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Path Embeddings with Prescribed Edge in the Balanced Hypercube Network

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Abstract: The balanced hypercube network, which is a novel interconnection network for parallel computation and data processing, is a newly-invented variant of the hypercube. The particular feature of the balanced hypercube is that each processor has its own backup processor and they are connected to the same neighbors. A Hamiltonian bipartite graph with bipartition $V_0 \cup V_1$ is Hamiltonian laceable if there exists a path between any two vertices $x \in V_0$ and $y \in V_1$. It is known that each edge is on a Hamiltonian cycle of the balanced hypercube. In this paper, we prove that, for an arbitrary edge e in the balanced hypercube, there exists a Hamiltonian path between any two vertices x and y in different partite sets passing through e with $e \neq xy$. This result improves some known results.

Keywords: interconnection network; balanced hypercube; Hamiltonian path; passing prescribed edge; data processing

1. Introduction

Interconnection networks play an essential role in the performance of parallel and distributed systems. In the event of practice, large multi-processor systems can also be adopted as tools to address complex management and big data problems. It is well-known that an interconnection network is generally modeled by an undirected graph, in which processors are represented by vertices and communication links between them are represented by edges. The hypercube network is recognized as one of the most popular interconnection networks, and it has gained great attention and recognition from researchers both in graph theory and computer science. Nevertheless, the hypercube also has some shortcomings. For example, its diameter is large. Therefore, many variants of the hypercube have been put forward [1–10] to improve performance of the hypercube in some aspects. Among these variants, the balanced hypercube has the following special properties: each vertex of the balanced has a backup (matching) vertex and they have the same neighborhood. Therefore, the backup vertex can undertake tasks that originally run on a faulty vertex. It has been proved that the diameter of an odd-dimensional balanced hypercube BH_n is $2n-1$ [10], which is smaller than that of the hypercube Q_{2n}.

With regard to the special properties discussed above, the balanced hypercube has been investigated by many researchers. Huang and Wu [11] studied the problem of resource placement of the balanced hypercube. Xu et al. [12] showed that the balanced hypercube is edge-pancyclic and Hamiltonian laceable. It is found that the balanced hypercube is bipanconnected for all $n \geq 1$ by Yang [13]. Huang et al. [14] discussed area efficient layout problems of the balanced hypercube. Yang [15] determined super (edge) connectivity of the balanced hypercube. Lü et al. studied (conditional) matching preclusion, hyper-Hamiltonian laceability, matching extendability and extra connectivity of the balanced hypercube in [16–19], respectively. Some symmetric properties of the balanced hypercube are presented in [20,21]. As stated above, the balanced hypercube possesses some desirable properties that the hypercube does not have, so it is interesting to explore other favorable properties that the balanced hypercube may have.

Since parallel applications such as image and signal processing are originally designed on array and ring architectures, it is important to have path and cycle embeddings in a network. Especially, Hamiltonian path and cycle embeddings and other properties of famous networks are extensively studied by many authors [12,13,22–26]. Xu et al. [12] proved that each edge of the balanced hypercube is on a cycle of even length from 4 to 4^n, that is, the balanced hypercube is *edge-bipancyclic*. They also showed that the balanced hypercube is Hamiltonian laceable for all integers $n \geq 1$. Recently, Lü et al. [17] further obtained that the balanced hypercube is hyper-Hamiltonian laceable for all integers $n \geq 1$.

The rest of this paper is organized as follows. Some necessary definitions are presented as preliminaries in Section 2. The main result of this paper is shown in Section 3. Finally, conclusions are given in Section 4.

2. Preliminaries

Let $G = (V, E)$ be a simple undirected graph, where V is a vertex-set of G and E is an edge-set of G. A *path* P from v_0 to v_n is a sequence of vertices $v_0 v_1 \cdots v_n$ from v_0 to v_n such that every pair of consecutive vertices are adjacent and all vertices are distinct except for v_0 and v_n. We also denote the path $P = v_0 v_1 \cdots v_n$ by $\langle v_0, P, v_n \rangle$. The *length* of a path P is the number of edges in P, denoted by $l(P)$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A graph is *bipartite* if its vertex-set can be partitioned into two subsets V_0 and V_1 such that each edge has its ends in different subsets. A graph is *Hamiltonian* if it possesses a spanning cycle. A graph is *Hamiltonian connected* if there exists a Hamiltonian path joining any two vertices of it. Obviously, any bipartite graph is not Hamiltonian connected. Simmons [27] proposed Hamiltonian laceability of bipatite graphs: a bipartite graph $G = (V_0 \cup V_1, E)$ is *Hamiltonian laceable* if there exists a Hamiltonian path between any two vertices x and y in different partite sets of G. A graph G is *hyper-Hamiltonian laceable* if it is Hamiltonian laceable and, for any vertex $v \in V_i (i \in \{0, 1\})$, there exists a Hamiltonian path in $G - v$ between any pair of vertices in V_{1-i}. For the graph definitions and notations not mentioned here, we refer the readers to [28,29].

Wu and Huang [10] gave the following definition of BH_n as follows.

