consider Equation (78), where the zeros $\lambda_{1,2}$ of polynomial (12) satisfy the condition in (46), and $(f_{-n})_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers. Then every solution to the equation on domain $\mathbb{Z} \setminus \mathbb{N}_2$ is bounded.*

Theorem 8. *Consider Equation (78), where the zeros $\lambda_{1,2}$ of polynomial (12) satisfy condition (46) and $(f_{-n})_{n\in\mathbb{N}}$ is a T-periodic sequence. Then the following statements hold.*

(a) There is a unique T-periodic solution to Equation (79) on domain $\mathbb{Z} \setminus \mathbb{N}_2$.
(b) All the solutions to Equation (79) on domain $\mathbb{Z} \setminus \mathbb{N}_2$, converge geometrically to the periodic one.

Proof. (a) If $(x_n)_{n\leq 1}$ is a T-periodic solution to Equation (78), then it must be

$$x_1 = x_{1-T} \quad \text{and} \quad x_0 = x_{-T}. \tag{93}$$

On the other hand, if (93) holds, then from (78) and since

$$f_{-n} = f_{-n-T}, \quad n \in \mathbb{N}, \tag{94}$$

we have

$$x_{-(T+1)} = -\frac{p}{q}x_{-T} - \frac{1}{q}x_{1-T} + \frac{f_{-(T+1)}}{q} = -\frac{p}{q}x_0 - \frac{1}{q}x_1 + \frac{f_{-1}}{q} = x_{-1}.$$

Using the same argument along with (78) and the method of induction it is proved that

$$x_{-mT-l} = x_{-l},$$

for every $m \in \mathbb{N}$ and $l \in \{-1, 0, 1, \ldots, T-2\}$, which shows that the solution to Equation (8) is T-periodic.

Case $p^2 \neq 4q$. This along with (86) shows that it is enough to prove that the linear system

$$\begin{aligned} x_0 =& \frac{\lambda_1^{-T}(\lambda_2 x_0 - x_1 + \sum_{j=1}^{T} f_{-j}\lambda_1^{j-1}) - \lambda_2^{-T}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{T} f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1} \\ x_1 =& \frac{\lambda_1^{1-T}(\lambda_2 x_0 - x_1 + \sum_{j=1}^{T-1} f_{-j}\lambda_1^{j-1}) - \lambda_2^{1-T}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{T-1} f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1}, \end{aligned} \tag{95}$$

has a unique solution in variables x_0 and x_1.

Note that system (95) can be written as follows

$$\begin{aligned} &(\lambda_1\lambda_2^{-T} - \lambda_2\lambda_1^{-T} + \lambda_2 - \lambda_1)x_0 + (\lambda_1^{-T} - \lambda_2^{-T})x_1 = \lambda_1^{-T}S_5 - \lambda_2^{-T}S_6 \\ &(\lambda_1\lambda_2^{1-T} - \lambda_2\lambda_1^{1-T})x_0 + (\lambda_1^{1-T} - \lambda_2^{1-T} + \lambda_2 - \lambda_1)x_1 = \lambda_1^{1-T}S_5 - \lambda_2^{1-T}S_6, \end{aligned} \tag{96}$$

where

$$S_5 := \sum_{j=1}^{T} f_{-j}\lambda_1^{j-1} \quad \text{and} \quad S_6 := \sum_{j=1}^{T} f_{-j}\lambda_2^{j-1}.$$

By some calculation is obtained

$$\begin{aligned} \Delta =& \begin{vmatrix} \lambda_1\lambda_2^{-T} - \lambda_2\lambda_1^{-T} + \lambda_2 - \lambda_1 & \lambda_1^{-T} - \lambda_2^{-T} \\ \lambda_1\lambda_2^{1-T} - \lambda_2\lambda_1^{1-T} & \lambda_1^{1-T} - \lambda_2^{1-T} + \lambda_2 - \lambda_1 \end{vmatrix} \\ =& (\lambda_1 - \lambda_2)^2(\lambda_1^{-T} - 1)(\lambda_2^{-T} - 1) \neq 0, \end{aligned} \tag{97}$$

due to (46) and $\lambda_1 \neq \lambda_2$.

