symmetry

MDPI

Article

Lie Symmetry Classification of the Generalized Nonlinear Beam Equation

Dingjiang Huang [1,2,*], Xiangxiang Li [1] and Shunchang Yu [1]

[1] Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China; TracytracyLi@126.com(X.L.); shuncyu@163.com (S.Y.)

[2] School of Data Science and Engineering, East China Normal University, Shanghai 200062, China

* Correspondence: djhuang@ecust.edu.cn or djhuang.math@gmail.com; Tel.: +86-21-6425-3147

Academic Editor: Roman M. Cherniha

Received: 22 March 2017; Accepted: 6 July 2017; Published: 11 July 2017

Abstract: In this paper we make a Lie symmetry analysis of a generalized nonlinear beam equation with both second-order and fourth-order wave terms, which is extended from the classical beam equation arising in the historical events of travelling wave behavior in the Golden Gate Bridge in San Francisco. We perform a complete Lie symmetry group classification by using the equivalence transformation group theory for the equation under consideration. Lie symmetry reductions of a nonlinear beam-like equation which are singled out from the classification results are investigated. Some classes of exact solutions, including solitary wave solutions, triangular periodic wave solutions and rational solutions of the nonlinear beam-like equations are constructed by means of the reductions and symbolic computation.

Keywords: lie symmetry classification; symmetry reduction; equivalence group; generalized nonlinear beam equation; exact solution

1. Introduction

In this paper, we study the group properties of higher-order nonlinear wave-type equations. As a basic model, we consider the fourth-order *generalized nonlinear beam equation* (GNBE) or *nonlinear wave equation* of the form

$$u_{tt} = -[K(u)u_{xxx}]_x + [D(u)u_x]_x + F(u), \tag{1}$$

where $K = K(u), D = D(u)$ and $F = F(u)$ are arbitrary smooth functions (hereafter the subscripts t and x denote differentiation with respect to these variables). Equation (1) generalizes a wide range of the known nonlinear wave equations arising in applications. The case with K identically zero and $D(u) \neq 0$ for almost all u, is the case of second-order nonlinear wave equation which has already been studied in many practical contexts including shallow water waves theory, dynamics of a finite nonlinear string and elastic-plastic materials, etc. (see Refs. [1], pp. 50–52 and [2]). Hereinafter we assume that $K(u)$ is not identically zero, so that the governing equation is of the fourth order, including a fourth-order wave term when K is non-negative.

The simplest equation of the form Equation (1), with $K(u) = 1$, $D(u) = 0$ and $F(u) = 0$, is the one dimensional *linear beam equation* or the *fourth-order linear wave equation* [3,4]

$$u_{tt} = -u_{xxxx}. \tag{2}$$

when $K(u) = 1$, $D(u) = 0$ and $F(u) = -ku^+ + 1, u^+ = \max\{u, 0\}$, Equation (1) is reduced to the following classical *nonlinear beam equation* or the *fourth-order nonlinear wave equation* of the form [5]

$$u_{tt} = -u_{xxxx} - ku^+ + 1, \tag{3}$$

which is an normalization of the original beam equations

$$u_{tt} = -u_{xxxx} - ku^+ + W(x) + \varepsilon f(t,x)$$

arising in the historical events of travelling wave behavior in the Golden Gate Bridge in San Francisco [6] and has been proposed as a model for a suspension bridge. When $k(u) = 1,\ D(u) = 0$, Equation (1) become

$$u_{tt} = -u_{xxxx} + F(u), \tag{4}$$

which is a slight generalization of the classical nonlinear beam Equation (3) and is also presented by McKenna and Walter [7] in studying travelling wave solutions.

Up to now, the mathematic structure and properties of the fourth-order beam Equations (2)–(4) have been widely investigated. For instance, the mathematical state of the art concerning the operator and semigroup theory for Equation (2) on a bounded x-interval has been studied in [4]. In [5,8], the existence of travelling wave solutions and standing wave oscillations of the nonlinear beam Equation (3) have been considered. Chen and McKenna gave a variational proof of the existence of travelling wave solutions for Equation (4) via the Mountain Pass Lemma and showed the existence of at least one nontrivial solution for the equation under consideration in [9,10]. In particular, for $F(u) = e^{u-1} - 1$, they did also obtain numerical solutions by applying the Mountain Pass algorithm to a finite subinterval of R. They claimed that solutions seem to exist in the range $0 < c < \sqrt{2}$. As the wave speed approaches $\sqrt{2}$, the solutions became highly oscillatory in nature, whereas when c approaches 0, they appear to go to infinity in amplitude. In [11–13], Equation (4) has been studied numerically by using either continuation methods or variational numerical methods in order to gain more information on the structure of the equation solutions set. In [14] Humphreys and McKenna considered the existence of multiple periodic solutions for the nonlinear beam Equation (4), while in [15,16] Doole and Hogan transformed the beam Equation (4) into an ordinary differential equation and treated it by using dynamical systems method. Recently, there are also some researches devoted to symmetry group structure and exact solutions of the fourth-order nonlinear beam Equation (4) [17,18].

However, the generalized nonlinear beam Equation (1) has been much less extensively investigated. It was only a few researches [19] that were devoted to qualitative mathematical properties such as quasi-periodic solutions of Equation (1) with $K(u) = D(u) = 1$, while the symmetry group properties and corresponding algebraic structure as well as explicit exact solutions of Equation (1) and its invariant models still remain open. Therefore, the aim of the present work is to find all possible Lie symmetries, which Equation (1) can admit depending on the function triplets (K, D, F), i.e., to solve the so-called group classification problem, which was formulated and solved for a class of nonlinear heat equations in the pioneering work by Ovsiannikov in 1959 [20] and now is the core stone of modern group analysis [21,22]. This problem for the second-order wave equation was probably first solved by Barone et al. in [23] and subsequently was extended to other general forms by many authors in the last two decades [2,24–43], but were all limited to second-order cases.

Ovsiannikov's method (also referred as the Lie-Ovsiannikov method) of Lie symmetry classification of differential equations [21] is based on the classical Lie scheme and a set of equivalence transformations of a given equation [44]. The formal application of this method to equations containing several arbitrary functions (Equation (1) contains three arbitrary functions) usually leads to a large number of equations admitting nontrivial Lie algebras of invariance [44].

Recent years, this method has been extended by many authors, in which they proposed a numbers of novel techniques, such as algebraic methods based on subgroup analysis of the equivalence group [45–48] and their generalizations [49–54], local transformations and form-preserving transformations [44,55–58], to solve group classification problem for numerous nonlinear partial differential equations. In this paper we extend the classical Lie-Ovsiannikov method based on equivalence transformations to the generalized nonlinear beam equation. We first carry out group classification of Equation (1) under the usual equivalence group. Then similar reductions of the classification models are performed and invariant solutions are also constructed. It is found that some similarity solutions are solutions with physical interest, including solitary wave solutions, triangular periodic wave solutions and rational solution.

The rest of paper is organized as follows. In Section 2, we derive the equivalence group and perform the group classification related to Equation (1), i.e., all possible Lie symmetries, which this equation can admit depending on the form the functions K, D and F, are found. In Section 3, the symmetry reductions and some exact solutions are constructed for particular case of Equation (1) that are likely to be useful in applications. Section 4 some concluding remarks are reported.

2. Lie Symmetry Classification

Background and procedures of the modern Lie group theory are well described in literature [21,22,35,37,50,59]. Without going into the details of the theory, we present only the results below.