Definition 1. *An n-dimensional balanced hypercube, denoted by BH_n, consists of 4^n vertices labelled by $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for each $0 \leq i \leq n-1$. Any vertex $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$ with $1 \leq i \leq n-1$ of BH_n has the following $2n$ neighbors:*

1. $((a_0 + 1) \mod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$,
   $((a_0 - 1) \mod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$, *and*
2. $((a_0 + 1) \mod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \mod 4, a_{i+1}, \ldots, a_{n-1})$,
   $((a_0 - 1) \mod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \mod 4, a_{i+1}, \ldots, a_{n-1})$.

In BH_n, the first coordinate a_0 of vertex $(a_0, \ldots, a_i, \ldots, a_{n-1})$ is called the *inner index* and the other coordinates are known as the a_i $(1 \leq i \leq n-1)$ *i-dimensional index*. Clearly, each vertex in BH_n has two *inner* neighbors, and $2n-2$ other neighbors. Note that all of the arithmetic operations on indices of vertices in BH_n are four-modulated.

BH_1 and BH_2 are illustrated in Figures 1 and 2, respectively.

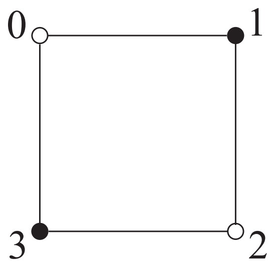

Figure 1. BH_1.

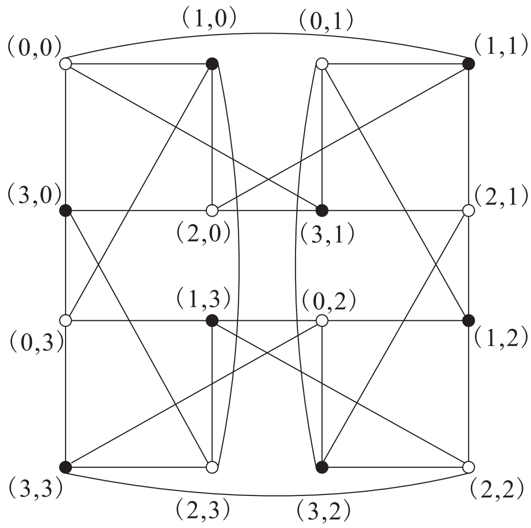

Figure 2. BH_2.

In the following, we give some basic properties of BH_n.

Proposition 1. *[10] The balanced hypercube is bipartite.*

Proposition 2. *[10,20] The balanced hypercube is vertex-transitive and edge-transitive.*

Proposition 3. *[10] The vertices $(a_0, a_1, \ldots, a_{n-1})$ and $((a_0 + 2) \bmod 4,\ a_1, \ldots, a_{n-1})$ of BH_n have the same neighborhood.*

3. Main Results

Firstly, we characterize edges of the BH_n. Let u and v be two adjacent vertices in BH_n. If u and v differ in only the inner index, then uv is said to be a 0-*dimensional* edge, and u is a 0-dimensional neighbor of v. If u and v differ in not only the inner index, but also some i-dimensional index $(i \neq 0)$ of the vertices, then uv is called an *i-dimensional* edge, and u is an i-dimensional neighbor of v. For convenience, we denote the set of all i-dimensional edges by ∂D_i $(0 \leq i \leq n-1)$. Let $BH_{n-1}^{(i)}$ $(0 \leq i \leq 3)$ be the subgraph of BH_n induced by the vertices of BH_n with the $(n-1)$-dimensional index i. That is, the $BH_{n-1}^{(i)}$'s can be obtained from BH_n by deleting all $(n-1)$-dimensional edges. Therefore, $BH_{n-1}^{(i)} \cong BH_{n-1}$ for each $0 \leq i \leq 3$.

By Proposition 1, we know that BH_n is bipartite. We can use V_0 and V_1 to denote the two partite sets of BH_n such that V_0 and V_1 consist of vertices of BH_n with an even inner index and an odd inner index, respectively. For convenience, the vertices of V_0 and V_1 are colored white and black, respectively. Throughout this paper, we use w_i and u_i (resp. b_i and v_i) to denote white (resp. black) vertices in $BH_{n-1}^{(i)}$ ($i \in \{0,1,2,3\}$).

Lemma 1. *[16] In BH_n, $\partial D_i (0 \leq i \leq n-1)$ can be divided into 4^{n-1} edge-disjoint 4-cycles for $n \geq 1$.*

Lemma 2. *[12] The balanced hypercube BH_n is Hamiltonian laceable and edge-bipancyclic for $n \geq 1$.*

Lemma 3. *[17] The balanced hypercube BH_n is hyper-Hamiltonian laceable for $n \geq 1$.*

Lemma 4. *[30] Assume u and x are two different vertices in V_0, and v and y are two different vertices in V_1. Then, there exist two vertex-disjoint paths P and Q such that P joins x to y, Q joins u to v and $V(P) \cup V(Q) = V(BH_n)$, where $n \geq 1$.*

Lemma 5. *Let $n \geq 2$ be an integer. Suppose that u, v, x and y are four distinct vertices differ only the inner index in BH_n. In addition, $u, x \in V_0$ and $v, y \in V_1$. Then, there exists a Hamiltonian path from u to v in $BH_n - x - y$.*