Also, we have

$$\begin{aligned} \Delta_1 =& \begin{vmatrix} \lambda_1^{-T}S_5 - \lambda_2^{-T}S_6 & \lambda_1^{-T} - \lambda_2^{-T} \\ \lambda_1^{1-T}S_5 - \lambda_2^{1-T}S_6 & \lambda_1^{1-T} - \lambda_2^{1-T} + \lambda_2 - \lambda_1 \end{vmatrix} \\ =& (\lambda_2 - \lambda_1)(\lambda_2^{-T}(\lambda_1^{-T} - 1)S_6 - \lambda_1^{-T}(\lambda_2^{-T} - 1)S_5), \end{aligned} \tag{98}$$

and

$$\begin{aligned} \Delta_2 =& \begin{vmatrix} \lambda_1\lambda_2^{-T} - \lambda_2\lambda_1^{-T} + \lambda_2 - \lambda_1 & \lambda_1^{-T}S_5 - \lambda_2^{-T}S_6 \\ \lambda_1\lambda_2^{1-T} - \lambda_2\lambda_1^{1-T} & \lambda_1^{1-T}S_5 - \lambda_2^{1-T}S_6 \end{vmatrix} \\ =& (\lambda_2 - \lambda_1)(\lambda_2^{1-T}(\lambda_1^{-T} - 1)S_6 - \lambda_1^{1-T}(\lambda_2^{-T} - 1)S_5) \end{aligned} \tag{99}$$

From (97)–(99), it follows that

$$x_0 = \frac{\lambda_2^{-T}(\lambda_1^{-T} - 1)S_6 - \lambda_1^{-T}(\lambda_2^{-T} - 1)S_5}{(\lambda_2 - \lambda_1)(\lambda_1^{-T} - 1)(\lambda_2^{-T} - 1)} \tag{100}$$

and

$$x_1 = \frac{\lambda_2^{1-T}(\lambda_1^{-T} - 1)S_6 - \lambda_1^{1-T}(\lambda_2^{-T} - 1)S_5}{(\lambda_2 - \lambda_1)(\lambda_1^{-T} - 1)(\lambda_2^{-T} - 1)}, \tag{101}$$

are the initial values for which is obtained the T-periodic solution to Equation (78) in this case.

Case $p^2 = 4q$. Using (87), we see that (93) becomes the linear system

$$\begin{aligned} x_0 =& \lambda^{-(T+1)}\left(\lambda x_0 + (\lambda x_0 - x_1)T + \sum_{j=1}^{T} f_{-j}(T - j + 1)\lambda^{j-1}\right) \\ x_1 =& \lambda^{-T}\left(\lambda x_0 + (\lambda x_0 - x_1)(T - 1) + \sum_{j=1}^{T-1} f_{-j}(T - j)\lambda^{j-1}\right). \end{aligned} \tag{102}$$

System (102) can be written as follows

$$\begin{aligned}&(\lambda^{T+1}-\lambda(T+1))x_0+Tx_1=S_7\\&\lambda Tx_0+(1-T-\lambda^T)x_1=S_8,\end{aligned}\tag{103}$$

where

$$S_7:=\sum_{j=1}^{T}f_{-j}(T-j+1)\lambda^{j-1}\quad\text{and}\quad S_8:=-\sum_{j=1}^{T-1}f_{-j}(T-j)\lambda^{j-1}.$$

After some calculation it is shown that the determinant of system (103) is

$$\Delta=\begin{vmatrix}\lambda^{T+1}-\lambda(T+1) & T\\ \lambda T & 1-T-\lambda^T\end{vmatrix}=-\lambda(\lambda^T-1)^2\neq 0,\tag{104}$$

due to (46). Also, we have

$$\Delta_1=\begin{vmatrix}S_7 & T\\ S_8 & 1-T-\lambda^T\end{vmatrix}=(1-T-\lambda^T)S_7-TS_8,\tag{105}$$

and

$$\Delta_2=\begin{vmatrix}\lambda^{T+1}-\lambda(T+1) & S_7\\ T\lambda & S_8\end{vmatrix}=(\lambda^{T+1}-\lambda(T+1))S_8-T\lambda S_7.\tag{106}$$

From (104)–(106), it follows that

$$x_0=\frac{(\lambda^T+T-1)S_7+TS_8}{\lambda(\lambda^T-1)^2}$$

and

$$x_1=\frac{TS_7-(\lambda^T-T-1)S_8}{(\lambda^T-1)^2},$$

are the initial values for which is obtained the T-periodic solution to Equation (78) in this case.