Let

$$\mathbf{v} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \phi(t,x,u)\frac{\partial}{\partial u}. \tag{5}$$

be a vector field or infinitesimal operator on the space of independent and dependent variables $t,\ x,\ u$. A local group of transformations G is a symmetry group of Equation (1) if and only if

$$\mathrm{pr}^{(4)}\mathbf{v}|(\triangle) = 0, \tag{6}$$

whenever $\triangle = u_{tt} + [K(u)u_{xxx}]_x - [D(u)u_x]_x - F(u) = 0$ for every generator of G, where $\mathrm{pr}^{(4)}\mathbf{v}$ is the fourth-order prolongation of $\mathbf{v}$ (Here we do not write the explicit prolongation formulas to avoid tediousness, one can see the book [59] or the paper [54] for details). Expanding Equation (6) we get

$$\begin{aligned} &\phi^{tt} + [K''(u)u_x u_{xxx} + K'(u)u_{xxxx} - D''(u)u_x^2 - D'(u)u_{xx} - F'(u)]\phi \\ &+[K'(u)u_{xxx} - 2D'(u)u_x]\phi^x - D(u)\phi^{xx} + K'(u)u_x\phi^{xxx} + K(u)\phi^{xxxx} = 0, \end{aligned} \tag{7}$$

which must be satisfied whenever Equation (1) is satisfied. Substituting the formulae of ϕ^{tt}, ϕ^x, ϕ^{xx}, ϕ^{xxx} and ϕ^{xxxx} into Equation (1) we get an equation of $t,\ x,\ u$ and the derivatives of τ, ξ, ϕ, u. Replacing u_{tt} by the right side of Equation (1) whenever it occurs, and equating the coefficients of the various independent monomials in the partial derivatives of u to zero, we obtain the determining equations

$$\begin{aligned} &\tau_x = \tau_u = \xi_t = \xi_u = \phi_{uu} = 0, \\ &2(2\phi_{xu} - 3\xi_{xx})K + \phi_x K_u = 0, \\ &2(\tau_t - 2\xi_x)K + \phi K_u = 0, \\ &(\phi_{xu} - \xi_{xx})K_u = 0, \\ &(4\phi_{xxxu} - \xi_{xxxx})K + \phi_{xxx}K_u + (\xi_{xx} - 2\phi_{xu})D - 2\phi_x D_u = 0, \\ &2\phi_{tu} - \tau_{tt} = 0, \\ &2(3\phi_{xxu} - 2\xi_{xxx})K + 2(\xi_x - \tau_t)D - \phi D_u = 0, \\ &\phi_{tt} + \phi_{xxxx}K - \phi_{xx}D + (\phi_u - 2\tau_t)F - \phi F_u = 0, \\ &K_u(\phi_u + 2\tau_t - 4\xi_x) + K_{uu}\phi = 0, \\ &D_u(2\xi_x - 2\tau_t - \phi_u) - D_{uu}\phi + K_u(3\phi_{xxu} - \xi_{xxx}) = 0. \end{aligned} \tag{8}$$

Investigating the compatibility of system Equation (8) we find that the last two equations of system Equation (8) are two identities (substituting the third, the fourth and the seventh equations of system Equation (8) into the last two one can yield this conclusion). Furthermore, with the aid of

$K(u) \neq 0$, the second equation and the derivative of the third equation with respect to x imply that $\xi_{xx} = 2\phi_{xu}$. Thus, the determining Equation (8) is reduced to

$$\begin{aligned}
&\tau_x = \tau_u = \xi_t = \xi_u = \phi_{uu} = 0, \quad \tau_{tt} = 2\phi_{tu},\\
&\phi_{xu}K_u = 0,\\
&\xi_{xx} = 2\phi_{xu},\\
&\phi K_u + 2(\tau_t - 2\xi_x)K = 0,\\
&\phi_{xxx}K_u - 2\phi_x D_u + 2\phi_{xxxu}K = 0,\\
&\phi D_u + 2(\tau_t - \xi_x)D + 2\phi_{xxu}K = 0,\\
&\phi_{tt} + \phi_{xxxx}K - \phi_{xx}D + (\phi_u - 2\tau_t)F - \phi F_u = 0.
\end{aligned} \tag{9}$$

The first two equations of system Equation (9) do not contain arbitrary elements. Integration of them yields

$$\tau = \tau(t), \quad \xi = \xi(x), \quad \phi = \phi^1(t,x)u + \phi^0(t,x), \quad \phi^1(t,x) = \frac{1}{2}\tau_t + \alpha(x), \tag{10}$$

Thus, group classification of Equation (1) reduces to solving the rest equations of of system Equation (9).

Splitting the rest of the system Equation (9) with respect to the arbitrary elements and their non-vanishing derivatives gives the equations $\tau_t = 0$, $\xi_x = 0$, $\phi = 0$ for the coefficients of the operators from $A^{\ker}$ of Equation (1). As a result, we obtain the following assertion.

Proposition 1. *The Lie algebra of the kernel of principal groups of Equation* (1) *is a two-dimensional algebra* $A^{\ker} = \langle \partial_t, \partial_x \rangle$.

In order to make the classification as simple as possible, we next look for equivalence transformations of class Equation (1), and then solve system Equation (9) under these transformations (Here we do not give detailed statements about equivalence transformation to avoid tediousness, one can see the book [21] or the paper [54] for details). An equivalence transformation is a nondegenerate change of the variables t, x and u taking any equation of the form Equation (1) into an equation of the same form, generally speaking, with different $K(u)$, $D(u)$ and $F(u)$. The set of all equivalence transformations forms the equivalence group $G^{\sim}$. To find the connected component of the unity of $G^{\sim}$, we have to investigate Lie symmetries of the system that consists of Equation (1) and some additional conditions, i.e.,

$$\begin{aligned}
&u_{tt} = -K_u u_x u_{xxx} - K u_{xxxx} + D_u u_x^2 + D u_{xx} + F,\\
&K_t = 0, \quad K_x = 0, \quad D_t = 0, \quad D_x = 0, \quad F_t = 0, \quad F_x = 0.
\end{aligned} \tag{11}$$

That is to say we must seek for an operator of the Lie algebra $A^{\sim}$ of $G^{\sim}$ in the form

$$\begin{aligned}
\mathbf{X} =& \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u\\
&+ \pi(t,x,u,K,D,F)\partial_K + \rho(t,x,u,K,D,F)\partial_D + \theta(t,x,u,K,D,F)\partial_F.
\end{aligned} \tag{12}$$

Here u and K, D, F are considered as different variables: u is on the space (t,x) and K, D, F is on the extended space (t,x,u). The coordinates τ, ξ, ϕ of the operator Equation (12) are sought as functions of t, x, u while the coordinates π, ρ, θ are sought as functions of t, x, u and K, D, F. Applying

$$\begin{aligned}
\mathrm{pr}^{(4)}\,\mathbf{X} =& v + \phi^{tt}\frac{\partial}{\partial u_{tt}} + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxxx}\frac{\partial}{\partial u_{xxxx}}\\
&+ \pi^t\frac{\partial}{\partial K_t} + \pi^x\frac{\partial}{\partial K_x} + \pi^u\frac{\partial}{\partial K_u} + \rho^t\frac{\partial}{\partial D_t} + \rho^x\frac{\partial}{\partial D_x} + \rho^u\frac{\partial}{\partial D_u} + \theta^t\frac{\partial}{\partial F_t} + \theta^x\frac{\partial}{\partial F_x}
\end{aligned}$$

to Equation (11) we get the infinitesimal criterion

$$\begin{aligned}
\phi^{tt} = & -u_x u_{xxx}\pi^u - K_u\phi^x u_{xxx} - K_u u_x \phi^{xxx} - \pi u_{xxxx} - K\phi^{xxxx} \\
& - {u_x}^2\rho^u + 2D_u u_x \phi^x + \rho u_{xx} + D\phi^{xx} + \theta \\
\pi^x = 0, \quad & \pi^t = 0, \quad \rho^x = 0, \quad \rho^t = 0, \quad \theta^x = 0, \quad \theta^t = 0,
\end{aligned} \tag{13}$$

which must be satisfied whenever Equation (11) is satisfied. Substituting the formulae of ϕ^{tt}, ϕ^x, ϕ^{xx}, ϕ^{xxx}, ϕ^{xxxx}, π^x, π^t, π^u, ρ^x, ρ^t, ρ^u, θ^x, and θ^t into Equation (13) we get equations of t, x, u, f, and the partial derivatives of ξ, τ, ϕ, π, ρ, θ. Replacing u_{tt}, K_x, K_t, D_x, D_t, F_x, and F_t by the right hand side of Equation (11) whenever they occur, and equating the coefficients of various independent monomials to zero, we obtain