Proof. We proceed with the proof by the induction on n. First, we consider $n = 2$. Clearly, u, v, x and y are in the same 4-cycle of ∂D_0. A Hamiltonian path of $BH_2 - x - y$ from u to v is shown in Figure 3. Thus, we suppose that the lemma holds for all integers $n-1$ with $n \geq 3$. Next, we consider BH_n. We split BH_n into four BH_{n-1}s by deleting $(n-1)$-dimensional edges. For convenience, we denote the four BH_{n-1}s by B_0, B_1, B_2 and B_3 according to the last position of vertices in BH_n, respectively. Without loss of generality, we may assume that u, v, x and y are in B_0. By an induction hypothesis, there exists a Hamiltonian path P_0 from u to v in $B_0 - x - y$. Let $u_0v_0 \in E(P_1)$, where u_0 (resp. v_0) are neither end-vertex of P_0. We denote the segment of P_0 from u to v_0 by P_{00}, and the segment of P_0 from u_0 to v by P_{10}. By Definition 1, u_0 (resp. v_0) has an $(n-1)$-dimensional neighbor v_1 (resp. u_3) in B_1 (resp. B_3). Moreover, there exist an edge v_3u_2 from B_3 to B_2, and an edge v_2u_1 from B_2 to B_1. Therefore, there exist a Hamiltonian path P_3 from u_3 to v_3 in B_3, a Hamiltonian path P_2 from u_2 to v_2 in B_2, and a Hamiltonian path P_1 from u_1 to v_1 of B_1. Hence, $\langle u, P_{00}, v_0, u_3, P_3, v_3, u_2, P_2, v_2, u_1, P_1, v_1, u_0, P_{10}, v \rangle$ is a Hamiltonian path of $BH_n - x - y$ (see Figure 4). □

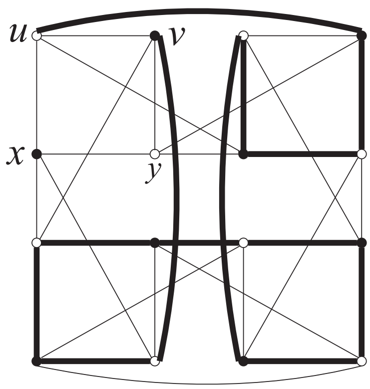

Figure 3. A Hamiltonian path of $BH_2 - x - y$.

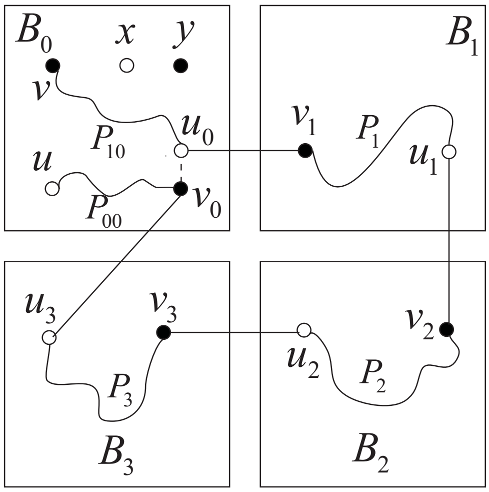

Figure 4. A Hamiltonian path of $BH_n - x - y$.

Next, we present the following lemma as a basis of our main theorem.

Lemma 6. *Let e be an arbitrary edge in* BH_2*. In addition, let* $x \in V_1$ *and* $y \in V_0$ *be any two vertices in* BH_2 *with* $e \neq xy$*. Then, there exists a Hamiltonian path between x and y passing through e.*

Proof. By Proposition 2, BH_2 is vertex-transitive and edge-transitive, and we may suppose that $e = (0,0)(1,0)$. Obviously, if $e = xy$, then there exists no Hamiltonian path of BH_2 from x to y passing e. Thus, at most, one of x and y is the end-vertex of e. We consider the following two cases:

Case 1: Neither x nor y is incident to e. By the relative positions of x and y, and Proposition 3, we consider the following: (1) $x \in V(B_0)$, $y \in V(B_0)$; (2) $x \in V(B_0)$, $y \in V(B_1)$; (3) $x \in V(B_0)$, $y \in V(B_2)$; (4) $x \in V(B_0)$, $y \in V(B_3)$; (5) $x \in V(B_1)$, $y \in V(B_1)$; (6) $x \in V(B_1)$, $y \in V(B_2)$; (7) $x \in V(B_1)$, $y \in V(B_3)$; (8) $x \in V(B_2)$, $y \in V(B_2)$; (9) $x \in V(B_2)$, $y \in V(B_3)$; (10) $x \in V(B_3)$, $y \in V(B_3)$. For simplicity, we list all Hamiltonian paths of the conditions above in Table 1.

Case 2: Either x or y is incident to e. Without loss of generality, suppose that x is incident to e, that is, $x = (1,0)$. By Proposition 3, we need only to consider four conditions of y: (1) $y \in V(B_0)$; (2) $y \in V(B_1)$; (3) $y \in V(B_2)$; and (4) $y \in V(B_3)$. Again, we list Hamiltonian paths of the conditions of x and y in this case in Table 2. □

Table 1. Hamiltonian paths passing through e with neither x nor y being incident to e.