(b) If $(\tilde{x}_n)_{n\le 1}$ is the T-periodic solution to Equation (78) and $(x_n)_{n\le 1}$ is any solution to the equation, then if $p^2\neq 4q$, from (86) we have

$$\begin{aligned}|\tilde{x}_{-n}-x_{-n}|=&\Bigg|\left(\frac{\lambda_1^{-n}(\lambda_2\tilde{x}_0-\tilde{x}_1+\sum_{j=1}^{n}f_{-j}\lambda_1^{j-1})-\lambda_2^{-n}(\lambda_1\tilde{x}_0-\tilde{x}_1+\sum_{j=1}^{n}f_{-j}\lambda_2^{j-1})}{\lambda_2-\lambda_1}\right)\\&-\left(\frac{\lambda_1^{-n}(\lambda_2x_0-x_1+\sum_{j=1}^{n}f_{-j}\lambda_1^{j-1})-\lambda_2^{-n}(\lambda_1x_0-x_1+\sum_{j=1}^{n}f_{-j}\lambda_2^{j-1})}{\lambda_2-\lambda_1}\right)\Bigg|\\\le&\frac{(|\lambda_2||\tilde{x}_0-x_0|+|\tilde{x}_1-x_1|)|\lambda_1|^{-n}+(|\lambda_1||\tilde{x}_0-x_0|+|\tilde{x}_1-x_1|)|\lambda_2|^{-n}}{|\lambda_2-\lambda_1|},\end{aligned}$$

for $n\in\mathbb{N}_0$, from which the statement follows in this case.

If $p^2 = 4q$, then from (87) we have

$$\begin{aligned}|\tilde{x}_{-n} - x_{-n}| =& \Bigg|\lambda^{-(n+1)}\left(\lambda\tilde{x}_0 + (\lambda\tilde{x}_0 - \tilde{x}_1)n + \sum_{j=1}^{n} f_{-j}(n-j+1)\lambda^{j-1}\right) \\ &- \lambda^{-(n+1)}\left(\lambda x_0 + (\lambda x_0 - x_1)n + \sum_{j=1}^{n} f_{-j}(n-j+1)\lambda^{j-1}\right)\Bigg| \\ \le& |\tilde{x}_0 - x_0||\lambda|^{-n} + (|\tilde{x}_1 - x_1| + |\lambda||\tilde{x}_0 - x_0|)n|\lambda|^{-(n+1)} \\ \le& M_1\left(\frac{1+|\lambda|}{2|\lambda|}\right)^n,\end{aligned}$$

for $n \in \mathbb{N}_0$, for some $M_1 = M_1(x_0, x_1, \tilde{x}_0, \tilde{x}_1, \lambda)$, from which the statement follows in this case. □

Theorem 9. *Consider Equation (78), where the zeros $\lambda_{1,2}$ of polynomial (12) satisfy condition (31), and $(f_{-n})_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers. Then, there is a unique bounded solution to the equation on domain $\mathbb{Z} \setminus \mathbb{N}_2$.*

Proof. If $p^2 \neq 4q$, then from (31) and (86) we see that a solution to Equation (79) is bounded if and only if

$$\lambda_2 x_0 - x_1 = -\sum_{j=1}^{\infty} f_{-j}\lambda_1^{j-1} \quad \text{and} \quad \lambda_1 x_0 - x_1 = -\sum_{j=1}^{\infty} f_{-j}\lambda_2^{j-1}, \tag{107}$$

(note that both sums are finite due to (31) and the boundedness of $(f_{-n})_{n\in\mathbb{N}}$), from which it follows that the solution is

$$x_{-n} = \frac{1}{\lambda_2 - \lambda_1}\sum_{j=n+1}^{\infty} f_{-j}(\lambda_2^{j-n-1} - \lambda_1^{j-n-1}). \tag{108}$$

By direct calculation is shown that (108) is a solution to Equation (78). Since

$$|x_{-n}| \le \|f\|_\infty \frac{2}{|\lambda_2 - \lambda_1|}\sum_{j=n+1}^{\infty} M^{j-n-1} = \frac{2\|f\|_\infty}{|\lambda_2 - \lambda_1|(1-M)},$$

for $n \in \mathbb{N}$, it follows that the solution is bounded. Since by (107), x_0 and x_1 are uniquely defined, the bounded solution is unique.

If $p^2 = 4q$, then from (31) and (87), similar to the corresponding part of the proof of Theorem 2 is obtained that Equation (78) has a unique bounded solution if and only if

$$x_0 = \sum_{j=2}^{\infty} f_{-j}(j-1)\lambda^{j-2} \quad \text{and} \quad x_1 = \sum_{j=1}^{\infty} f_{-j}j\lambda^{j-1}, \tag{109}$$

(note that both sums are also finite due to (31) and the boundedness of $(f_{-n})_{n\in\mathbb{N}}$), from which it follows that the solution is

$$x_{-n} = \lambda^{-(n+1)}\sum_{j=n+1}^{\infty} f_{-j}(j-n-1)\lambda^{j-1}. \tag{110}$$