$$\begin{aligned}
&\tau_x = 0,\ \tau_u = 0, \quad \xi_t = 0,\ \xi_u = 0,\ \phi_{uu} = 0, \quad \tau_{tt} - 2\phi_{tu} = 0, \\
&K_u(\phi_{xu} - \xi_{xx}) = 0, \quad K(4\xi_x - 2\tau_t) - \pi = 0, \\
&K_u(4\xi_x - 2\tau_t) - \pi_u - K_u\pi_K - D_u\pi_D - F_u\pi_F = 0, \\
&2D_u(\phi_u + \tau_t - \xi_x) - K_u(3\phi_{xxu} - \xi_{xxx}) - \rho_u - K_u\rho_K - D_u\rho_D - F_u\rho_F = 0, \\
&2D(\tau_t - \xi_x) - K(6\phi_{xxu} - 4\xi_{xxx}) + \rho = 0, \\
&D(2\phi_{xu} - \xi_{xx}) - K_u\phi_{xxx} - K(4\phi_{uxxx} - \xi_{xxxx}) + 2D_u\phi_x = 0, \\
&\phi_{xx}D - \phi_{xxxx}K + \theta - \phi_{tt} - F(\phi_u - 2\tau_t) = 0, \\
&\pi_t - K_u\phi_t = 0, \pi_x - K_u\phi_x = 0, \quad \rho_t - D_u\phi_t = 0, \rho_x - D_u\phi_x = 0, \quad \theta_t - F_u\phi_t = 0, \theta_x - F_u\phi_x = 0,
\end{aligned} \tag{14}$$

which can be reduced to

$$\begin{aligned}
&\tau = c_1 t + c_4, \quad \xi = c_2 x + c_5, \quad \phi = c_3 u + c_6, \\
&\pi = 2(2c_2 - c_1)K(u), \quad \rho = 2(c_2 - c_1)D(u), \quad \theta = (c_3 - 2c_1)F(u),
\end{aligned}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

Thus the Lie algebra of $G^{\sim}$ for class Equation (11) is

$$A^{\sim} = \langle \partial_t, \partial_x, \partial_u, t\partial_t - 2K\partial_K - 2D\partial_D - 2F\partial_F, x\partial_x + 4K\partial_K + 2D\partial_D, u\partial_u + F\partial_F \rangle.$$

Continuous equivalence transformations of class Equation (11) are generated by the operators from $A^{\sim}$. In fact, $G^{\sim}$ contains the following continuous transformations:

$$\begin{aligned}
&\tilde{t} = t\varepsilon_1 + \varepsilon_4, \qquad \tilde{x} = x\varepsilon_2 + \varepsilon_5, \qquad \tilde{u} = u\varepsilon_3 + \varepsilon_6, \\
&\tilde{K} = K\varepsilon_1^{-2}\varepsilon_2^4, \qquad \tilde{D} = D\varepsilon_1^{-2}\varepsilon_2^2, \qquad \tilde{F} = F\varepsilon_1^{-2}\varepsilon_3,
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6$ are arbitrary constants.

Solve the system of determining Equation (9) under the above equivalence group $G^{\sim}$, we can obtain sixteen inequivalent equations of class Equation (1) with respect to the transformations from $G^{\sim}$.

Theorem 1. *A complete set of $G^{\sim}$-inequivalent extensions of $A^{\max} \neq A^{\ker}$ for Equation (1) is exhausted by ones given in Table 1.*

Table 1. Group classification of class Equation (1).

N	$K(u)$	$D(u)$	$F(u)$	Basis of $A^{\max}$
1.	$\forall$	$\forall$	$\forall$	$\partial_t,\ \partial_x$
2	$\forall$	0	0	$\partial_t, \partial x, x\partial_x + 2t\partial t$
3	$e^{\mu u}$	0	0	$\partial_t,\ \partial_x,\ -2t\partial_t - x\partial_x,\ \frac{\mu}{4}x\partial_x + \partial_u$
4	$e^{\mu u}$	0	$fe^{\gamma u}(f \neq 0)$	$\partial_t,\ \partial_x,\ -\frac{\gamma}{2}t\partial_t + \frac{\mu-\gamma}{4}x\partial_x + \partial_u$
5	$e^{\mu u}$	$de^{\nu u}(d \neq 0)$	$fe^{(2\nu-\mu)u}$	$\partial_t, \partial_x,\ \frac{\mu-2\nu}{2}t\partial_t + \frac{\mu-\nu}{2}x\partial_x + \partial_u$
6	u^{-4}	du^{-4}	fu^{-3}	$\partial_t,\ \partial_x,\ t\partial_t + \frac{1}{2}u\partial_u,\ t^2\partial_t + tu\partial_u$
7	u^{-4}	du^{-4}	$\frac{1}{4}\omega u + fu^{-3}(\omega > 0)$	$\partial_t,\ \partial_x,\ e^{\sqrt{\omega}t}\partial_t + \frac{\sqrt{\omega}}{2}e^{\sqrt{\omega}t}u\partial_u,\ e^{-\sqrt{\omega}t}\partial_t - \frac{\sqrt{\omega}}{2}e^{-\sqrt{\omega}t}u\partial_u$
8	u^{-4}	du^{-4}	$\frac{1}{4}\omega u + fu^{-3}(\omega < 0)$	$\partial_t,\ \partial_x,\ \sin(\sqrt{-\omega}t)\partial_t + \frac{\sqrt{-\omega}}{2}\cos(\sqrt{-\omega}t)u\partial_u,$ $\cos(\sqrt{-\omega}t)\partial_t - \frac{\sqrt{-\omega}}{2}\sin(\sqrt{-\omega}t)u\partial_u$
9	u^{-4}	0	0	$\partial_t,\ \partial_x,\ t\partial_t + \frac{1}{2}u\partial_u,\ t^2\partial_t + tu\partial_u,\ -x\partial_x + u\partial_u$
10	u^{-4}	0	$fu(f > 0)$	$\partial_t,\ \partial_x,\ e^{2\sqrt{f}t}\partial_t + \sqrt{f}e^{2\sqrt{f}t}u\partial_u,\ e^{-2\sqrt{f}t}\partial_t - \sqrt{f}e^{-2\sqrt{f}t}u\partial_u,$ $-x\partial_x + u\partial_u$
11	u^{-4}	0	$fu(f < 0)$	$\partial_t,\ \partial_x,\ \sin(2\sqrt{-f}t)\partial_t + \sqrt{-f}\cos(2\sqrt{-f}t)u\partial_u,$ $\cos(2\sqrt{-f}t)\partial_t - \sqrt{-f}\sin(2\sqrt{-f}t)u\partial_u,\ -x\partial_x + u\partial_u$
12	u^{μ}	0	0	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x, -\frac{\mu}{2}t\partial_t + u\partial_u$
13	u^{μ}	0	$fu^{\gamma}(f \neq 0)$	$\partial_t,\ \partial_x,\ \frac{1-\gamma}{2}t\partial_t + \frac{1-\gamma+\mu}{4}x\partial_x + u\partial_u$
14	u^{μ}	$du^{\nu}(d \neq 0)$	$fu^{1-\mu+2\nu}$	$\partial_t,\ \partial_x,\ \frac{1}{2}(\mu - 2\nu)t\partial_t + \frac{1}{2}(\mu - \nu)x\partial_x + u\partial_u$
15	1	0	0	$\partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x,\ u\partial_u,\ \mathbf{v}^{\infty} = b(x,t)\partial_u$ where $b_{tt} + b_{xxxx} = 0$
16	1	d	fu	$\partial_t, \partial_x, u\partial_u, \mathbf{v}^{\infty} = b(x,t)\partial_u$ where $b_{tt} - db_{xx} + b_{xxxx} - fb(t,x) = 0$

Here, for Case 16, d, f can not be zero simultaneously.

Proof. To obtain the classification result we need to solve the system Equation (9) using the compatibility method [49,50]. The basic idea of this method is based on the fact that the substitution of the coefficients of any operator from $A^{\max} \setminus A^{\ker}$ into the classifying equations results in nonidentity equations for arbitrary elements (see [49,50] for more details and exhaustive examples of applications). In our case the procedure of looking for the possible cases mostly depends on the fifth equation of system Equation (9). For any symmetry operator, equations $\phi K_u + 2(\tau_t - 2\xi_x)K = 0$ give some equations on K of the general form

$$(au + b)K_u + cK = 0,$$

where a, b and c are constants. For all operators from $A^{\max}$ the number k of such independent equations is not greater than 2; otherwise they form an incompatible system on K. k is an invariant value for the transformations from $G^{\sim}$. Therefore, there exist three inequivalent cases for the value of k: (i) $k = 0$: $K(u)$ is arbitrary; (ii) $k = 1$: $K(u) = e^{\mu u}$ or $K(u) = u^{\mu}$ $(\mu \neq 0)$ mod $G^{\sim}$, and (iii) $k = 2$: $K(u) = 1$ mod $G^{\sim}$. Therefore, to complete the classification we have to consider all possible cases of the values of $K(u)$. We attempted to present our calculations in reasonable detail so that verification would be feasible.