	x	y	**Hamiltonian Paths Passing through e with Neither x nor y Being Incident to e**
(1)	(3,0)	(2,0)	(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,2)(1,2)(2,1)(3,1)(0,1)(1,1)(0,0)(1,0)(2,0)
(2)	(3,0)	(0,1)	(3,0)(0,0)(1,0)(2,3)(3,3)(0,3)(1,3)(0,2)(3,2)(2,2)(1,2)(2,1)(3,1)(2,0)(1,1)(0,1)
(3)	(3,0)	(2,2)	(3,0)(0,3)(3,3)(2,3)(1,0)(0,0)(3,1)(2,0)(1,1)(0,1)(1,2)(2,1)(3,2)(0,2)(1,3)(2,2)
(4)	(3,0)	(0,3)	(3,0)(0,0)(1,0)(2,0)(3,1)(0,1)(1,1)(2,1)(1,2)(2,2)(3,2)(0,2)(1,3)(2,3)(3,3)(0,3)
(5)	(1,1)	(2,1)	(1,1)(0,1)(3,1)(2,0)(1,0)(0,0)(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,2)(1,2)(2,1)
(6)	(1,1)	(2,2)	(1,1)(0,1)(3,1)(2,0)(1,0)(0,0)(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,1)(1,2)(2,2)
(7)	(1,1)	(2,3)	(1,1)(0,0)(3,1)(0,1)(1,2)(2,1)(3,2)(2,2)(1,3)(0,2)(3,3)(0,3)(1,0)(2,0)(3,0)(2,3)
(8)	(1,2)	(2,2)	(1,2)(2,1)(1,1)(0,1)(3,1)(2,0)(1,0)(0,0)(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,2)
(9)	(1,2)	(2,3)	(1,2)(2,1)(1,1)(0,1)(3,1)(2,0)(1,0)(0,0)(3,0)(0,3)(1,3)(2,2)(3,2)(0,2)(3,3)(2,3)
(10)	(1,3)	(2,3)	(1,3)(0,3)(3,0)(0,0)(1,0)(2,0)(1,1)(2,1)(3,1)(0,1)(3,2)(2,2)(1,2)(0,2)(3,3)(2,3)

Table 2. Hamiltonian paths passing through e with x or y being incident to e.

	x	y	**Hamiltonian Paths Passing through e with x or y Being Incident to e**
(1)	(1,0)	(2,0)	(1,0)(0,0)(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,2)(1,2)(2,1)(1,1)(0,1)(3,1)(2,0)
(2)	(1,0)	(0,1)	(1,0)(0,0)(3,0)(0,3)(3,3)(2,3)(1,3)(0,2)(3,2)(2,2)(1,2)(2,1)(1,1)(2,0)(3,1)(0,1)
(3)	(1,0)	(0,2)	(1,0)(0,0)(3,0)(0,3)(1,3)(2,3)(3,3)(2,2)(3,2)(2,1)(1,1)(2,0)(3,1)(0,1)(1,2)(0,2)
(4)	(1,0)	(0,3)	(1,0)(0,0)(3,0)(2,0)(3,1)(0,1)(1,1)(2,1)(1,2)(2,2)(3,2)(0,2)(1,3)(2,3)(3,3)(0,3)

Now, we are ready to state the main theorem of this paper.

Theorem 1. *Let $n \geq 2$ be an integer and e be an arbitrary edge in BH_n. In addition, let $x \in V_1$ and $y \in V_0$ be any two vertices in BH_n with $e \neq xy$. Then, there exists a Hamiltonian path of BH_n between x and y passing through e.*

Proof. We prove this theorem by induction on n. By Lemma 6, we know that the theorem is true for $n = 2$. Therefore, we suppose that the theorem holds for $n-1$ with $n \geq 3$. Next, we consider BH_n. Firstly, we divide BH_n into $BH_{n-1}^{(i)}$ $(0 \leq i \leq 3)$ by deleting all $(n-1)$-dimensional edges. For convenience, we denote $BH_{n-1}^{(i)}$ by B_i according to the last position of the vertices in BH_n for each $i \in \{0,1,2,3\}$. Similarly, suppose that $e \in E(B_0)$. Let $x \in V_1$ and $y \in V_0$ be two distinct vertices in BH_n. By relative positions of x and y, we consider the following cases:

Case 1: $x \in V(B_0)$, $y \in V(B_0)$. By an induction hypothesis, there exists a Hamiltonian path P_0 from x to y of B_0 passing through e. Thus, there is an edge u_0v_0 on P_0 such that u_0v_0 is not adjacent to e and u_0v_0 divides P_0 into two sections P_{00} and P_{10}, where P_{00} connects x to u_0 and P_{10} connects v_0 to y. Let v_1 (resp. u_3) be an $(n-1)$-dimensional neighbor of u_0 (resp. v_0). By Definition 1, there exist an edge u_1v_2 from B_1 to B_2, and an edge u_2v_3 from B_2 to B_3. Thus, by Lemma 2, there exist a Hamiltonian path P_1 from v_1 to u_1 in B_1, a Hamiltonian path P_2 from v_2 to u_2 in B_2, and a Hamiltonian path P_3 from v_3 to u_3 in B_3. Hence, $\langle x, P_{00}, u_0, v_1, P_1, u_1, v_2, P_2, u_2, v_3, P_3, u_3, v_0, P_{10}, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 5).

