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On Extended Representable Uninorms and Their Extended Fuzzy Implications (Coimplications)

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Received: 4 August 2017; Accepted: 14 August 2017; Published: 18 August 2017

Abstract: In this work, by Zadeh's extension principle, we extend representable uninorms and their fuzzy implications (coimplications) to type-2 fuzzy sets. Emphatically, we investigate in which algebras of fuzzy truth values the extended operations are type-2 uninorms and type-2 fuzzy implications (coimplications), respectively.

Keywords: type-2 fuzzy sets; extended operations; uninorms; fuzzy implications; fuzzy coimplications

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1. Introduction

Type-2 fuzzy sets, which were introduced by Zadeh [1] in 1975, are an extension of the ordinary (type-1) fuzzy sets since truth values of the latter are precise on the unit interval $[0,1]$, while the former are equipped with fuzzy truth value mappings from $[0,1]$ to itself. Type-2 fuzzy sets are used mainly in different control systems [2–8] and other related fields [9–15].

There is some literature studying operations on type-2 fuzzy sets, such as type-2 aggregations [16], type-2 t-(co)norms [17–20], type-2 negations [21] and type-2 fuzzy implications [22], and other operations [23–29] and so on. All of the results obtained in the above work are based on continuous type-1 operations. On the other hand, uninorms, which are a generalization of t-norms and t-conorms, are not continuous if their neutral elements are in the open interval $(0,1)$. Fuzzy implications (coimplications) [30,31] also are important operations in fuzzy logic and applied in related fields [32–34]. By using uninorms and other fuzzy logic operations, we can construct fuzzy implications (coimplications), such as (U,N)- and RU-implications (coimplications) [32,35] (Their concepts can be seen from Definitions 9 and 10 in this work, respectively). The well-known classes of uninorms are the $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ classes [36], representable uninorms [36], idempotent uninorms [37,38] and uninorms continuous in $(0,1)^2$ [39]. Xie in Ref. [40] introduced the concept of type-2 uninorm, and extended uninorms, which belong to $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ classes, to type-2 fuzzy sets and discussed under which conditions they are type-2 uninorms. Now, in this work, we will extend representable uninorms and fuzzy implications (coimplications) derived from them to type-2 fuzzy sets. The paper also discusses in which algebra of fuzzy truth values they are classified in, i.e., type-2 uninorms and fuzzy implications (coimplications), respectively.

The rest of this paper is organized as follows. In Section 2, we recall some fundamental concepts and related properties and introduce the definitions of type-2 uninorms and fuzzy implications (coimplications). In Section 3, we investigate extended representable uninorms. Especially, we study their distributivity over type-2 meet and uninon and hence present conditions under which extended representable uninorms are type-2 uninorms. In Sections 4 and 5, we consider extended (U,N), (RU)-implications (coimplications) derived from representable uninorms, and study in which algebras of fuzzy truth values they are type-2 fuzzy implications (coimplications), and discuss their properties on type-2 fuzzy sets.

2. Preliminaries

Some concepts and facts will be listed in this section. For the sake of convenience, we use $\mathscr{I}$ to denote the unit interval $[0,1]$.

Definition 1. *In References [41,42], a binary function $U : \mathscr{I}^2 \to \mathscr{I}$ is called a uninorm if it is commutative, associative, non-decreasing in each place and there exists some element $e \in \mathscr{I}$ (called neutral element of U) such that $U(x,e) = x$ for all $x \in \mathscr{I}$.*

Obviously, the function U is a t-norm if $e = 1$, and a t-conorm if $e = 0$. Fodor and Yager [36] proved that $U(0,1) \in \{0,1\}$. U is said to be conjunctive if $U(1,0) = 0$, and disjunctive if $U(1,0) = 1$. We use U_c and U_d to denote the sets of conjunctive uninorms and disjunctive uninorms, respectively.

The usual classes of uninorms are the $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ classes [36], representable uninorms [36], idempotent uninorms [37,38] and uninorms continuous in $(0,1)^2$ [39]. Because representable uninorms are needed in this work, we only review definitions of representable uninorms. For the left three kinds of uninorms, one can refer to [36,37,39].

Definition 2. *A uninorm U with neutral element $e \in (0,1)$ is said to be* representable *if there exists a strictly increasing and continuous function $h : [0,1] \to [0,+\infty]$ with $h(0) = 0$, $h(e) = 1$ and $h(1) = +\infty$ such that U is given by $U(x,y) = h^{-1}(h(x) + h(y))$ for all $(x,y) \in [0,1]^2 \backslash \{(0,1),(1,0)\}$, and either $U(0,1) = U(1,0) = 1$ or $U(0,1) = U(1,0) = 0$.*

Here, h is called an additive generator of U.