Case 1: $k = 0$**.** In this case, the fifth equation of system Equation (9) imply that $\phi = 0$ and $\tau_t - 2\xi_x = 0$. With this condition, the system Equation (9) can be reduced to

$$\tau_{tt} = 0, \quad \xi_{xx} = 0, \quad 2(\tau_t - 2\xi_x) = 0, \quad (\tau_t - \xi_x)D = 0, \quad \tau_t F = 0. \tag{15}$$

If $D \neq 0$ or $F \neq 0$, which is corresponding to the Case 1. If $D = F = 0$, we immediately arrive at Case 2 of Table 1 from system Equation (15).

Case 2: $k = 1$**.** Here $K \in \{e^{\mu u}, u^{\mu}, \mu \neq 0\}$ mod $\hat{G}^{\sim}$ and there exists $\mathbf{v} \in A^{\max}$ with $\phi \neq 0$, otherwise there is no additional extension of the maximal Lie invariance algebra in comparison with the case $k = 0$.

Case 2.1: Let us investigate the first possibility $K = e^{\mu u}$. In this case, the third equation and the fourth equation imply $\phi_{xu} = 0, \xi_{xx} = 0$. Thus, from the fifth equation of system Equation (9), we can get $\phi_x K_u = 0$, which implies $\phi_x = 0$, and thus leads to $\alpha(x) = c_3, \phi^0(t,x) = \phi^0(t)$, where c_3 is a constant. Furthermore, substituting $K = e^{\mu u}$ into the fifth equation of system Equation (9) and using Equation (10), we can obtain $\phi^1 = 0, \mu\phi^0 + 2(\tau_t - 2\xi_x) = 0$, and thus leads to $\frac{1}{2}\tau_t + c_3 = 0,\ \phi^0_t = 0$. Therefore, the system Equation (9) can be reduced to

$$\begin{aligned} &\tau_t = -2c_3, \xi_{xx} = 0, \phi = c_4 \\ &\mu\phi + 2(\tau_t - 2\xi_x) = 0, \\ &\phi D_u + 2(\tau_t - \xi_x)D = 0, \\ &-2\tau_t F - \phi F_u = 0. \end{aligned} \tag{16}$$

where c_4 is a constant. From the last two equations of system Equation (16), we can obtain $D = de^{\nu u}$ and $F = fe^{\gamma u}$ mod $G^{\sim}$, where $d,\ f$ are two arbitrary constants. Substituting them into the system Equation (16) we can obtain

$$\begin{aligned} &\tau_t = -2c_3, \xi_{xx} = 0, \phi = c_4, \\ &\mu\phi + 2(\tau_t - 2\xi_x) = 0, \\ &d[\nu\phi + 2(\tau_t - \xi_x)] = 0, \\ &f[\phi\gamma + 2\tau_t] = 0. \end{aligned} \tag{17}$$

The last two equations of the above system are two classifying conditions, which can be decomposed into four cases:

$$(i)\ \ d = 0, f = 0; \quad (ii)\ \ d = 0, f \neq 0; \quad (iii)\ \ d \neq 0, f \neq 0; (iv)\ \ d \neq 0, f = 0.$$

(i) For this case the system Equation (17) can be reduced to

$$\tau_t = -2c_3, \quad \xi_{xx} = 0, \quad \phi = c_4, \quad \mu\phi + 2(\tau_t - 2\xi_x) = 0.$$

Solving this system, we can obtain $\tau = -2c_3 t + c_1, \xi = (\frac{\mu}{4}c_4 - c_3)x + c_2, \phi = c_4$, which is corresponding to Case 3.

(ii) For $d = 0, f \neq 0$, the system Equation (17) can be reduced to

$$\tau_t = -2c_3, \quad \xi_{xx} = 0, \quad \phi = c_4, \quad \mu\phi + 2(\tau_t - 2\xi_x) = 0, \quad \gamma\phi + 2\tau_t = 0,$$

from which we can obtain $c_3 = \frac{\gamma}{4}c_4, \tau = -\frac{\gamma}{2}c_4 t + c_1, \xi = \frac{\mu-\gamma}{4}c_4 x + c_2, \phi = c_4$, which is corresponding to Case 4.

(iii) When $d \neq 0, f \neq 0$, the system Equation (17) can be reduced to

$$\begin{aligned} &\tau_t = -2c_3, \quad \xi_{xx} = 0, \quad \phi = c_4, \quad \mu\phi + 2(\tau_t - 2\xi_x) = 0, \\ &\nu\phi + 2(\tau_t - \xi_x) = 0, \quad \phi\gamma + 2\tau_t = 0. \end{aligned}$$

Solving the above system, we can obtain $\gamma = 2\nu - \mu, c_3 = \frac{\gamma}{4}c_4, \tau = -\frac{\gamma}{2}c_4 t + c_1, \xi = \frac{\mu-\gamma}{4}c_4 x + c_2, \phi = c_4$, which is corresponding to Case 5.

(iv) For $d \neq 0, f = 0$, the system Equation (17) can be reduced to

$$\tau_t = -2c_3, \quad \xi_{xx} = 0, \quad \phi = c_4, \quad \mu\phi + 2(\tau_t - 2\xi_x) = 0, \quad \nu\phi + 2(\tau_t - \xi_x) = 0,$$

from which we can obtian $c_3 = \frac{2\nu-\mu}{4}c_4, \tau = \frac{\mu-2\nu}{2}c_4 t + c_1, \xi = \frac{\mu-\nu}{2}c_4 x + c_2, \phi = c_4$, which is corresponding to Case 5.

Case 2.2: Consider the case $K = u^\mu(\mu \neq 0)$. The third equation of system Equation (9) implies $\mu\phi_{xu}u^{\mu-1} = 0$, thus we have $\phi_{xu} = 0$. Then, from the fourth equation and the fifth equation we can obtain $\xi_{xx} = 0, \phi_x = 0$, which implies $\alpha(x) = c_3, \phi^0(t,x) = \phi^0(t), \xi(x) = c_1 x + c_2$, and thus the system Equation (9) can be reduced to

$$\begin{array}{l} \xi(x) = c_1 x + c_2, \\ \phi = (\frac{1}{2}\tau_t + c_3)u + \phi^0(t), \\ \mu\phi + 2(\tau_t - 2\xi_x)u = 0, \\ \phi D_u + 2(\tau_t - \xi_x)D = 0, \\ \frac{1}{2}\tau_{ttt}u + \phi^0_{tt} + (\phi_u - 2\tau_t)F - \phi F_u = 0. \end{array} \tag{18}$$

The second and the third equations of system Equation (18) implies

$$[(\tfrac{\mu}{2}+2)\tau_t + \mu c_3 - 4\xi_x]u + \mu\phi^0(t) = 0, \quad (\tfrac{\mu}{2}+2)\tau_{tt} = 0. \tag{19}$$

Thus we have $\phi^0(t) = 0$, and there exist two cases: (I) $\tau_{tt} \neq 0$ or (II) $\tau_{tt} = 0$.

(I) For $\tau_{tt} \neq 0$, from system Equation (19) we have $\mu = -4, c_1 = -c_3$. Thus the system Equation (18) can be reduced to

$$\begin{array}{l} \phi = (\frac{1}{2}\tau_t + c_3)u, \\ \phi D_u + 2(\tau_t + c_3)D = 0, \\ \frac{1}{2}\tau_{ttt}u + (\phi_u - 2\tau_t)F - \phi F_u = 0. \end{array} \tag{20}$$

Substituting the first equation of system Equation (20) into the second equation of system Equation (20) and then take the derivative for it with respect to the variable t, we have

$$(uD_u + 4D)\tau_{tt} = 0, \quad c_3(uD_u + 2D) = 0. \tag{21}$$

Solving the first equation of the above system we can obtain $D(u) = du^{-4}$, where d is an arbitrary constant. Substituting it into the second equation of the system Equation (21), we can obtain a classifying condition $dc_3 = 0$, which imply there exist two cases: (i) $c_3 = 0$ or (ii) $c_3 \neq 0$.