By direct calculation is shown that (110) is a solution to Equation (79). Since

$$|x_{-n}| \le \|f\|_\infty \sum_{j=n+2}^{\infty}(j-n-1)M^{j-n-2} = \frac{\|f\|_\infty}{(1-M)^2},$$

for $n \in \mathbb{N}$, it follows that the solution is bounded. Since due to (109), x_0 and x_1 are uniquely defined, the bounded solution is unique. □

Theorem 10. *Consider Equation (78), where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, the zeros $\lambda_{1,2}$ of the polynomial (12) are distinct and satisfy condition (31) and $(f_{-n})_{n\in\mathbb{N}}$ is a T-periodic sequence. Then, the unique bounded solution to Equation (79) is T-periodic.*

Proof. Let $(\tilde{x}_n)_{n\leq 1}$ be the unique bounded solution to Equation (79). From Theorem 9 we see that the unique bounded solution to Equation (79) is given by (108) if $p^2 \neq 4q$, and it is given by (110) if $p^2 = 4q$.

If $p^2 \neq 4q$, then we have

$$\begin{aligned}\tilde{x}_{-(n+T)} =& \frac{1}{\lambda_2-\lambda_1}\sum_{j=n+T+1}^{\infty} f_{-j}(\lambda_2^{j-n-T-1}-\lambda_1^{j-n-T-1})\\ =& \frac{1}{\lambda_2-\lambda_1}\sum_{k=n+1}^{\infty} f_{-k-T}(\lambda_2^{k-n-1}-\lambda_1^{k-n-1})\\ =& \frac{1}{\lambda_2-\lambda_1}\sum_{k=n+1}^{\infty} f_{-k}(\lambda_2^{k-n-1}-\lambda_1^{k-n-1}) = \tilde{x}_{-n},\end{aligned} \tag{111}$$

for $n \geq -1$, while if $p^2 = 4q$, we have

$$\begin{aligned}\tilde{x}_{-(n+T)} =& \lambda^{-(n+T+1)}\sum_{j=n+T+1}^{\infty} f_{-j}(j-n-T-1)\lambda^{j-1}\\ =& \lambda^{-(n+T+1)}\sum_{k=n+1}^{\infty} f_{-k-T}(k-n-1)\lambda^{k+T-1}\\ =& \lambda^{-(n+1)}\sum_{k=n+1}^{\infty} f_{-k}(k-n-1)\lambda^{k-1} = \tilde{x}_{-n},\end{aligned} \tag{112}$$

for $n \geq -1$. From (111) and (112) the result follows. □

Theorem 11. *Consider Equation (78), where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, the zeros $\lambda_{1,2}$ of polynomial (12) satisfy condition (59), and $(f_{-n})_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers. Then, the following statements are true.*

(a) If $|\lambda_1| < 1 < |\lambda_2|$, then a solution to Equation (78) is bounded if and only if

$$x_1 - \lambda_2 x_0 = \sum_{j=1}^{\infty} f_{-j}\lambda_1^{j-1}. \tag{113}$$

(b) If $|\lambda_2| < 1 < |\lambda_1|$, then a solution to Equation (78) is bounded if and only if

$$x_1 - \lambda_1 x_0 = \sum_{j=1}^{\infty} f_{-j}\lambda_2^{j-1}. \tag{114}$$

Proof. (a) Since $|\lambda_2| > 1$, we have that

$$\begin{aligned}\left|\lambda_2^{-n}\left(\lambda_1 x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda_2^{j-1}\right)\right| \leq& (|\lambda_1||x_0|+|x_1|)|\lambda_2|^{-n} + \|f\|_\infty \sum_{j=1}^{n}|\lambda_2|^{-n+j-1}\\ \leq& |\lambda_1||x_0|+|x_1| + \frac{\|f\|_\infty}{|\lambda_2|-1} < \infty.\end{aligned} \tag{115}$$

From (86), (115), and since $|\lambda_1| < 1$, it follows that the boundedness of a solution x_n to Equation (79) implies (113).

Now assume that (113) holds. Then from (86) and (113) it follows that the solution in the case must be

$$x_{-n} = \frac{\lambda_1^{-n}(-\sum_{j=n+1}^{\infty} f_{-j}\lambda_1^{j-1}) - \lambda_2^{-n}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1}. \tag{116}$$

Since $|\lambda_1| < 1$, we have

$$\left|\lambda_1^{-n}\left(-\sum_{j=n+1}^{\infty} f_{-j}\lambda_1^{j-1}\right)\right| \le \|f\|_\infty \sum_{k=n+1}^{\infty} |\lambda_1|^{-n+j-1} = \frac{\|f\|_\infty}{1-|\lambda_1|} < \infty. \tag{117}$$

Using (115) and (117) in (116), we have

$$|x_{-n}| \le \frac{1}{|\lambda_2 - \lambda_1|}\left(|\lambda_1||x_0| + |x_1| + \frac{\|f\|_\infty}{|\lambda_2| - 1} + \frac{\|f\|_\infty}{1-|\lambda_1|}\right),$$

from which it follows that the solution to Equation (79) is bounded.