Definition 3. *In reference [31], a function $I : \mathscr{I}^2 \to \mathscr{I}$ is called a* fuzzy implication *if it is decreasing in its first variable and increasing in its second variable and satisfies $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.*

Definition 4. *In reference [30], a function $J : \mathscr{I}^2 \to \mathscr{I}$ is called a* fuzzy coimplication *if it is decreasing in its first variable and is increasing in its second variable and satisfies $J(0,0) = J(1,1) = 0$ and $J(0,1) = 1$.*

Definition 5. *In references [22,24],* fuzzy truth values *are mappings of $\mathscr{I}$ onto itself. The set of fuzzy truth values is denoted by $\mathscr{F}$.*

Example 1. *Two special fuzzy truth values are the following:*

$$\mathbf{0}(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise}. \end{cases} \qquad \mathbf{1}(x) = \begin{cases} 1, & x = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Generally, for any constant $e \in [0,1]$, we define fuzzy truth value $\mathbf{e}$ *as*

$$\mathbf{e}(x) = \begin{cases} 1, & x = e, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 6. *In reference [20], a fuzzy truth value $f \in \mathscr{F}$ is said to be*

(i) normal *if there exists some $x_0 \in [0,1]$ such that $f(x_0) = 1$. The set of all normal fuzzy truth values is denoted by $\mathscr{F}_N$.*

(ii) convex *if for all $x \le z \le y$, $f(z) \ge f(x) \wedge f(y)$. The set of all convex fuzzy truth values is denoted by $\mathscr{F}_C$.*

Let $\mathscr{F}_{CN}$ denote the set of all convex and normal fuzzy truth values.

According to Zadeh's extension principle, a two-place function $* : \mathscr{I}^2 \to \mathscr{I}$ can be extended to $\rhd_* : \mathscr{F}^2 \longrightarrow \mathscr{F}$ by the convolution of $*$ with respect to $\wedge$ and $\vee$. Let $f,g \in \mathscr{F}$, then

$$(f \rhd_* g)(z) = \bigvee_{z = x * y} (f(x) \wedge g(y)).$$

Here, $\rhd_*$ is called the extended $*$, or extend operation of $*$.

Example 2. *(i) If $*$ is t-norm $T_M = \min$ or t-conorm $S_M = \max$, then we get*

$$(f \rhd_{T_M} g)(z) = \bigvee_{z=x\wedge y} (f(x) \wedge g(y)), \tag{1}$$
$$(f \rhd_{S_M} g)(z) = \bigvee_{z=x\vee y} (f(x) \wedge g(y)). \tag{2}$$

The forms of (1) and (2) are rewritten as $f \sqcap g$ and $f \sqcup g$, respectively.

(ii) If $*$ is uninorm U, then we have extended uninorm by

$$(f \rhd_U g)(z) = \bigvee_{z=U(x,y)} (f(x) \wedge g(y)).$$

The operations $\sqcap$ and $\sqcup$ above define two partial orders $\sqsubseteq$ and $\preceq$ on $\mathscr{F}$ [20]. In particular, $f \sqsubseteq g$ if and only if $f \sqcap g = f$, and $f \preceq g$ if and only if $f \sqcup g = g$. In general, the two partial orders are not the same and neither implies the other. However, the two partial orders coincide in $\mathscr{F}_{CN}$.

For any $f \in \mathscr{F}$, let

$$f^R(x) = \bigvee_{y\geq x} f(y), \;\; f^L(x) = \bigvee_{y\leq x} f(y), \;\; f^{LR} = \bigvee_{x\in[0,1]} (f(x)).$$

Remark 1. *In reference [20], the following holds. (i) For any fuzzy truth value f, f^L is increasing and f^R is decreasing.*

(ii) A fuzzy truth value f is convex if and only if $f = f^L \wedge f^R$.

(iii) For any fuzzy truth values f and g, it holds that

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L),$$
$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R).$$

Proposition 1. *Let $f, g \in \mathscr{F}$. If f is convex and g is normal, then*

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = f.$$

Theorem 1. *In reference [20], let T be a t-norm and S be a t-conorm. The following hold for all $f, g \in \mathscr{F}$ if and only if h is convex:*

$$(i) \;\; (f \sqcap g) \rhd_T h = (f \rhd_T h) \sqcap (g \rhd_T h), \qquad (f \sqcup g) \rhd_T h = (f \rhd_T h) \sqcup (g \rhd_T h),$$
$$(ii) \;\; (f \sqcap g) \rhd_S h = (f \rhd_S h) \sqcap (g \rhd_S h), \qquad (f \sqcup g) \rhd_S h = (f \rhd_S h) \sqcup (g \rhd_S h).$$

Definition 7. *For any $f \in \mathscr{F}$, let*

$$\begin{aligned} f^r(x) &= \bigvee_{y>x} f(y),\ x<1, \\ f^l(x) &= \bigvee_{y<x} f(y),\ x>0, \\ f^{lr} &= \bigvee_{y\in(0,1)} f(y). \end{aligned}$$

Type-1 uninorms and fuzzy implications (coimplications) are defined in the algebra $\mathbf{I} = (\mathscr{I}, \vee, \wedge, \leq, 0, 1)$. We will define type-2 uninorms and fuzzy implications (coimplications) analogously to their respective type-1 counterparts. The underlying set of truth values is generalized from $\mathscr{I}$ to a subset of $\mathscr{F}$, and since it may not be a lattice, the two partial orders defined by $\sqsubseteq$ and $\preceq$ are considered instead of $\leq$.

Definition 8. *Let $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, where $\mathscr{A} \subseteq \mathscr{F}$.*

(i) A function $\bullet : \mathscr{F}^2 \longrightarrow \mathscr{F}$ is called a type-2 uninorm over $\mathscr{A}$, if it is commutative, associative, non-decreasing in each variable with at least one of the partial orders $\sqsubseteq$ and $\preceq$, and there exists $\mathbf{e} \in \mathscr{F}$, called the neutral element of $\bullet$, such that $f \bullet \mathbf{e} = f$ for all $f \in \mathscr{F}$.