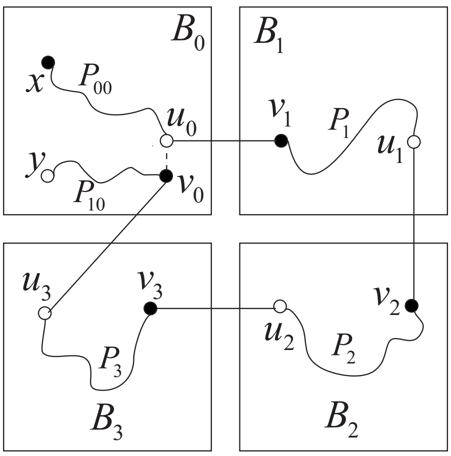

Figure 5. Illustration for Case 1.

Case 2: $x \in V(B_0)$, $y \in V(B_1)$. Let $u_0 \in V(B_0)$ be a white vertex such that u_0 is not incident to e. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from x to u_0 passing through e. Supposing that v_0 is a black vertex adjacent to u_0 on P_0, we denote the segment of the path P_0 from x to v_0 by P_{00}. Let the two $(n-1)$-dimensional neighbors of u_0 be b_1 and v_1. By Lemma 2, there exists a Hamiltonian path P_1 of B_1 from b_1 to y. Let u_1 be the neighbor of v_1 in the section of P_1 from b_1 to v_1. Then $P_1 - u_1v_1$ consists of two subpaths P_{01} and P_{11}, which connect u_1 to b_1 and v_1 to y, respectively.

Let u_3 (resp. v_2) be an $(n-1)$-dimensional neighbor of v_0 (resp. u_1). Furthermore, there exists an edge v_3u_2 from B_3 to B_2. Then, there exist a Hamiltonian path P_2 from u_2 to v_2 in B_2, and a Hamiltonian path P_3 from u_3 to v_3 in B_3. Hence, $\langle x, P_{00}, v_0, u_3, P_3, v_3, u_2, P_2, v_2, u_1, P_{01}, b_1, u_0, v_1, P_{11}, y \rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 6).

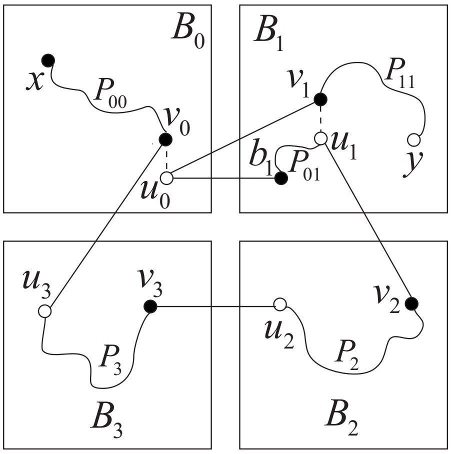

Figure 6. Illustration for Case 2.

Case 3: $x \in V(B_0)$, $y \in V(B_2)$. Let u_0 be a white vertex in B_0 not incident to e, and b_1 and v_1 be two $(n-1)$-dimensional neighbors of u_0. In addition, assume that w_1 is an arbitrary white vertex in B_1. There exists a Hamiltonian path of B_1 from b_1 to w_1. Thus, there exists an edge $u_1v_1 \in E(P_1)$ whose removal will lead to two disjoint subpaths P_{01} and P_{11}, where P_{01} connects u_1 to b_1 and P_{11} connects v_1 to w_1. Let v_2 (resp. b_2) be an $(n-1)$-dimensional neighbor of u_1 (resp. w_1). There also exists a Hamiltonian path P_2 of B_2 from y to b_2 via the edge v_2u_2. Deleting v_2u_2 results in two disjoint paths P_{02} and P_{12}, where P_{02} connects u_2 to b_2 and P_{12} connects v_2 to y. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from x to u_0 via the edge v_0u_0. For convenience, denote $P_0 - u_0$ by P_{00}, that is, P_{00} connects x to v_0. Let u_3 (resp. v_3) be an $(n-1)$-dimensional neighbor of v_0 (resp. u_2). Again, there exists a Hamiltonian path P_3 of B_3 from u_3 to v_3. Hence, $\langle x, P_{00}, v_0, u_3, P_3, v_3, u_2, P_{02}, b_2, w_1, P_{11}, v_1, u_0, b_1, P_{01}, u_1, v_2, P_{12}, y \rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 7).

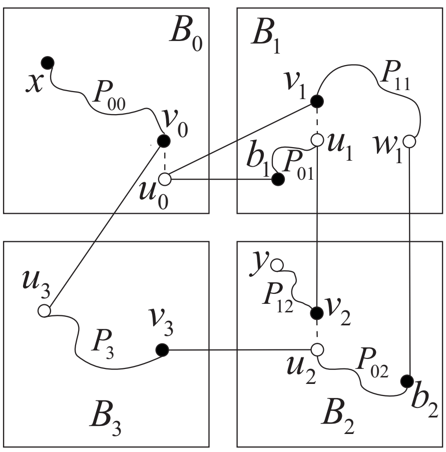

Figure 7. Illustration for Case 3.

Case 4: $x \in V(B_0)$, $y \in V(B_3)$. Let u_0 (resp. v_3) be a white (resp. black) vertex in B_0 (resp. B_3). There exist an edge u_0v_1 from B_0 to B_1, an edge u_1v_2 from B_1 to B_2, and an edge u_2v_3 from B_2 to B_3.