(b) The proof of the statement is similar/dual to the one in (a). Hence, it is omitted. □

Theorem 12. *Consider Equation (78), where $p \in \mathbb{C}$, $q \in \mathbb{C} \setminus \{0\}$, the zeros $\lambda_{1,2}$ of the polynomial (12) satisfy condition (59) and $(f_{-n})_{n\in\mathbb{N}}$ is a T-periodic sequence. Then, the following statements are true.*

(a) There is a unique T-periodic solution to Equation (78) on domain $\mathbb{Z} \setminus \mathbb{N}_2$.
(b) All bounded solutions to Equation (78) on domain $\mathbb{Z} \setminus \mathbb{N}_2$, converge geometrically to the periodic one.

Proof. (a) We may assume that the condition holds $|\lambda_1| < 1 < |\lambda_2|$, since the other case is essentially the same. By Theorem 11, we see that a solution to Equation (78) is bounded if and only if (113) holds, and that bounded solutions to Equation (78) have the form in (116). If a solution to the equation is T-periodic, then it must be $x_1 = x_{1-T}$, that is,

$$x_1 = \frac{\lambda_1^{1-T}(-\sum_{j=T}^{\infty} f_{-j}\lambda_1^{j-1}) - \lambda_2^{1-T}(\lambda_1 x_0 - x_1 + \sum_{j=1}^{T-1} f_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1},$$

from which, along with (116) and by some calculation is obtained

$$x_1 = \frac{\lambda_1(\lambda_2^{-T} - \lambda_1^{-T})\sum_{j=1}^{\infty} f_{-j}\lambda_1^{j-1} + \sum_{j=1}^{T-1} f_{-j}(\lambda_1^{j-T} - \lambda_2^{j-T})}{(\lambda_2 - \lambda_1)(1-\lambda_2^{-T})}. \tag{118}$$

By using equalities (113) and (118) in (116) and after some calculation it is shown that for such chosen x_1 is obtained a T-periodic solution to Equation (78). Since initial value x_1 is uniquely defined by (118), and consequently by (113) initial value x_0 is uniquely defined, the T-periodic solution is unique too, as claimed.

(b) If $(\tilde{x}_n)_{n\le 1}$ is the T-periodic solution to Equation (78) and $(x_n)_{n\le 1}$ is any bounded solution to the equation, then from (116) and some simple estimates, we have

$$|\tilde{x}_{-n} - x_{-n}| = \frac{1}{|\lambda_2-\lambda_1|}\Bigg|\lambda_1^{-n}\Bigg(-\sum_{j=n+1}^{\infty} f_{-j}\lambda_1^{j-1}\Bigg) - \lambda_2^{-n}\Bigg(\lambda_1\tilde{x}_0 - \tilde{x}_1 + \sum_{j=1}^{n} f_{-j}\lambda_2^{j-1}\Bigg)$$
$$- \lambda_1^{-n}\Bigg(-\sum_{j=n+1}^{\infty} f_{-j}\lambda_1^{j-1}\Bigg) - \lambda_2^{-n}\Bigg(\lambda_1 x_0 - x_1 + \sum_{j=1}^{n} f_{-j}\lambda_2^{j-1}\Bigg)\Bigg|$$
$$\le \frac{|\lambda_1||\tilde{x}_0 - x_0| + |\tilde{x}_1 - x_1|}{|\lambda_2-\lambda_1||\lambda_2|^n},$$

from which the statement easily follows. □

Remark 5. Since the sequence $(f_{-n})_{n\in\mathbb{N}}$ in Theorem 12 is T-periodic, then the expression for x_1 in (118) can be written in a somewhat nicer way. Namely, since the series $\sum_{j=1}^{\infty} f_{-j}\lambda_1^{j-1}$ is absolutely convergent, we have

$$\begin{aligned}\sum_{j=mT+1}^{\infty} f_{-j}\lambda_1^{j-1} &= \sum_{k=m}^{\infty}\sum_{j=kT+1}^{(k+1)T} f_{-j}\lambda_1^{j-1} = \sum_{k=m}^{\infty}\sum_{i=1}^{T} f_{-i-kT}\lambda_1^{kT+i-1}\\ &= \sum_{k=m}^{\infty}\lambda_1^{kT}\sum_{i=1}^{T} f_{-i}\lambda_1^{i-1} = \frac{\lambda_1^{mT}}{1-\lambda_1^{T}}\sum_{i=1}^{T} f_{-i}\lambda_1^{i-1},\end{aligned} \tag{119}$$

for every $m \in \mathbb{N}_0$.