(ii) A function $\circ : \mathscr{F}^2 \longrightarrow \mathscr{F}$ is called a type-2 fuzzy implication over $\mathbf{A}$, if it satisfies

$$\mathbf{0} \circ \mathbf{0} = \mathbf{1} \circ \mathbf{1} = \mathbf{0} \circ \mathbf{1} = \mathbf{1},\ \mathbf{1} \circ \mathbf{0} = \mathbf{0},$$

and it is antitone in the first argument and monotone in the second argument w.r.t. at least one of the partial orders $\sqsubseteq$ and $\preceq$.

(iii) A function $\diamond : \mathscr{F}^2 \longrightarrow \mathscr{F}$ is called a type-2 fuzzy coimplication over $\mathbf{A}$, if it satisfies

$$\mathbf{0} \diamond \mathbf{0} = \mathbf{1} \diamond \mathbf{1} = \mathbf{0},\ \mathbf{0} \diamond \mathbf{1} = \mathbf{1},$$

and it is antitone in the first and monotone in the second argument w.r.t. at least one of the partial orders $\sqsubseteq$ and $\preceq$.

Remark 2. *It is worth pointing out that extended fuzzy implications (coimplications) or uninorms are not necessary type-2 fuzzy implications (coimplications) or uninorms. We will try to find the conditions under which extended fuzzy implications (coimplications) or uninorms are type-2 fuzzy implications (coimplications) or uninorms.*

3. Extended Representable Uninorms

Lemma 1. *Let $\mathscr{A} \subseteq \mathscr{F}$, U be a type-1 uninorm with neutral element $e \in (0,1)$, and $\rhd_U$ be its extension. Then $\rhd_U$ is commutative, associative and has neutral element $\mathbf{e}$.*

Proof. It is easy to check that $\rhd_U$ satisfies commutative, associative properties,
and $(f \rhd_U \mathbf{e})(z) = \bigvee_{U(x,y)=z} (f(x) \wedge \mathbf{e}(y)) = \bigvee_{U(x,e)=z} (f(x)) = f(z)$. □

In the following, we first will consider the case that U is a conjunctive representable uninorm, i.e., it satisfies $U(0,1) = U(1,0) = 0$.

Proposition 2. *Let $\mathscr{A} \subseteq \mathscr{F}$, U be a type-1 conjunctive representable uninorm with neutral element $e \in (0,1)$, and $\rhd_U$ be its extension. Then, $((f \rhd_U h) \sqcap (g \rhd_U h)) = ((f \sqcap g) \rhd_U h)$ for any $f, g \in \mathscr{A}$ if and only if h is convex on $\mathscr{I}$.*

Proof. Let

$$(I) = ((f \sqcap g) \rhd_U h)(z) = \bigvee_{U(p \wedge q, y) = z} (f(p) \wedge g(q) \wedge h(y))$$

and

$$(II) = ((f \rhd_U h) \sqcap (g \rhd_U h))(z) = \bigvee_{U(p,s) \wedge U(q,t) = z} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)).$$

($\Leftarrow$) Suppose h is convex on $\mathscr{I}$.

It can be proved that $(I) = (II)$ always holds for $z = 0$ or 1. In fact, if $z = 0$, then

$$\begin{aligned}
(I) &= \bigvee_{U(p\wedge q,y)=0} (f(p) \wedge g(q) \wedge h(y)) \\
&= \bigvee_{p\wedge q=0 \text{ or } y=0} (f(p) \wedge g(q) \wedge h(y)) \\
&= \Big(\bigvee_{p=0,\ q\geq 0,\ y\geq 0} (f(p) \wedge g(q) \wedge h(y)) \Big) \vee \Big(\bigvee_{q=0,\ p\geq 0,\ y\geq 0} (f(p) \wedge g(q) \wedge h(y)) \Big) \\
&\vee \Big(\bigvee_{p,\ q\geq 0,\ y=0} (f(p) \wedge g(q) \wedge h(y)) \Big) \\
&= (f(0) \wedge g^{LR} \wedge h^{LR}) \vee (g(0) \wedge f^{LR} \wedge h^{LR}) \vee (h(0) \wedge g^{LR} \wedge f^{LR}),
\end{aligned}$$

and

$$\begin{aligned}
(II) &= \bigvee_{U(p,s)\wedge U(q,t)=0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \\
&= \Big(\bigvee_{U(p,s)=0,\ U(q,t)\geq 0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \Big) \\
&\vee \Big(\bigvee_{U(p,s)\geq 0,\ U(q,t)=0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \Big) \\
&= \Big(\bigvee_{(p=0, \text{ or } s=0),\ q,t\geq 0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \Big) \\
&\vee \Big(\bigvee_{p,s\geq 0,\ (q=0, \text{ or } t=0)} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \Big) \\
&= (f(0) \wedge g^{LR} \wedge h^{LR} \wedge h^{LR}) \vee (f^{LR} \wedge g^{LR} \wedge h(0) \wedge h^{LR}) \\
&\vee (f^{LR} \wedge g(0) \wedge h^{LR} \wedge h^{LR}) \vee (f^{LR} \wedge g^{LR} \wedge h^{LR} \wedge h(0)) \\
&= (f(0) \wedge g^{LR} \wedge h^{LR}) \vee (f^{LR} \wedge g^{LR} \wedge h(0)) \\
&\vee (f^{LR} \wedge g(0) \wedge h^{LR}) \vee (f^{LR} \wedge g^{LR} \wedge h(0)) \\
&= (f(0) \wedge g^{LR} \wedge h^{LR}) \vee (g(0) \wedge f^{LR} \wedge h^{LR}) \vee (h(0) \wedge g^{LR} \wedge f^{LR}).
\end{aligned}$$

thus, $(I) = (II)$ for $z = 0$.