(i) For $c_3 = 0$, we have $c_1 = 0$, and thus the last equations of system Equation (20) can be reduced to

$$uF_u + 3F - \frac{\tau_{ttt}}{\tau_t}u = 0, \tag{22}$$

which implies $\frac{\tau_{ttt}}{\tau_t} = \omega$, where ω is a constant. When $\omega = 0$, from Equation (22) we have $F(u) = fu^{-3}$, where f is a constant. Furthermore, $\tau_{ttt} = 0$ and $c_1 = 0$ imply $\tau = c_6 t^2 + c_5 t + c_4$, $\xi = c_2$, and thus $\phi = (c_6 t + \frac{c_5}{2})u$, which is corresponding to Case 6. When $\omega > 0$, we can get $\tau = c_4 + c_5 e^{\sqrt{\omega}t} + c_6 e^{-\sqrt{\omega}t}, \xi = c_2, \phi = \frac{1}{2}(c_5\sqrt{\omega}e^{\sqrt{\omega}t} - c_6\sqrt{\omega}e^{-\sqrt{\omega}t})u$, and $F(u) = \frac{1}{4}\omega u + fu^{-3}$. which is corresponding to Case 7. When $\omega < 0$, we can get $\tau = c_4 + c_5\sin(\sqrt{-\omega}t) + c_6\cos(\sqrt{-\omega}t)$, $\xi = c_2$, $\phi = \frac{1}{2}(c_5\sqrt{-\omega}\cos(\sqrt{-\omega}t) - c_6\sqrt{-\omega}\sin(\sqrt{-\omega}t))u$, and $F(u) = \frac{1}{4}\omega u + fu^{-3}$. Which is corresponding to Case 8.

(ii) For $c_3 \neq 0$, we have $d = 0$ and $D(u) = 0$, and thus the last equations of system Equation (20) can be reduced to

$$\tfrac{1}{2}\tau_t(uF_u + 3F) - \tfrac{1}{2}\tau_{ttt}u + c_3(uF_u - F) = 0, \tag{23}$$

Taking the derivative of Equation (23) with respect to the variable t, we have

$$uF_u + 3F - \frac{\tau_{tttt}}{\tau_{tt}}u = 0.$$

which implies $\frac{\tau_{tttt}}{\tau_{tt}} = \lambda$.

When $\lambda = 0$, we have $\tau_{tttt} = 0$ and $F(u) = fu^{-3}$, where f is an arbitrary constant. Substituting $F(u) = fu^{-3}$ into Equation (23), it can be reduced to

$$4c_3 f u^{-3} + \frac{1}{2}\tau_{ttt} u = 0.$$

which implies $\tau_{ttt} = 0$ and $f = 0$. Then we can get $\tau = c_6 t^2 + c_5 t + c_4, \xi = -c_3 x + c_2$, and thus $\phi = (c_6 t + \frac{c_5}{2} + c_3)u$, which is corresponding to Case 9.

When $\lambda \neq 0$, we can get $\tau_{tttt} \neq 0$, and the Equation (23) can be separated into

$$\tfrac{1}{2}\tau_t(uF_u + 3F) - \tfrac{1}{2}\tau_{ttt} u = 0, \quad c_3(uF_u - F) = 0, \tag{24}$$

From the second equation of Equation (24), we can get $F(u) = fu$, where f is an arbitrary constant. Substituting $F(u) = fu$ into the first equation of Equation (24), we can obtain

$$4fu - \frac{\tau_{ttt}}{\tau_t} u = 0.$$

which implies $\frac{\tau_{ttt}}{\tau_t} = 4f$ and $f \neq 0$. When $f > 0$, we can get $\tau = c_4 + c_5 e^{2\sqrt{f}t} + c_6 e^{-2\sqrt{f}t}$, $\xi = -c_3 x + c_2, \phi = (c_5\sqrt{f}e^{2\sqrt{f}t} - c_6\sqrt{f}e^{-2\sqrt{f}t} + c_3)u$, and $F(u) = fu(f > 0)$, which is corresponding to Case 10. When $f < 0$, we can get $\tau = c_4 + c_5 \sin(2\sqrt{-f}t) + c_6\cos(2\sqrt{-f}t)$, $\xi = -c_3 x + c_2$, $\phi = (c_5\sqrt{-f}\cos(2\sqrt{-f}t) - c_6\sqrt{-f}\sin(2\sqrt{-f}t) + c_3)u$, and $F(u) = fu(f < 0)$, which is corresponding to Case 11.

(II) $\tau_{tt} = 0$ implies τ_t is a constant. Let $\frac{1}{2}\tau_t + c_3 \triangleq c$, the system Equation (18) can be reduced to

$$\begin{aligned}
&\tau_{tt} = 0, \xi(x) = c_1 x + c_2, \phi = cu,\\
&\mu\phi + 2(\tau_t - 2\xi_x)u = 0,\\
&\phi D_u + 2(\tau_t - \xi_x)D = 0,\\
&\phi F_u - (\phi_u - 2\tau_t)F = 0.
\end{aligned} \tag{25}$$

As one can see, for any symmetry operator, equations $\phi D_u + 2(\tau_t - \xi_x)D = 0$ and $\phi F_u + (\phi_u - 2\tau_t)F = 0$ give some equations (not greater than 2 each) on D and F of the general form

$$cuD_u + rD = 0, \quad cuF_u + sF = 0,$$

where c, r and s are constants. Solving this system, we can obtain $D = du^{\nu}$ and $F = fu^{\gamma}$ mod $G^{\sim}$, where d, f are two arbitrary constants. Substituting them into the system Equation (25) we can get

$$\begin{aligned}
&\tau_{tt} = 0, \xi(x) = c_1 x + c_2, \phi = cu,\\
&c\mu + 2(\tau_t - 2\xi_x) = 0,\\
&d[c\nu + 2(\tau_t - \xi_x)] = 0,\\
&f[c\gamma - (c - 2\tau_t)] = 0.
\end{aligned} \tag{26}$$

The last two equations of the above system are two classifying conditions, which can be decomposed into four cases:

$$(i)\ \ d = 0, f = 0; \quad (ii)\ \ d = 0, f \neq 0; \quad (iii)\ \ d \neq 0, f \neq 0; (iv)\ \ d \neq 0, f = 0.$$

(i) For this case, the system Equation (26) can be reduced to

$$\tau_{tt} = 0, \quad \xi(x) = c_1 x + c_2, \quad \phi = cu, \quad c\mu + 2(\tau_t - 2\xi_x) = 0.$$

Solving the above system, we can obtain $\tau = (2c_1 - \frac{\mu}{2}c)t + c_0, \xi = c_1 x + c_2, \phi = cu$, which is corresponding to Case 12.

(ii) When $d = 0, f \neq 0$, the system Equation (26) can be reduced to

$$\tau_{tt} = 0, \quad \xi(x) = c_1 x + c_2, \quad \phi = cu, \quad c\mu + 2(\tau_t - 2\xi_x) = 0, \quad c\gamma - (c - 2\tau_t) = 0,$$

from which we can obtain $\tau = \frac{1-\gamma}{2}ct + c_0, \xi = \frac{1-\gamma+\mu}{4}cx + c_2, \phi = cu$, which is corresponding to Case 13.

(iii) For $d \neq 0, f \neq 0$, the system Equation (26) can be reduced to

$$\begin{aligned} &\tau_{tt} = 0, \quad \xi(x) = c_1 x + c_2, \quad \phi = cu, \quad c\mu + 2(\tau_t - 2\xi_x) = 0, \\ &c\nu + 2(\tau_t - \xi_x) = 0, \quad c\gamma - (c - 2\tau_t) = 0. \end{aligned}$$

Solving the above system, we can obtain $\gamma = 1 - \mu + 2\nu, \tau = \frac{\mu-2\nu}{2}ct + c_0, \xi = \frac{\mu-\nu}{2}cx + c_2, \phi = cu$, which is corresponding to Case 14.