Using (119) in (118) for $m = 0$ and after some calculation it follows that

$$x_1 = \frac{(\lambda_1^{-T}-1)\sum_{j=1}^{T} f_{-j}\lambda_2^{j-T} - (\lambda_2^{-T}-1)\sum_{j=1}^{T} f_{-j}\lambda_1^{j-T}}{(\lambda_2-\lambda_1)(\lambda_1^{-T}-1)(\lambda_2^{-T}-1)}.$$

From this, (113) and some calculation we get

$$x_0 = \frac{\lambda_1(\lambda_1^{-T}-1)\sum_{j=1}^{T} f_{-j}\lambda_2^{j-T} - \lambda_2(\lambda_2^{-T}-1)\sum_{j=1}^{T} f_{-j}\lambda_1^{j-T}}{\lambda_1\lambda_2(\lambda_2-\lambda_1)(\lambda_1^{-T}-1)(\lambda_2^{-T}-1)}.$$

Note that the initial values match with those in (100) and (101).

The following result considers Equation (8) on domain $\mathbb{Z}$. The result is essentially a consequence of some of above mentioned results on domains $\mathbb{N}_0$ and $\mathbb{Z}\setminus\mathbb{N}_2$, so it is formulated as a corollary, although it is, in fact, an important results concerning the equation.

Corollary 4. *Consider Equation* (8) *where $p \in \mathbb{C}$ and $q \in \mathbb{C}\setminus\{0\}$. Let $\lambda_{1,2}$ be the zeros of polynomial* (12)*, $(f_n)_{n\in\mathbb{Z}}$ be a bounded sequence of complex numbers, and one of the conditions* (31)*,* (46)*,* (59) *holds. Then, the equation has a unique bounded solution on $\mathbb{Z}$.*

Proof. First, assume that (46) holds, then by Theorem 2 there is a unique bounded solution to Equation (8) on $\mathbb{N}_0$, while by Corollary 3 all the solutions to the equation on $\mathbb{Z}\setminus\mathbb{N}_2$ are bounded, from which the result follows in the case. If (31) holds, then by Corollary 2 we have that all the solutions to the equation on $\mathbb{N}_0$ are bounded, while by Theorem 9 there is a unique bounded solution to the equation on $\mathbb{Z}\setminus\mathbb{N}_2$, from which the result follows in the case. Now, assume that (59) holds. Then, if $|\lambda_1| < 1 < |\lambda_2|$ by Theorems 4 (a) and 11 (a), it follows that a solution to Equation (8) on $\mathbb{Z}$ is bounded if and only if (60) and (113) hold, which is a two-dimensional linear system in x_0 and x_1 with the determinant different from zero, from which it follows that there is a unique pair of initial values x_0, x_1, such that the solution to Equation (8) is bounded. If $|\lambda_2| < 1 < |\lambda_1|$, then by Theorems 4 (b) and 11 (b), it follows that a solution to (8) on $\mathbb{Z}$ is bounded if and only if (61) and (114) hold, which is also a two-dimensional linear system in x_0 and x_1 with the determinant different from zero, from which it follows that there is a unique pair of initial values x_0, x_1, such that the solution to Equation (8) is bounded, in this case, finishing the proof of the theorem. □

The following two results are the main results in this section and deal with the general Equation (5) on domain $\mathbb{Z} \setminus \mathbb{N}_2$. They correspond to Theorems 6 and 7, and are proved similarly. We present their proofs for the completeness and benefit of the reader.

Theorem 13. *Assume that $(p_{-n})_{n\in\mathbb{N}}$ and $(q_{-n})_{n\in\mathbb{N}}$ are sequences of complex numbers such that*

$$\hat{q}_3 := \sup_{n\in\mathbb{N}} \frac{|p_{-n}/q_{-n} + r_1^{-1} + r_2^{-1}| + |(r_1r_2)^{-1} - 1/q_{-n}|}{|r_1 - r_2|(1 - r_M)} < \frac{1}{2}, \tag{120}$$

for some nonzero numbers r_1 and r_2, such that $r_M := \max\{|r_1|, |r_2|\} < 1$,

$$\inf_{n\in\mathbb{N}} |q_{-n}| > \delta > 0, \tag{121}$$

and $(f_{-n})_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers. Then, Equation (5) has a unique bounded solution on $\mathbb{Z} \setminus \mathbb{N}_2$.