If $z = 1$, then

$$\begin{aligned}
(I) &= \bigvee_{U(p\wedge q,y)=1} (f(p) \wedge g(q) \wedge h(y)) \\
&= \Big(\bigvee_{p\wedge q=1,\ y>0} (f(p) \wedge g(q) \wedge h(y)) \Big) \vee \Big(\bigvee_{p\wedge q>0,\ y=1} (f(p) \wedge g(q) \wedge h(y)) \Big) \\
&= \Big(\bigvee_{p=q=1,\ y>0} (f(p) \wedge g(q) \wedge h(y)) \Big) \vee \Big(\bigvee_{p,q>0,\ y=1} (f(p) \wedge g(q) \wedge h(y)) \Big) \\
&= (f(1) \wedge g(1) \wedge h^r(0)) \vee (f^r(0) \wedge g^r(0) \wedge h(1)),
\end{aligned}$$

and

$$
\begin{aligned}
(II) &= \bigvee_{U(p,s)\wedge U(q,t)=1} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \\
&= (\bigvee_{U(p,s)=1=U(q,t)} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))) \\
&= (\bigvee_{p=1,\ s>0,\ q=1,\ t>0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))) \\
&\bigvee (\bigvee_{p=1,\ s>0,\ q>0,\ t=1} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))) \\
&\bigvee (\bigvee_{s=1,\ p>0,\ q=1,\ t>0} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))) \\
&\bigvee (\bigvee_{s=1,\ p>0,\ q>0,\ t=1} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))) \\
&= (f(1) \wedge g(1) \wedge h^r(0) \wedge h^r(0)) \vee (f(1) \wedge g^r(0) \wedge h(1) \wedge h^r(0)) \\
&\vee (f^r(0) \wedge g(1) \wedge h(1) \wedge h^r(0)) \vee (f^r(0) \wedge g^r(0) \wedge h(1) \wedge h(1)) \\
&= (f(1) \wedge g(1) \wedge h^r(0)) \vee (f(1) \wedge g^r(0) \wedge h(1)) \\
&\vee (f^r(0) \wedge g(1) \wedge h(1)) \vee (f^r(0) \wedge g^r(0) \wedge h(1)) \\
&= (f(1) \wedge g(1) \wedge h^r(0)) \vee (f^r(0) \wedge g^r(0) \wedge h(1)).
\end{aligned}
$$

thus, $(I) = (II)$ for $z = 1$.

Now, it is enough to consider $z \in (0,1)$. It is clear that $(I) \leq (II)$. In the following, we will show that $(II) \leq (I)$.

Let $U(p,s) \wedge U(q,t) = z \in (0,1)$ in (II).

(i) Suppose $U(p,s) = U(q,t) = z$. Then, let $y = s \vee t$. So $U(p \wedge q, y) = z$ and $f(p) \wedge g(q) \wedge h(y) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

(ii) Suppose $U(p,s) = z < U(q,t)$. In this case, if $U(q,s) \geq z = U(p,s)$, then $q \geq p$. We can take $y = s$ and get that $U(p \wedge q, y) = U(p,s) = z$ and $f(p) \wedge g(q) \wedge h(s) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

If $U(q,s) < z = U(p,s)$, then $q < p$ and $U(q,s) < z < U(q,t)$. We can prove that $q \in (0,1)$. Otherwise, if $q = 0$, then $U(q,t) = 0$, which contradicts $U(q,s) < z < U(q,t)$ and $z \in (0,1)$. If $q = 1$, from $U(q,s) < z < U(q,t)$, we can obtain $s = 0$ and $t > 0$. However, $z = U(p,s) = U(p,0) = 0$, which is a contradiction with $z \in (0,1)$. As a result, $q \in (0,1)$. Since $U(q,\cdot)$ is continuous, there exists some $c \in (s,t)$ such that $U(q,c) = z$. Again, because h is convex, it holds $h(c) \geq h(s) \wedge h(t)$. That is to say, $U(p \wedge q, c) = U(q,c) = z$ and $f(p) \wedge g(q) \wedge h(c) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

(iii) Suppose $U(q,t) = z < U(p,s)$. It is similar to (ii).

Summing up the above, we can obtain that, for any $p, q, s, t \in \mathscr{I}$ fulfilling $U(p,s) \wedge U(q,t) = z$, there always exists some $y \in \mathscr{I}$ such that $U(p \wedge q, y) = z$ and $f(p) \wedge g(q) \wedge h(y) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$. Thus, $(II) \leq (I)$ for $z \in (0,1)$.