(iv) When $d \neq 0, f = 0$, the system Equation (26) can be reduced to

$$\tau_{tt} = 0, \quad \xi(x) = c_1 x + c_2, \quad \phi = cu, \quad c\mu + 2(\tau_t - 2\xi_x) = 0, \quad c\nu + 2(\tau_t - \xi_x) = 0,$$

from which we can obtain $\tau = \frac{\mu-2\nu}{2}ct + c_0, \xi = \frac{\mu-\nu}{2}cx + c_2, \phi = cu$, which is corresponding to Case 14.

Case 3: $k = 2$. The assumption of two independent equations of form of the fifth equation of system Equation (9) for K yields $K = \text{const}$, i.e. $K = 1 \bmod G^{\sim}$. From the fifth equation of system Equation (9) we have $\tau_t = 2\xi_x$. Thus, we can get $\tau_{tt} = 0, \xi_{xx} = 0, \phi_{tu} = \phi_{xu} = 0$, which implies system Equation (9) can be reduced to

$$\begin{aligned} &\tau_{tt} = \xi_{xx} = \phi_{tu} = \phi_{xu} = 0, \quad \tau_t - 2\xi_x = 0, \quad 2\phi_x D_u = 0, \\ &\phi D_u + 2(\tau_t - \xi_x)D = 0, \quad \phi F_u - (\phi_u - 2\tau_t)F + \phi_{xx} D - \phi_{xxxx} - \phi_{tt} = 0. \end{aligned} \tag{27}$$

In a way similar to the proof in case 2.2, from the fourth equation of system Equation (27) we obtain the following different values of $D(u)$: $D(u) = \forall$, or $de^{\nu u}$, or $du^{\nu}(d \neq 0, \nu \neq 0)$, or d. Furthermore, taking derivative of the last equation of system Equation (27) with respect to u, we can get $\phi F_{uu} + 2\tau_t F_u = 0$, which give some equations (not greater than 3) on F of the general form

$$(mu + n)F_{uu} + wF_u = 0,$$

for any symmetry operator, where m, n and w are constants. Solving this equation up to $G^{\sim}$, we obtain the following different values of $F(u)$: $F(u) = fe^{\gamma u}$, or $fu^{\gamma}(f \neq 0, \gamma \neq 0)$, or $f \ln u \bmod G^{\sim}$. Therefore, to complete the classification we have to consider all possible combinations of the values of $D(u)$ and $F(u)$. Substituting all the different combinations of $D(u)$ and $K(u)$ to the system Equation (27), we can find three nontrivial inequivalent combinations: (I) $D(u) = de^{\nu u}(d \neq 0, \nu \neq 0), F(u) = fe^{\gamma u}(f \neq 0, \gamma \neq 0)$; (II) $D = du^{\nu}(d \neq 0, \nu \neq 0), F = fu^{\gamma}(f \neq 0, \gamma \neq 0)$ and (III) $D = d, F = fu$.

(I) For the first combination, substituting $D(u) = de^{\nu u}(d \neq 0, \nu \neq 0)$ into the fourth equation of the system Equation (27), we can obtain

$$\phi_x D_u = 0, \quad \phi_t D_u = 0.$$

Thus we have $\phi = cu + c_3$, and the system Equation (27) can be reduced to

$$\begin{aligned}&\tau_{tt} = 0, \quad \xi_{xx} = 0, \quad \phi = cu + c_3, \quad \tau_t - 2\xi_x = 0,\\&(cu + c_3)\nu + 2(\tau_t - \xi_x) = 0, \quad (cu + c_3)\gamma - (c - 2\tau_t) = 0.\end{aligned}$$

The above system implies $\tau = -c_3\nu t + c_1$, $\xi = -\frac{1}{2}c_3\nu x + c_2$, $\phi = c_3$ and $\gamma = 2\nu$, where $c_i(i = 1, \ldots, 3)$ are arbitrary constants, which is corresponding to Case 5 with $\mu = 0$.

(II) For the second combination, substituting $D = du^\nu (d \neq 0, \nu \neq 0)$ into the fourth equation of the system Equation (27), we can obtain

$$\phi_x D_u = 0, \quad \phi_t D_u = 0.$$

Thus we have $\phi = cu + c_3$, and the system Equation (27) can be reduced to

$$\begin{aligned}&\tau_{tt} = 0, \quad \xi_{xx} = 0, \quad \phi_x = 0, \quad \phi = cu + c_3, \quad \tau_t - 2\xi_x = 0,\\&c_3\nu + [c\nu + 2(\tau_t - \xi_x)]u = 0, \quad (cu + c_3)\gamma - (c - 2\tau_t)u = 0.\end{aligned}$$

The above system implies $\tau = -c\nu t + c_1$, $\xi = -\frac{1}{2}c\nu x + c_2$, $\phi = cu$ and $\gamma = 1 + 2\nu$, where $c_i(i = 1, \ldots, 3)$ are arbitrary constants, which is corresponding to Case 14 with $\mu = 0$.

(III) For the last combination, substituting them into the original system Equation (27), we can obtain

$$\begin{aligned}&\tau_{tt} = 0, \quad \xi_{xx} = 0, \quad \phi = cu + b(t,x), \quad \tau_t - 2\xi_x = 0, \quad d(\xi_x - \tau_t) = 0,\\&b_{tt} + b_{xxxx} - db_{xx} + f[(c - 2\tau_t)u - (cu + b(t,x))] = 0.\end{aligned} \tag{28}$$

The fifth equation of the above system is a classifying condition, which leads to two cases: (i) $d = 0$; (ii) $d \neq 0$.

(i) For $d = 0$, the last equation of the system Equation (28) can be reduced to

$$b_{tt} + b_{xxxx} - fb(t,x) - 2\tau_t fu = 0,$$

from which for $f = 0$ we can obtian $\tau = 2c_3 t + c_1, \xi = c_3 x + c_2, \phi = cu + b(t,x)$, and $b(t,x)$ is satisfied with the equation:

$$b_{tt} + b_{xxxx} = 0$$

which is corresponding to Case 15. For $f \neq 0$, we can obtain $\tau = c_1, \xi = c_2, \phi = cu + b(t,x)$, and $b(t,x)$ is satisfied with the equation:

$$b_{tt} + b_{xxxx} - fb(t,x) = 0$$

which is corresponding to Case 16 with $d = 0$.

(ii) For $d \neq 0$, the system Equation (28) implies $\tau = c_1$, $\xi = c_2$, $\phi = cu + b(t,x)$, where $c_i(i = 1, \ldots, 3)$ are arbitrary constants, and $b(t,x)$ is satisfied with the equation:

$$b_{tt} - db_{xx} + b_{xxxx} - fb(t,x) = 0.$$

which is corresponding to Case 16. □

3. Symmetry Reduction and Exact Solutions

The Lie symmetry operators in the Table 1 found as a result of the group classification problem can be applied to construction exact solutions of the corresponding equations by means of symmetry reduction. The method of symmetry reduction with respect to subalgebras of Lie invariance algebras is well-known and quite algorithmic to use in most cases, we refer to the standard textbooks on the subject [21,59]. Below, we perform symmetry reduction for one classification model.

We consider Equation (1) with the fixed values of the parameter-functions $K = u^{\mu}$, $D = u^{\nu}$ and $F = u^{1-\mu+2\nu}$, i.e.,

$$u_{tt} = -[u^{\mu}u_{xxx}]_x + [u^{\nu}u_x]_x + u^{1-\mu+2\nu}, \tag{29}$$

where μ, ν are two arbitrary constants. As shown in case 14 with $d = f = 1$ of the Table 1, the Equation (29) admits the three-dimensional Lie invariance algebra $\mathfrak{g}$ generated by the operators

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = \frac{1}{2}(\mu - 2\nu)t\partial_t + \frac{1}{2}(\mu - \nu)x\partial_x + u\partial_u,$$

which is equivalent to

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + \frac{\mu-\nu}{\mu-2\nu}x\partial_x + \frac{2}{\mu-2\nu}u\partial_u.$$

Because these operators satisfy the commutation relations

$$[Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = \frac{\mu-\nu}{\mu-2\nu}Q_2,$$

and thus the corresponding symmetry algebra $\mathfrak{g}$ is a realization of the algebra $A^{\alpha}_{3,5}$ [60], where $0 < |\frac{\mu-\nu}{\mu-2\nu}| < 1$. A complete list of inequivalent non-zero one-dimensional subalgebras of $\mathfrak{g}$ is exhausted by the algebras [60]

$$\langle Q_1\rangle, \quad \langle Q_2\rangle, \quad \langle Q_3\rangle, \quad \langle Q_1 + \alpha Q_2\rangle,$$

where $\alpha = \pm 1$. Lie reduction of Equation (29) to ordinary differential equations (ODEs) can be made with the one-dimensional subalgebra $\langle Q_1\rangle$, $\langle Q_2\rangle$, $\langle Q_3\rangle$ and $\langle Q_1 + \alpha Q_2\rangle$. The associated ansatz and the reduced ODEs are listed in Table 2.