By Lemma 2, there exist a Hamiltonian path P_1 of B_1 from v_1 to u_1, a Hamiltonian path P_2 of B_2 from v_2 to u_2, and a Hamiltonian path P_3 of B_3 from v_3 to u_3. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from x to u_0 passing through e. Hence, $\langle x, P_0, u_0, v_1, P_1, u_1, v_2, P_2, u_2, v_3, P_3, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 8).

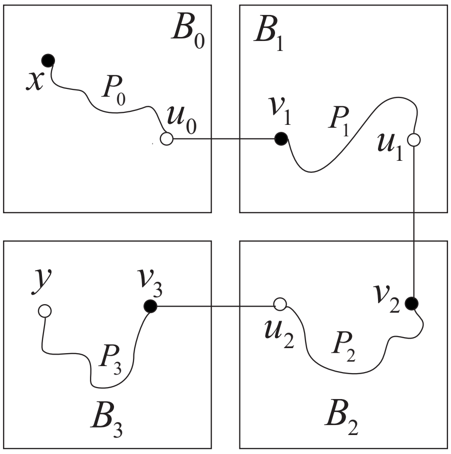

Figure 8. Illustration for Case 4.

Case 5: $x \in V(B_1), y \in V(B_1)$. Let $v_1 \neq x$ be a black vertex in B_1. By Lemma 3, there exists a Hamiltonian path P_1 of $B_1 - y$ from x to v_1. Furthermore, there exist an edge v_1u_0 from B_1 to B_0, an edge v_0u_3 from B_0 to B_3, an edge v_3u_2 from B_3 to B_2, and an edge v_2y from B_2 to B_1. Moreover, there exist a Hamiltonian path P_0 of B_0 from u_0 to v_0 passing through e, a Hamiltonian path P_3 of B_3 from u_3 to v_3, and a Hamiltonian path P_2 of B_2 from u_2 to v_2. Hence, $\langle x, P_1, v_1, u_0, P_0, v_0, u_3, P_3, v_3, u_2, P_2, v_2, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 9).

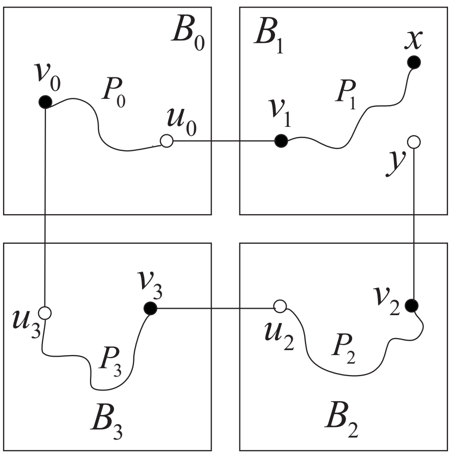

Figure 9. Illustration for Case 5.

Case 6: $x \in V(B_1), y \in V(B_2)$. Let $v_1 \neq x$ (resp. u_1) be a black (resp. white) vertex in B_1. By Lemma 3, there exists a Hamiltonian path P_1 of $B_1 - u_1$ from x to v_1. In addition, suppose that v_2 and b_2 are two $(n-1)$-dimensional neighbors of u_1. By Lemma 2, there exists a Hamiltonian P_2 of B_2 from v_2 to y via the edge u_2b_2. Thus, P_2 can be divided into three sections: P_{02}, u_2v_2 and P_{12}, where P_{02} connects u_2 to v_2 and P_{12} connects b_2 to y. Furthermore, there exist an edge v_1u_0 from B_1 to B_0, an edge v_0u_3 from B_0 to B_3, and an edge v_3u_2 from B_3 to B_2. Therefore, there exist a Hamiltonian path P_0 of B_0 from u_0 to v_0 passing through e, and a Hamiltonian path P_3 of B_3 from u_3 to v_3. Hence, $\langle x, P_1, v_1, u_0, P_0, v_0, u_3, P_3, v_3, u_2, P_{02}, v_2, u_1, b_2, P_{12}, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 10).

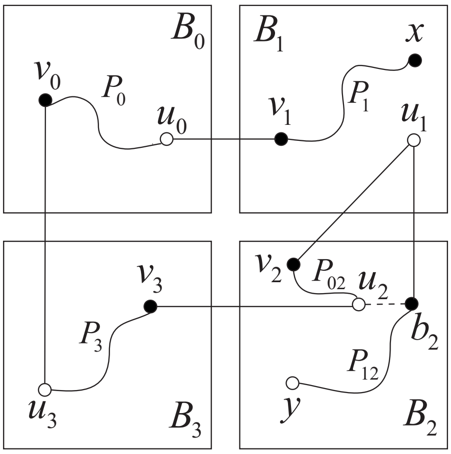

Figure 10. Illustration for Case 6.

Case 7: $x \in V(B_1), y \in V(B_3)$. Let v_3 and b_3 be two black vertices in B_3. Suppose that u_2 and w_2 are $(n-1)$-dimensional neighbors of v_2 and b_2, respectively. By Lemma 3, there exists a Hamiltonian path P_3 of $B_3 - y$ from b_3 to v_3. By Definition 1, there exist two edges v_2u_1 and b_2w_1 from B_2 to B_1, an edge v_1u_0 from B_1 to B_0, and an edge v_0y from B_0 to B_3, where $x \neq v_1$. By Lemma 4, there exist two vertex-disjoint paths P_{01} and P_{11} such that P_{01} joins v_1 and u_1, P_{11} joins x and w_1, and $V(P_{01}) \cup V(P_{11}) = V(B_1)$. Similarly, there exist two vertex-disjoint paths P_{02} and P_{12} such that P_{02} joins v_2 and u_2, P_{12} joins b_2 and w_2, and $V(P_{02}) \cup V(P_{12}) = V(B_2)$. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from u_0 to v_0 passing through e. Hence, $\langle x, P_{11}, w_1, b_2, P_{12}, w_2, b_3, P_3, v_3, u_2, P_{02}, v_2, u_1, P_{01}, v_1, u_0, P_0, v_0, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 11).