($\Rightarrow$) Suppose that $(I) = (II)$. Let $f = \mathbf{e}$ and $g(q) = \begin{cases} 1, & q \geq e, \\ 0, & \text{otherwise.} \end{cases}$

For any $z \in (0,1)$,

$$
\begin{aligned}
(I) &= \bigvee_{U(p\wedge q,y)=z} (f(p) \wedge g(q) \wedge h(y)) \\
&= \bigvee_{U(e,y)=z} (h(y)) = h(z),
\end{aligned}
$$

and

$$
\begin{aligned}
(II) &= \bigvee_{U(p,s)\wedge U(q,t)=z} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \\
&= \bigvee_{s\wedge U(q,t)=z,\ q\geq e} (h(s) \wedge h(t)) \\
&\geq \bigvee_{U(q,t)=z,\ s\geq z,\ q\geq e} (h(s) \wedge h(t)) \\
&= h^R(z) \wedge (\bigvee_{U(q,t)=z,\ q\geq e} (h(t))).
\end{aligned}
$$

It can be proved that $\bigvee_{U(q,t)=z,\ q\geq e}(h(t)) = \bigvee_{t\leq z}(h(t))$. In fact, if $U(q,t)=z$ and $q\geq e$, then $t\leq z$ and hence $\bigvee_{U(q,t)=z,\ q\geq e}(h(t)) \leq \bigvee_{t\leq z}(h(t))$. On the contrary, if $t\leq z$, there always exists some $q = h^{-1}(h(z)-h(t)) \geq e$ such that $U(q,t)=z$ and so $\bigvee_{U(q,t)=z,\ q\geq e}(h(t)) \geq \bigvee_{t\leq z}(h(t))$. From the above, we know that $\bigvee_{U(q,t)=z,\ q\geq e}(h(t)) = \bigvee_{t\leq z}(h(t))$. Following this fact, we can get that

$$\begin{aligned}
(II) &= \bigvee_{U(p,s)\wedge U(q,t)=z}(f(p)\wedge g(q)\wedge h(s)\wedge h(t))\\
&= \bigvee_{s\wedge U(q,t)=z,\ q\geq e}(h(s)\wedge h(t))\\
&\geq \bigvee_{U(q,t)=z,\ s\geq z,\ q\geq e}(h(s)\wedge h(t))\\
&= h^R(z)\wedge\Big(\bigvee_{U(q,t)=z,\ q\geq e}(h(t))\Big)\\
&= h^R(z)\wedge\Big(\bigvee_{t\leq z}(h(t))\Big)\\
&= h^R(z)\wedge h^L(z).
\end{aligned}$$

Consequently, $h(z)\geq h^R(z)\wedge h^L(z)$. Since $h(z)\leq h^R(z)\wedge h^L(z)$ always holds, then $h(z) = h^R(z)\wedge h^L(z)$ holds for any $z\in(0,1)$. Because $h(0)=h^L(0)$ and $h(1)=h^R(1)$, then $h(z)=h^R(z)\wedge h^L(z)$ always holds for $z=1$ or 0. Consequently, h is convex on $\mathscr{I}$. □

Theorem 2. *Let $\mathscr{A}\subseteq\mathscr{F}$, $\mathbf{A}=(\mathscr{A},\mathbf{0},\mathbf{1},\sqsubseteq,\preceq)$, U be a type-1 conjunctive representable uninorm with neutral element $e\in(0,1)$ and $\rhd_U$ be its extension. Then, $\rhd_U$ is a type-2 uninorm on $\mathbf{A}$ with neutral element $\mathbf{e}$ if and only if $\mathscr{A}\subseteq\mathscr{F}_C$. Moreover,*

$$(f\rhd_U g)(z) = \begin{cases} (f(0)\wedge g^{LR})\vee(g(0)\wedge f^{LR}) & z=0,\\ (f(1)\wedge g^r(0))\vee(g(1)\wedge f^r(0)) & z=1,\\ \bigvee_{x\in(0,1)}(f(x)\wedge g\circ h^{-1}(h(z)-h(x)))\ or & \\ \bigvee_{y\in(0,1)}(g(y)\wedge f\circ h^{-1}(h(z)-h(y))) & otherwise,\end{cases} \tag{3}$$

where h is an additive generator of U.

Proof. ($\Leftarrow$) Lemma 1 shows that $\rhd_U$ is associative, commutative and has neutral element $\mathbf{e}$. Suppose $f_1,f_2,f_3\in\mathscr{A}$ and $f_1\sqsubseteq f_2$. Then $f_1\sqcap f_2=f_1$. From the above proposition, we obtain that $(f_1\rhd_U f_3)\sqcap(f_2\rhd_U f_3)=(f_1\sqcap f_2)\rhd_U f_3 = f_1\rhd_U f_3$, which implies that $f_1\rhd_U f_3\sqsubseteq f_2\rhd_U f_3$. That is to say, $\rhd_U$ is increasing with the partial order $\sqsubseteq$.

Consequently, $\rhd_U$ is a type-2 uninorm on $\mathbf{A}$.

($\Rightarrow$) For any $f_1,f_2,f_3\in\mathscr{A}$ with $f_1\sqsubseteq f_2$, we have $f_1\rhd_U f_3\sqsubseteq f_2\rhd_U f_3$, which means that $(f_1\rhd_U f_3)\sqcap(f_2\rhd_U f_3) = f_1\rhd_U f_3 = (f_1\sqcap f_2)\rhd_U f_3$. Thus, $(f_1\rhd_U f_3)\sqcap(f_1\rhd_U f_3)=(f_1\sqcap f_2)\rhd_U f_3$. Again from Proposition 2, we have that f_3 is convex. Thus, $\mathscr{A}\subseteq\mathscr{F}_C$.