Table 2. Reduced ODEs for Equation (29) ($\alpha = \pm 1$).

N	*Snbalgebra*	Ansatz for u	y	Reduced ODE
1	$< Q_1 >$	$h(y)$	x	$-\mu h^{\mu-1}h'h''' - h^{\mu}h'''' + \nu h^{\nu-1}h'^2 + h^{\nu}h'' + h^{1-\mu+2\nu} = 0$
2	$< Q_2 >$	$h(y)$	t	$h'' = h^{1-\mu+2\nu}$
3	$< Q_3 >$	$h(y)t^{\frac{2}{\mu-2\nu}}$	$xt^{-\frac{\mu-\nu}{\mu-2\nu}}$	$(\mu-\nu)^2y^2h'' + (\mu-\nu)(2\mu-3\nu-4)yh' + 2(2-\mu+2\nu)h =$ $(\mu-2\nu)^2(-\mu h^{\mu-1}h'h''' - h^{\mu}h'''' + \nu h^{\nu-1}h'^2 + h^{\nu}h'' + h^{1-\mu+2\nu})$
4	$< Q_1 + \alpha Q_2 >$	$h(y)$	$x - \alpha t$	$h'' = -\mu h^{\mu-1}h'h''' - h^{\mu}h'''' + \nu h^{\nu-1}h'^2 + h^{\nu}h'' + h^{1-\mu+2\nu}$

Solving the reduced ODEs in Table 2, we can obtain some exact solutions of the corresponding Equation (29) by using ansatz. For example, solving the ODE corresponding to the case 4 of Table 2 with $\mu = -1, \nu = -1$ by using the extended tanh function method [61,62], we can obtain the following exact travelling wave solutions for the nonlinear beam-like Equation (29) with $\mu = -1, \nu = -1$:

Case 1. Solitary wave solutions:

$$u = 12R\tanh^2[\sqrt{-R}(x - \alpha t)] - 8R + 1, \quad u = 12R\coth^2[\sqrt{-R}(x - \alpha t)] - 8R + 1;$$

where $R < 0$.

Case 2. Rational solutions:

$$u = -12(x - \alpha t)^{-2} + 1.$$

Case 3. Traingular periodic wave solutions:

$$u = -12R\tan^2[\sqrt{R}(x - \alpha t)] - 8R + 1, \quad u = -12R\cot^2[\sqrt{R}(x - \alpha t)] - 8R + 1,$$

where $R > 0$.

4. Conclusions and Discussion

In this paper we present a complete group classification of the generalized nonlinear beam Equation (1) by using the classical Lie-Ovsiannikov method based on equivalence transformations. The main results on classification are collected in Table 1 where we list sixteen inequivalent cases of extensions with the corresponding Lie invariance algebras. It should be noted that the right-hand side of Equation (1) is exactly the same as the generalized thin film Equation (1.1) in [58]. Compare the classification results of Equation (1.1) in [58], we know that the generalized nonlinear beam Equation (1) have more inequivalent classification cases than the generalized thin film equation (eight cases). Furthermore, the maximal extension of Lie invariance algebra for the generalized nonlinear beam Equation (1) is five dimension, while for the generalized thin film equation it is four dimension. Therefore, the classification results of the two equations are different.

For one classification models from the list we perform corresponding Lie symmetry reduction which are presented in Table 2. This enabled to obtain some exact solutions for the equation under consideration, including solitary wave solutions, triangular periodic wave solutions and rational solutions. These results may lead to further applications in physics and engineering such as tests in numerical solutions of Equation (1) and as trial functions for application of variational approach in the analysis of different perturbed versions of Equation (1). Other topics including nonclassical symmetry, non-Lie exact solutions and physical applications of class Equation (1) will be studied in subsequent publication.

Acknowledgments: This work was partially supported by the National Natural Science Foundation of China (Grant No. 11501204), the Natural Science Foundation of Shanghai (Grant No. 15ZR1408300), and Shanghai Key Laboratory of Intelligent Information Processing (Grant No. IIPL-2014-001).

Author Contributions: All the authors D.J.H., X.X.L. and S.C.Y.(check all the computations) make contributions to Sections 2 and 3. D.J.H make contribution to Sections 1 and 4.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ames, W.F. *Nonlinear Partial Differential Equations in Engineering*; Academic: New York, NY, USA, 1972; Volume II, pp. 50–52.
2. Ames, W.F.; Adams, E.; Lohner, R.G. Group properties of $u_{tt} = [f(u)u_x]_x$. *Int. J. Non-Linear Mech.* **1981**, *16*, 439–447.
3. Galaktionov, V.A. The formation of shocks and fundamental solution of a fourth-order quasilinear Boussinesq-type equation. *Nonlinearity* **2009**, *22*, 239–257.
4. Favini, A.; Goldstein, G.R.; Goldstein, J.A.; Romanelli, S. Classification of general Wentzell boundary conditions for fourth order operators in one space dimension. *J. Math. Anal. Appl.* **2007**, *335*, 219–235.
5. McKenna, P.J.; Walter, W. Travelling waves in a suspension bridges. *SIAM J. Appl. Math.* **1990**, *50*, 703–715.
6. Ammann, O.H.; Karman, T.V.; Woodruff, G.B. *The Failure of the Tacoma Narrow Bridge*; Federal Works Agency: Washington, DC, USA, 1941.
7. Champreys, A.R.; McKenna, P.J.; Zegeling, P.A. Solitary waves in nonlinear beam equations: Stability, fission and fusion. *Nonlinear Dynam.* **2000**, *21*, 31–53.
8. McKenna, P.J.; Walter, W. Nonlinear oscillations in a suspension bridge. *Arch. Ration. Mech. Anal.* **1987**, *98*, 167–177.
9. Chen, Y.; McKenna, P.J. Traveling waves in a nonlinearly suspended beam: Theoretical results and numerical observations. *J. Differ. Equ.* **1997**, *136*, 325–355.
10. Chen, Y.; McKenna, P.J. Traveling waves in a nonlinearly suspended beam: Some computational results and four open questions. *Phil. Trans. R. Soc. Lond. A* **1997**, *355*, 2175–2184.
11. Choy, Y.S.; Jen, K.S.; McKenna, P.J. The structure of the solution set for periodic oscillations in a suspension bridge model. *IMA J. Appl. Math.* **1991**, *47*, 283–306.
12. Humphreys, L.D. Numerical mountain pass solutions of a suspension bridge equation. *Nonlinear Anal. TMA* **1997**, *35*, 1811–1826.