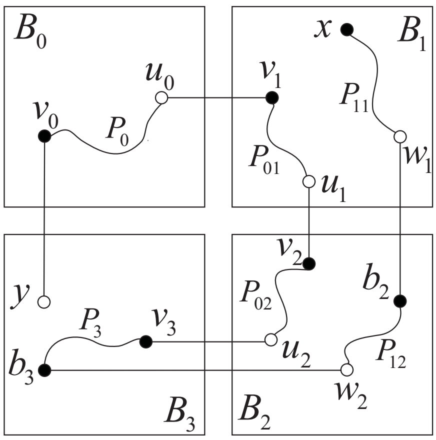

Figure 11. Illustration for Case 7.

Case 8: $x \in V(B_2), y \in V(B_2)$. Let $u_2 \in V(B_2)$ be an arbitrary white vertex. By Lemma 3, there exists a Hamiltonian path P_2 of $B_2 - x$ from u_2 to y. By Definition 1, there exist an edge xu_1 from B_2 to B_1, an edge v_1u_0 from B_1 to B_0, an edge v_0u_3 from B_0 to B_3, and an edge v_3u_2 from B_3 to B_2. Following Lemma 2, we can obtain a Hamiltonian path P_1 of B_1 from u_1 to v_1, and a Hamiltonian path P_3 of B_3 from u_3 to v_3. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from u_0 to v_0 passing through e. Therefore, $\langle x, u_1, P_1, v_1, u_0, P_0, v_0, u_3, P_3, v_3, u_2, P_2, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 12).

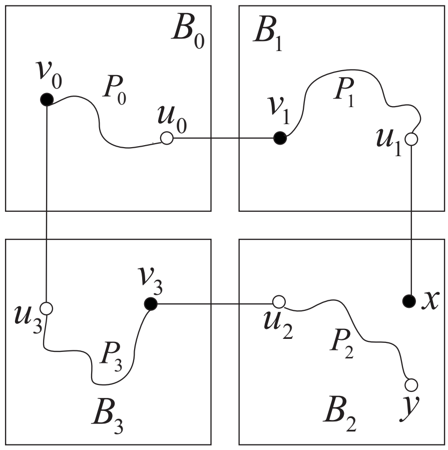

Figure 12. Illustration for Case 8.

Case 9: $x \in V(B_2)$, $y \in V(B_3)$. Let u_2 and w_2 be two distinct white vertices in B_2, and v_3 and b_3 be $(n-1)$-dimensional neighbors of u_2 and w_2, respectively. By Lemma 3, there exists a Hamiltonian path P_2 of $B_2 - x$ from u_2 to w_2. By Lemma 2, there exists a Hamiltonian path P_3 of B_3 from v_3 to y via the edge u_3b_3. By deleting u_3b_3, we can obtain two disjoint subpaths: P_{03} and P_{13}, where P_{03} connects u_3 to v_3 and P_{13} connects b_3 to y. Furthermore, there exist an edge xu_1 from B_2 to B_1, an edge v_1u_0 from B_1 to B_0, and an edge v_0u_3 from B_0 to B_3. By Lemma 2, there exists a Hamiltonian path P_1 of B_1 from u_1 to v_1. By an induction hypothesis, there exists a Hamiltonian path P_0 of B_0 from u_0 to v_0 passing through e. Hence, $\langle x, u_1, P_1, v_1, u_0, P_0, v_0, u_3, P_{03}, v_3, u_2, P_2, w_2, b_3, P_{13}, y\rangle$ is a Hamiltonian path of BH_n from x to y passing through e (see Figure 13).

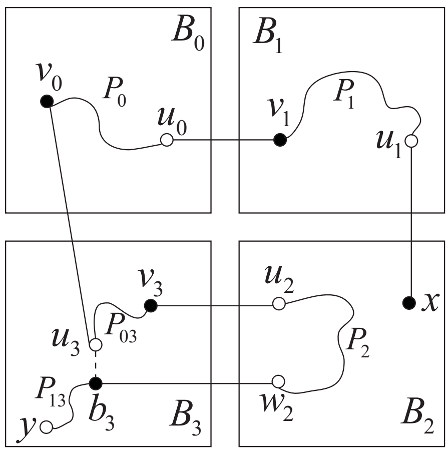

Figure 13. Illustration for Case 9.

Case 10: $x \in V(B_3)$, $y \in V(B_3)$. The proof is analogous to that of Case 5, and we omit it. □

4. Conclusions

In this paper, we study a type of path embedding of the balanced hypercube, and show that, for an arbitrary edge $e \neq xy$, there exists a Hamiltonian path between any two vertices x and y in different partite sets passing through e. This result also implies that each edge is on a Hamiltonian cycle of the balanced hypercube, which is part of the results of edge bipancyclicity of the balanced hypercube.