For any $f,g\in\mathscr{A}$, it holds that $(f\rhd_U g)(z)=\bigvee_{U(x,y)=z}(f(x)\wedge g(y))$.

$z=0\Leftrightarrow x=0$ or $y=0$. Then

$$\begin{aligned}
\bigvee_{U(x,y)=0}(f(x)\wedge g(y)) &= \Big(\bigvee_{x=0,\ y\in[0,1]}(f(x)\wedge g(y))\Big)\vee\Big(\bigvee_{y=0,\ x\in[0,1]}(f(x)\wedge g(y))\Big)\\
&= (f(0)\wedge g^{LR})\vee(g(0)\wedge f^{LR}).
\end{aligned}$$

$z = 1 \Leftrightarrow x = 1$ and $y \in (0, 1]$, or $y = 0$ and $x \in (0, 1]$. Then

$$\begin{aligned}\bigvee_{U(x,y)=1} (f(x) \wedge g(y)) &= \Big(\bigvee_{x=1,\ y\in(0,1]} (f(x) \wedge g(y)) \Big) \vee \Big(\bigvee_{y=1,\ x\in(0,1]} (f(x) \wedge g(y)) \Big) \\ &= (f(1) \wedge g^r(0)) \vee (g(1) \wedge f^r(0)).\end{aligned}$$

If $z \in (0, 1)$, then $x, y \in (0, 1)$ and $U(x, y) = z \Rightarrow y = h^{-1}(h(z) - h(x))$ or $x = h^{-1}(h(z) - h(y))$. So $\bigvee_{U(x,y)=z} (f(x) \wedge g(y)) = \bigvee_{h^{-1}(h(x)+h(y))=z} (f(x) \wedge g(y)) = \bigvee_{x\in(0,1)} (f(x) \wedge g \circ h^{-1}(h(z) - h(x)))$ or $\bigvee_{y\in(0,1)} (g(y) \wedge f \circ h^{-1}(h(z) - h(y)))$. □

Similar to the above, we have the following facts for disjunctive representable uninorms.

Proposition 3. *Let $\mathscr{A} \subseteq \mathscr{F}$, U be a type-1 disjunctive representable uninorm with neutral element $e \in (0, 1)$ and $\rhd_U$ be its extension. Then, $((f \rhd_U h) \sqcup (g \rhd_U h)) = ((f \sqcup g) \rhd_U h)$ for any $f, g \in \mathscr{A}$ if and only if h is convex on $\mathscr{I}$.*

Theorem 3. *Let $\mathscr{A} \subseteq \mathscr{F}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, U be a type-1 disjunctive representable uninorm with neutral element $e \in (0, 1)$ and $\rhd_U$ be its extension. Then, $\rhd_U$ is a type-2 uninorm on $\mathbf{A}$ with neutral element $\mathbf{e}$ if and only if $\mathscr{A} \subseteq \mathscr{F}_C$. Moreover,*

$$(f \rhd_U g)(z) = \begin{cases} (f(1) \wedge g^{LR}) \vee (g(1) \wedge f^{LR}) & z = 1, \\ (f(0) \wedge g^l(1)) \vee (g(0) \wedge f^l(1)) & z = 0, \\ \bigvee_{x\in(0,1)} (f(x) \wedge g \circ h^{-1}(h(z) - h(x))) \text{ or} & \\ \bigvee_{y\in(0,1)} (g(y) \wedge f \circ h^{-1}(h(z) - h(y))) & \text{otherwise}, \end{cases} \quad (4)$$

where h is an additive generator of U.

4. Extended (U,N)-Implications ((U,N)-Coimplications) and Their Properties

Definition 9. *A function $I_{U,N} : \mathscr{I}^2 \to \mathscr{I}$ is called a (U,N)-operation if there exists a uninorm U and a strong negation N such that*

$$I_{U,N}(x, y) = U(N(x), y),\ x, y \in \mathscr{I}.$$

Baczyński and Jayaram in Reference [32] have proved that $I_{U,N}$ is a type-1 fuzzy implication if and only if U is a disjunctive uninorm.

By the same way, we can define (U,N)-coimplications $J_{U,N}$ from a conjunctive uninorm U and a strong negation N, that is

$$J_{U,N}(x, y) = U(N(x), y),\ x, y \in \mathscr{I}.$$

For a (U,N)-implication $I_{U,N}$ derived from a disjunctive uninorm U and a strong negation N, its extended operation is given by

$$\begin{aligned}(f \rhd_{I_{U,N}} g)(z) &= \bigvee_{U(N(x),y)=z} (f(x) \wedge g(y)) \\ &= \bigvee_{U(x,y)=z} (f \circ N(x) \wedge g(y)) \\ &= ((f \circ N) \rhd_U g)(z).\end{aligned}$$

For a (U,N)-coimplication $J_{U,N}$ derived from a disjunctive uninorm U and a strong negation N, its extended counterpart is given by

$$\begin{aligned}(f \rhd_{J_{U,N}} g)(z) &= \bigvee_{U(N(x),y)=z} (f(x) \wedge g(y)) \\ &= \bigvee_{U(x,y)=z} (f \circ N(x) \wedge g(y)) \\ &= ((f \circ N) \rhd_U g)(z).\end{aligned}$$