13. Lazer, A.C.; McKenna, P.J. Large Amplitude periodic oscillation in suspension bridges: Some new connections with nonlinear analysis. *SIAM Rev.* **1990**, *32*, 537–578.
14. Humphreys, L.D.; McKenna, P.J. Multiple periodic solutions for a nonlinear suspension bridge equation. *IMA J. Appl. Math.* **1999**, *63*, 37–49.
15. Doole, S.H.; Hogan, S.J. The nonlinear dynamics of suspension bridges under harmonic forcing. *Appl. Nonlinear Math. Rep.* **1996**, *76*, 127-128.
16. Doole, S.H.; Hogan, S.J. A piecewise linear suspension bridge model nonlinear dynamics and orbit continuation. *Dynam. Stabil. Syst.* **1996**, *11*, 19–47.
17. Bruzón, M.S.; Camacho, J.C.; Gandarias, M.L. Similarity reductions of a nonlinear model for vibrations of beams. *PAMM Proc. Appl. Math. Mech.* **2007**, *7*, 2040063–2040064.
18. Camacho, J.C.; Bruzón, M.S.; Ramírez, J.; Gandarias, M.L. Exact travelling wave solutions of a beam equation. *J. Nonlinear Math. Phys.* **2011**, *18*, 33–49.
19. Gao, Y.X. Quasi-periodic Solutions of the General Nonlinear Beam Equations. *Commun. Math. Res.* **2012**, *28*, 51–64.
20. Ovsiannikov, L.V. Group properties of the nonlinear heat-conduction equation. *Dokl. Akad. Nauk SSSR*, **1959**, *V.125*, 492–495. (In Russian)
21. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
22. Bluman, G.; Cheviakov, A.; Anco, S. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: Berlin, Germany, 2010.
23. Barone, A.; Esposito, F.; Magee, C.G.; Scott, A.C. Theory and applications of the sine-Gordon equation. *Riv. Nuovo Cimento* **1971**, *1*, 227–267.
24. Arrigo, D.J. Group properties of $u_{xx} - u_y^m u_{yy} = f(u)$. *Int. J. Non-Linear Mech.* **1991**, *26*, 619–629.
25. Pucci, E.; Salvatori, M.C. Group properties of a class of semilinear hyperbolic equations. *Int. J. Non-Linear Mech.* **1986**, *21*, 147–155.
26. Torrisi, M.; Valenti, A. Group properties and invariant solutions for infinitesimal transformations of a nonlinear wave equation. *Int. J. Non-Linear Mech.* **1985**, *20*, 135–144.
27. Donato, A. Similarity analysis and nonlinear wave propagation. *Int. J. Non-Linear Mech.* **1987**, *22*, 307–314.
28. Ibragimov, N.H.; Torrisi, M.; Valenti, A. Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$. *J. Math. Phys.* **1991**, *32*, 2988–2995.
29. Ibragimov, N.H. (Ed.) *Lie Group Analysis of Differential Equations—Symmetries, Exact Solutions and Conservation Laws, V.1.*; CRC Press: Boca Raton, FL, USA, 1994.
30. Oron, A.; Rosenau, P. Some symmetries of the nonlinear heat and wave equations. *Phys. Lett. A* **1986**, *118*, 172–176.
31. Chikwendu, S.C., Non-linear wave propagation solutions by Fourier transform perturbation. *Int. J. Non-Linear Mech.* **1981**, *16*, 117–128.
32. Gandarias, M.L.; Torrisi, M.; Valenti, A. Symmetry classification and optimal systems of a non-linear wave equation. *Int. J. Non-Linear Mech.* **2004**, *39*, 389–398.
33. Pucci, E. Group analysis of the equation $u_{tt} + \lambda u_{xx} = g(u, u_x)$. *Riv. Mat. Univ. Parma* **1987**, *12*, 71–87.
34. Bluman, G.W.; Cheviakov, A.F. Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation. *J. Math. Anal. Appl.* **2007**, *333*, 93–111.
35. Bluman, G. W.; Kumei, S. *Symmetries and Differential Equations*: Springer: Berlin, Germany, 1989.
36. Bluman, G.W.; Temuerchaolu; Sahadevan, R. Local and nonlocal symmetries for nonlinear telegraph equation. *J. Math. Phys.* **2005**, *46*, 023505.
37. Huang, D.J.; Ivanova, N.M. Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations. *J. Math. Phys.* **2007**, *48*, 073507.
38. Huang, D.J.; Zhou, S.G. Group properties of generalized quasi-linear wave equations. *J. Math. Anal. Appl.* **2010**, *366*, 460–472.
39. Huang, D.J.; Zhou, S.G. Group-theoretical analysis of variable coefficient nonlinear telegraph equations. *Acta Appl. Math.* **2012**, *117*, 135–183.
40. Lahno, V.; Zhdanov R.; Magda, O. Group classification and exact solutions of nonlinear wave equations. *Acta Appl. Math.* **2006**, *91*, 253–313.
41. Sophocleous, C.; Kingston, J.G. Cyclic symmetries of one-dimensional non-linear wave equations. *Int. J. Non-Linear Mech.* **1999**, *34*, 531–543.

42. Suhubi, E.S.; Bakkaloglu, A. Group properties and similarity solutions for a quasi-linear wave equation in the plane. *Int. J. Non-Linear Mech.* **1991**, *26*, 567–584.
43. Vasilenko, O.F.; Yehorchenko, I.A. Group classification of multidimensional nonlinear wave equations. *Proc. Inst. Math. NAS Ukr.* **2001**, *36*, 63–66.
44. Cherniha, R.; Serov, M.; Rassokha, I. Lie symmetries and form-preserving transformations of reaction-diffusion-convection equations. *J. Math. Anal. Appl.* **2008**, *342*, 1363–1379.
45. Basarab-Horwath, P.; Lahno, V.I.; Zhdanov, R.Z. The structure of Lie algebras and the classification problem for partial differential equations. *Acta Appl. Math.* **2001**, *69*, 43–94.
46. Zhdanov, R.Z.; Lahno, V.I. Group classification of heat conductivity equations with a nonlinear source. *J. Phys. A* **1999**, *32*, 7405–7418.
47. Gazeau, J.P.; Winternitza, P. Symmetries of variable coefficient Korteweg-de Vries equations. *J. Math. Phys.* **1992**, *33*, 4087–4102.
48. Popovych, R.O.; Kunzinger, M.; Eshraghi, H. Admissible point transformations and normalized classes of nonlinear Schrödinger equations. *Acta Appl. Math.* **2010**, *109*, 315–359.
49. Nikitin, A.G.; Popovych, R.O. Group classification of nonlinear Schrödinger equations. *Ukr. Math. J.* **2001**, *53*, 1053–1060.
50. Popovych, R.O.; Ivanova, N.M. New results on group classification of nonlinear diffusion-convection equations. *J. Phys. A* **2004**, *37*, 7547–7565.
51. Ivanova, N.M.; Popovych, R.O.; Sophocleous, C. Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification. *Lobachevskii J. Math.* **2010**, *31*, 100–122.
52. Huang, D.J.; Yang, Q.M.; Zhou, S.G. Lie symmetry classification and equivalence transformation of variable coefficient nonlinear wave equations with power nonlinearities. *Chin. J. Contemp. Math.* **2012**, *33*, 205–214.
53. Huang, D.J.; Yang, Q.M.; Zhou, S.G. Conservation law classification of variable coefficient nonlinear wave equation with power Nonlinearity. *Chin. Phys. B* **2011**, *20*, 070202.
54. Huang, D.J.; Ivanova, N.M. Algorithmic framework for group analysis of differential equations and its application to generalized Zakharov-Kuznetsov equations. *J. Differ. Equ.* **2016**, *260*, 2354–2382.
55. Cherniha, R.; King, J.R. Lie symmetries of nonlinear multidimensional reaction-diffusion systems: I. *J. Phys. A* **2000**, *33*, 267–282, 7839–7841.
56. Cherniha, R.; King, J.R. Lie symmetries of nonlinear multidimensional reaction-diffusion systems: II. *J. Phys. A* **2003**, *36*, 405–425.
57. Cherniha, R.; King, J.R. Lie symmetries and conservation laws of nonlinear multidimensional reaction-diffusion systems with variable diffusivities. *IMA J. Appl. Math.* **2006**, *71*, 391–408.
58. Cherniha, R.; Myroniuk, L. Lie Symmetries and Exact Solutions of the Generalized Thin Film Equation. *J. Phys. Math.* **2010**, *2*, P100508:1–P100508:19.
59. Olver, P.J. *Application of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
60. Patera, J.; Winternitz, P. Subalgebras of real three- and four-dimensional Lie algebras. *J. Math. Phys.* **1977**, *18*, 1449–1455.
61. Malfliet, W.; Hereman, W. The tanh method. I. Exact solutions of nonlinear evolution and wave equations. *Phys. Scr.* **1996**, *54*, 563–568.
62. Fan, E.G. Extended tanh-function method and its applications to nonlinear equations. *Phys. Lett. A* **2000**, *277*, 212–218.