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References

1. Abraham, S.; Padmanabhan, K. The twisted cube topology for multiprocessors: A study in network asymmetry. *J. Parall. Distrib. Comput*. **1991**, *13*, 104–110, doi:10.1016/0743-7315(91)90113-N.
2. Choudum, S.A.; Sunitha, V. Augmented cubes. *Networks* **2002**, *40*, 71–84.
3. Cull, P.; Larson, S.M. The Möbius cubes. *IEEE Trans. Comput*. **1995**, *44*, 647–659.
4. Dally, W.J. Performance analysis of *k*-ary *n*-cube interconnection networks. *IEEE Trans. Comput.* **1990**, *39*, 775–785.
5. Efe, K. The crossed cube architecture for parallel computation. *IEEE Trans. Parall. Distr. Syst.* **1992**, *3*, 513–524.
6. El-amawy, A.; Latifi, S. Properties and performance of folded hypercubes. *IEEE Trans. Parall. Distrib. Syst*. **1991**, *2*, 31–42.
7. Li, T.K.; Tan, J.J.M.; Hsu, L.H.; Sung, T.Y. The shuffle-cubes and their generalization. *Inform. Process. Lett*. **2001**, *77*, 35–41.
8. Preparata, F.P.; Vuillemin, J. The cube-connected cycles: A versatile network for parallel computation. *Comput. Arch. Syst.* **1981**, *24*, 300–309.
9. Xiang, Y.; Stewart, I.A. Augmented *k*-ary *n*-cubes. *Inform. Sci.* **2011**, *181*, 239–256.
10. Wu, J.; Huang, K. The balanced hypercube: A cube-based system for fault-tolerant applications. *IEEE Trans. Comput*. **1997**, *46*, 484–490.
11. Huang, K.; Wu, J. Fault-tolerant resource placement in balanced hypercubes. *Inform. Sci.* **1997**, *99*, 159–172.
12. Xu, M.; Hu, H.; Xu, J. Edge-pancyclicity and Hamiltonian laceability of the balanced hypercubes. *Appl. Math. Comput.* **2007**, *189*, 1393–1401.
13. Yang, M. Bipanconnectivity of balanced hypercubes. *Comput. Math. Appl.* **2010**, *60*, 1859–1867.
14. Huang, K.; Wu, J. Area efficient layout of balanced hypercubes. *Int. J. High Speed Electr. Syst.* **1995**, *6*, 631–645.
15. Yang, M. Super connectivity of balanced hypercubes. *Appl. Math. Comput*. **2012**, *219*, 970–975.
16. Lü, H.; Li, X.; Zhang, H. Matching preclusion for balanced hypercubes. *Theor. Comput. Sci.* **2012**, *465*, 10–20, doi:10.1016/j.tcs.2012.09.020.
17. Lü, H.; Zhang, H. Hyper-Hamiltonian laceability of balanced hypercubes. *J. Supercomput.* **2014**, *68*, 302–314, doi:10.1007/s11227-013-1040-6.
18. Lü, H.; Gao, X.; Yang, X. Matching extendability of balanced hypercubes. *Ars Combinatoria* **2016**, *129*, 261–274.
19. Lü, H. On extra connectivity and extra edge-connectivity of balanced hypercubes. *Int. J. Comput. Math.* **2017**, *94*, 813–820.
20. Zhou, J.-X.; Wu, Z.-L.; Yang, S.-C.; Yuan, K.-W. Symmetric property and reliability of balanced hypercube. *IEEE Trans. Comput*. **2015**, *64*, 876–881.
21. Zhou, J.-X.; Kwak, J.; Feng, Y.-Q.; Wu, Z.-L. Automorphism group of the balanced hypercube. *Ars Math. Contemp.* **2017**, *12*, 145–154.
22. Jha, P.K.; Prasad, R. Hamiltonian decomposition of the rectangular twisted torus. *IEEE Trans. Comput*. **2012**, *23*, 1504–1507
23. Chang, N.-W.; Tsai, C.-Y.; Hsieh, S.-Y. On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes. *IEEE Trans. Comput*. **2014**, *63*, 1594–1600.
24. Hsieh, S.-Y.; Yu, P.-Y. Fault-free mutually independent Hamiltonian cycles in hypercubes with faulty edges. *J. Combin. Optim.* **2007**, *13*, 153–162.

25. Park, C.; Chwa, K. Hamiltonian properties on the class of hypercube-like networks. *Inform. Process. Lett.* **2004**, *91*, 11–17.
26. Wang, S.; Li, J.; Wang, R. Hamiltonian paths and cycles with prescribed edges in the 3-ary n-cube. *Inform. Sci.* **2011**, *181*, 3054–3065
27. Simmons, G. Almost all n-dimensional rectangular lattices are Hamilton–Laceable. *Congr. Numer.* **1978**, *21*, 103–108.
28. West, D.B. *Introduction to Graph Theory*, 2nd ed.; Prentice Hall: Englewood Cliffs, NJ, USA, 2001.
29. Xu, J.M. *Topological Structure and Analysis of Interconnection Networks*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2001.
30. Cheng, D.; Hao, R.-X.; Feng, Y.-Q. Two node-disjoint paths in balanced hypercubes. *Appl. Math. Comput*. **2014**, *242*, 127–142.