Lemma 2. *Let $\mathscr{A} \subseteq \mathscr{F}$, $I_{U,N}$ be a (U,N)-implication derived from a disjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{I_{U,N}}$ be the extended operation of $I_{U,N}$. For any $f, g \in \mathscr{A}$, if $f, g \in \mathscr{F}_N$, then $f \rhd_{I_{U,N}} g$ is normal.*

Proof. If f, g is normal, then there exist x_0, y_0 such that $f(x_0) = 1$ and $g(y_0) = 1$. It can be proved that there correspondingly exists some z_0 such that $U(N(x_0), y_0) = z_0$. In fact, if $x_0 = 1$ and $y_0 \in [0, 1)$, then $z_0 = 0$; if $x_0 = 1$ and $y_0 = 1$, then $z_0 = 1$; if $x_0 = 0$ and $y_0 \in [0, 1]$, then $z_0 = 1$; if $x_0 \in (0, 1)$ and $y_0 = 0$, then $z_0 = 0$; if $x_0 \in (0, 1)$ and $y_0 = 1$, then $z_0 = 1$; if $x_0 \in (0, 1)$ and $y_0 \in (0, 1)$, then take $z_0 = h^{-1}(h(N(x_0)) + h(y_0)) \in (0, 1)$, where h is an additive generator of representable uninorm U.

Consequently, we have that

$$\begin{aligned}(f \rhd_{I_{U,N}} g)(z_0) &= \bigvee_{U(N(x),y)=z_0} (f(x) \wedge g(y)) \\ &= \Big(\bigvee_{U(N(x_0),y_0)=z_0} (f(x_0) \wedge g(y_0))\Big) \vee \Big(\bigvee_{U(N(x),y)=z_0,\ x\neq x_0,\ y\neq y_0} (f(x) \wedge g(y))\Big) \\ &= 1.\end{aligned}$$

Namely, $f \rhd_{I_{U,N}} g$ is normal. □

Lemma 3. *Let $\mathscr{A} \subseteq \mathscr{F}$, $I_{U,N}$ be a (U,N)-implication derived from a disjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{I_{U,N}}$ be the extended operation of $I_{U,N}$. For any $f, g \in \mathscr{A}$, if $f, g \in \mathscr{F}_C$, $f \rhd_{I_{U,N}} g \in \mathscr{F}_C$.*

Proof. Assume that $f, g \in \mathscr{F}_C$ and $0 < x \leq z \leq y < 1$. Then,

$$(f \rhd_{I_{U,N}} g)(x) \wedge (f \rhd_{I_{U,N}} g)(y) = \bigvee_{I_{U,N}(x_1,x_2)=x,\ I_{U,N}(y_1,y_2)=y} (f(x_1) \wedge f(y_1) \wedge g(x_2) \wedge g(y_2)).$$

Since $0 < x \leq z \leq y < 1$, it holds that $0 < x_1, x_2, y_1, y_2 < 1$. Let $a^- = x_1 \wedge y_1$, $a^+ = x_1 \vee y_1$, $b^- = x_2 \wedge y_2$, $b^+ = x_2 \vee y_2$. Then, $x, y \in I_{U,N}([a^-, a^+], [b^-, b^+])$. Again because $I_{U,N}$ is continuous in $(0, 1)^2$, there exist $z_1 \in [a^-, a^+]$ and $z_2 \in [b^-, b^+]$ such that $z = I_{U,N}(z_1, z_2)$. Because f and g is convex, then $f(x_1) \wedge f(y_1) \leq f(z_1)$ and $g(x_2) \wedge g(y_2) \leq g(z_2)$ and hence

$$\begin{aligned}(f \rhd_{I_{U,N}} g)(x) \wedge (f \rhd_{I_{U,N}} g)(y) &= \bigvee_{I_{U,N}(x_1,x_2)=x,\ I_{U,N}(y_1,y_2)=y} (f(x_1) \wedge f(y_1) \wedge g(x_2) \wedge g(y_2)) \\ &\leq \bigvee_{I_{U,N}(z_1,z_2)=z} (f(z_1) \wedge g(z_2)) \\ &= (f \rhd_{I_{U,N}} g)(z).\end{aligned}$$

Thus, $(f \rhd_{I_{U,N}} g = (f \rhd_{I_{U,N}} g)^L \wedge (f \rhd_{I_{U,N}} g)^R$ for any $z \in (0, 1)$. Again because $(f \rhd_{I_{U,N}} g)(0) = (f \rhd_{I_{U,N}} g)^L(0) = (f \rhd_{I_{U,N}} g)^L(0) \wedge (f \rhd_{I_{U,N}} g)^R(0)$ and $(f \rhd_{I_{U,N}} g)(1) = (f \rhd_{I_{U,N}} g)^R(1) = (f \rhd_{I_{U,N}} g)^L(1) \wedge (f \rhd_{I_{U,N}} g)^R(1)$, $f \rhd_{I_{U,N}} g = (f \rhd_{I_{U,N}} g)^L \wedge (f \rhd_{I_{U,N}} g)^R$ always holds for any $z \in \mathscr{I}$. Namely, $f \rhd_{I_{U,N}} g \in \mathscr{F}_C$. □

Remark 3. *The above proof that $\rhd_{I_{U,N}}$ is convex on $z \in (0,1)$ is similar to that of Proposition 3.6 in Ref. [29]. However, for the consistency of this proof, we give it again.*

Lemma 4. *Let $\mathscr{A} \subseteq \mathscr{F}$, $I_{U,N}$ be a (U,N)-implication derived from a disjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{I_{U,N}}$ be the extended operation of $I_{U,N}$. Then,*

$$(f \sqcap g) \rhd_{I_{U,N}} h = (f \rhd_{I_{U,N}} h) \sqcup (g \rhd_{I_{U,N}} h)$$

or

$$h \rhd_{I_{U,N}} (f \sqcup g) = (h \rhd_{I_{U,N}} f) \sqcup (h \rhd_{I_{U,N}} g)$$

for any $f, g \in \mathscr{F}$ if and only if h is convex on $\mathscr{I}$.