Proof. Write Equation (5) in the following form

$$\begin{aligned} & x_{-n} - (r_1^{-1} + r_2^{-1})x_{-(n-1)} + (r_1r_2)^{-1}x_{-(n-2)} \\ & = -(p_{-n}/q_{-n} + r_1^{-1} + r_2^{-1})x_{-(n-1)} + ((r_1r_2)^{-1} - 1/q_{-n})x_{-(n-2)} + f_{-n}/q_{-n}, \end{aligned} \tag{122}$$

for $n \in \mathbb{N}$.

Let A be the following operator defined on the class of all sequences

$$A(u) = \left(\sum_{j=n+1}^{\infty} \frac{r_2^{j-n-1} - r_1^{j-n-1}}{r_2 - r_1} \left(\left(\frac{1}{r_1r_2} - \frac{1}{q_{-j}} \right) u_{2-j} - \left(\frac{p_{-j}}{q_{-j}} + r_1^{-1} + r_2^{-1} \right) u_{1-j} + \frac{f_{-j}}{q_{-j}} \right) \right)_{n\le 1}. \tag{123}$$

If $u \in l^\infty$, then from (121), (123), and some simple inequalities, we have

$$\begin{aligned} \|A(u)\|_\infty &= \sup_{n\in\mathbb{N}_0} \left| \sum_{j=n+1}^{\infty} \frac{r_2^{j-n-1} - r_1^{j-n-1}}{r_2 - r_1} \left(\left(\frac{1}{r_1r_2} - \frac{1}{q_{-j}} \right) u_{2-j} - \left(\frac{p_{-j}}{q_{-j}} + r_1^{-1} + r_2^{-1} \right) u_{1-j} + \frac{f_{-j}}{q_{-j}} \right) \right| \\ &\le 2 \sup_{n\in\mathbb{N}_0} \sum_{j=n+1}^{\infty} r_M^{j-n-1} \frac{\|u\|_\infty(|(r_1r_2)^{-1} - q_{-j}^{-1}| + |p_{-j}/q_{-j} + r_1^{-1} + r_2^{-1}|) + \|f\|_\infty\delta^{-1}}{|r_2 - r_1|} \\ &\le \frac{\|u\|_\infty|r_1 - r_2|(1 - r_M) + 2\|f\|_\infty\delta^{-1}}{|r_2 - r_1|(1 - r_M)} < \infty, \end{aligned}$$

which means that operator A maps the Banach space l^∞ into itself.

On the other hand, for every $u, v \in l^\infty$ we have

$$\begin{aligned} & \|A(u) - A(v)\|_\infty \\ &= \sup_{n\in\mathbb{N}_0} \left| \sum_{j=n+1}^{\infty} \frac{r_2^{j-n-1} - r_1^{j-n-1}}{r_2 - r_1} \left(\left(\frac{1}{r_1r_2} - \frac{1}{q_{-j}} \right) (u_{2-j} - v_{2-j}) - \left(\frac{p_{-j}}{q_{-j}} + r_1^{-1} + r_2^{-1} \right) (u_{1-j} - v_{1-j}) \right) \right| \\ &\le \|u - v\|_\infty 2 \sup_{n\in\mathbb{N}} \sum_{j=n+1}^{\infty} r_M^{j-n-1} \frac{|(r_1r_2)^{-1} - 1/q_{-j}| + |p_{-j}/q_{-j} + r_1^{-1} + r_2^{-1}|}{|r_2 - r_1|} \\ &\le 2\hat{q}_3\|u - v\|_\infty, \end{aligned}$$

from which along with (120) it follows that $A : l^\infty \to l^\infty$ is a contraction.

By the Banach fixed point theorem we get that the operator has a unique fixed point, say $x^* = (x_n^*)_{n\le 1} \in l^\infty$, that is, $A(x^*) = x^*$ or equivalently

$$x_{-n}^* = \sum_{j=n+1}^{\infty} \frac{r_2^{j-n-1} - r_1^{j-n-1}}{r_2 - r_1} \left(\left(\frac{1}{r_1 r_2} - \frac{1}{q_{-j}} \right) x_{2-j}^* - \left(\frac{p_{-j}}{q_{-j}} + r_1^{-1} + r_2^{-1} \right) x_{1-j}^* + \frac{f_{-j}}{q_{-j}} \right), \tag{124}$$

for $n \ge -1$.

A direct calculation shows that this bounded sequence satisfies difference Equation (5), from which the theorem follows. □

Theorem 14. *Assume that $(p_{-n})_{n\in\mathbb{N}}$ and $(q_{-n})_{n\in\mathbb{N}}$ are sequences of complex numbers such that*

$$\hat{q} := \sup_{n\in\mathbb{N}} \frac{|p_{-n}/q_{-n} + 2r^{-1}| + |r^{-2} - 1/q_{-n}|}{(1-r)^2} < 1, \tag{125}$$

for some number $r < 1$, that condition (121) holds, and $(f_{-n})_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers. Then, Equation (5) has a unique bounded solution on $\mathbb{Z} \setminus \mathbb{N}_2$.

Proof. Write Equation (5) in the following form

$$x_{-n} - \frac{2}{r} x_{-(n-1)} + \frac{1}{r^2} x_{-(n-2)} = -\left(\frac{p_{-n}}{q_{-n}} + \frac{2}{r} \right) x_{-(n-1)} + \left(\frac{1}{r^2} - \frac{1}{q_{-n}} \right) x_{-(n-2)} + \frac{f_{-n}}{q_{-n}}, \tag{126}$$

for $n \in \mathbb{N}$.

Let A be the following operator defined on the class of all sequences

$$A(u) = \left(\sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left(\frac{1}{r^2} - \frac{1}{q_{-j}} \right) u_{2-j} - \left(\frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right) u_{1-j} + \frac{f_{-j}}{q_{-j}} \right) \right)_{n\in\mathbb{N}_0}. \tag{127}$$

If $u \in l^\infty$, then from (127) it follows that

$$\begin{aligned} \|A(u)\|_\infty &= \sup_{n\in\mathbb{N}_0} \left| \sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left(\frac{1}{r^2} - \frac{1}{q_{-j}} \right) u_{2-j} - \left(\frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right) u_{1-j} + \frac{f_{-j}}{q_{-j}} \right) \right| \\ &\le \sup_{n\in\mathbb{N}_0} \sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left| \frac{1}{r^2} - \frac{1}{q_{-j}} \right| |u_{2-j}| + \left| \frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right| |u_{1-j}| + \frac{|f_{-j}|}{\delta} \right) \\ &\le \frac{(1-r)^2 \|u\|_\infty + \|f\|_\infty \delta^{-1}}{(1-r)^2} < \infty, \end{aligned}$$

which means that operator A maps the Banach space l^∞ into itself.

On the other hand, for every $u, v \in l^\infty$ we have

$$\begin{aligned} &\|A(u) - A(v)\|_\infty \\ &= \sup_{n\in\mathbb{N}_0} \left| \sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left(\frac{1}{r^2} - \frac{1}{q_{-j}} \right) (u_{2-j} - v_{2-j}) - \left(\frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right) (u_{1-j} - v_{1-j}) \right) \right| \\ &\le \|u - v\|_\infty \sup_{n\in\mathbb{N}_0} \sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left| \frac{1}{r^2} - \frac{1}{q_{-j}} \right| + \left| \frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right| \right) \\ &\le \hat{q} \|u - v\|_\infty. \end{aligned} \tag{128}$$

From (128) and condition (125) it follows that the operator $A : l^\infty \to l^\infty$ is a contraction.

By the Banach fixed point theorem we get that the operator has a unique fixed point, say $x^* = (x^*_n)_{n\le 1} \in l^\infty$, that is, $A(x^*) = x^*$ or equivalently

$$x^*_{-n} = \sum_{j=n+2}^{\infty} (j-n-1) r^{j-n-2} \left(\left(\frac{1}{r^2} - \frac{1}{q_{-j}} \right) x^*_{2-j} - \left(\frac{p_{-j}}{q_{-j}} + \frac{2}{r} \right) x^*_{1-j} + \frac{f_{-j}}{q_{-j}} \right), \tag{129}$$

for $n \ge -1$.

A direct calculation shows that this bounded sequence satisfies difference Equation (5), from which the theorem follows. □

4. Discussion

Motivated by some recent investigations of ours here we present how some solvability methods along with the contraction mapping principle can be employed in studying of a nonhomogeneous linear second-order difference equation, with a special attention on the existence of bounded solutions and their relationship with other solutions. Beside the study of the equation on the usual domain $\mathbb{N}_0$, it is also studied on the neglected ones $\mathbb{Z} \setminus \mathbb{N}_2$ and $\mathbb{Z}$. A natural problem is to develop the methods and ideas in the paper so that they can be applied in the study of other related difference equations of second as well as of higher orders on all these domains, which will be one of our further directions in the investigation of difference equations and systems.

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