Proof.

$$\begin{aligned} ((f \sqcap g) \rhd_{I_{U,N}} h)(z) &= \bigvee_{U(N(p\wedge q),y)=z} (f(p) \wedge g(q) \wedge h(y)) \\ &= \bigvee_{U(N(p)\vee N(q),y)=z} (f(p) \wedge g(q) \wedge h(y)) \\ &= \bigvee_{U(p\vee q,y)=z} (f \circ N(p) \wedge g \circ N(q) \wedge h(y)), \end{aligned}$$

and

$$\begin{aligned} ((f \rhd_{I_{U,N}} h) \sqcup (g \rhd_{I_{U,N}} h))(z) &= \bigvee_{U(N(p),s)\vee U(N(q),t)=z} (f(p) \wedge g(q) \wedge h(s) \wedge h(t)) \\ &= \bigvee_{U(p,s)\vee U(q,t)=z} (f \circ N(p) \wedge g \circ N(q) \wedge h(s) \wedge h(t)) \end{aligned};$$

$$\begin{aligned} ((h \rhd_{I_{U,N}} f) \sqcup (h \rhd_{I_{U,N}} g))(z) &= \bigvee_{x\vee y=z} ((h \rhd_{I_{U,N}} f)(x) \wedge (h \rhd_{I_{U,N}} g)(y)) \\ &= \bigvee_{U(p,s)\vee U(q,t)=z} (f(p) \wedge g(q) \wedge h \circ N(s) \wedge h \circ N(t)), \end{aligned}$$

and

$$\begin{aligned} (h \rhd_{I_{U,N}} (f \sqcup g))(z) &= \bigvee_{U(N(y),x)=z} \Big(h(y) \wedge ((f \sqcup g)(x))\Big) \\ &= \bigvee_{U(p\vee q,y)=z} (f(p) \wedge g(q) \wedge h \circ N(y)). \end{aligned}$$

Similarly to the proof of Proposition 2, we can prove $(f \sqcap g) \rhd_{I_{U,N}} h = (f \rhd_{I_{U,N}} h) \sqcup (g \rhd_{I_{U,N}} h)$ or $(h \rhd_{I_{U,N}} f) \sqcup (h \rhd_{I_{U,N}} g) = h \rhd_{I_{U,N}} (f \sqcup g)$ for any $f, g \in \mathscr{F}$ if and only if h is convex. □

Theorem 4. *Let $\mathscr{A} \subseteq \mathscr{F}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, $I_{U,N}$ be a (U,N)-implication derived from a disjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{I_{U,N}}$ be the extended operation of $I_{U,N}$. If $\mathscr{A} \subseteq \mathscr{F}_{CN}$, then $\rhd_{I_{U,N}}$ is a type-2 fuzzy implication. In addition,*

$$(f \rhd_{I_{U,N}} g)(z) = \begin{cases} (g(0) \wedge f^r(0)) \vee (f(1) \wedge g^l(1)) & z = 0, \\ (f(0) \wedge g^{LR}) \vee (g(1) \wedge f^{LR}) & z = 1, \\ \bigvee_{x\in(0,1)} \Big(f(x) \wedge \big(g \circ h^{-1}(h(z) - h(N(x)))\big)\Big) \text{ or} & \\ \bigvee_{y\in(0,1)} \Big(g(y) \wedge \big(f \circ N \circ h^{-1}(h(z) - h(y))\big)\Big) & \text{otherwise}, \end{cases} \tag{5}$$

where h is an additive generator of U.

Proof. From Equation (4), one can easily obtain that

$$\begin{aligned}(\mathbf{0} \rhd_{I_{U,N}} \mathbf{0})(z) &= \mathbf{1},\\ (\mathbf{0} \rhd_{I_{U,N}} \mathbf{1})(z) &= \mathbf{1},\\ (\mathbf{1} \rhd_{I_{U,N}} \mathbf{1})(z) &= \mathbf{1},\\ (\mathbf{1} \rhd_{I_{U,N}} \mathbf{0})(z) &= \mathbf{0}.\end{aligned}$$

Let $f, g \in \mathscr{A}$ with $f \sqsubseteq g$. Then, $f \sqcap g = f$. In the following, we will prove that $g \rhd_{I_{U,N}} h \sqsubseteq f \rhd_{I_{U,N}} h$ for any $h \in \mathscr{A}$. In fact, according to Lemma 4, it holds that

$$\begin{aligned}(g \rhd_{I_{U,N}} h) \sqcap (f \rhd_{I_{U,N}} h) &= (g \rhd_{I_{U,N}} h) \sqcap ((f \sqcap g) \rhd_{I_{U,N}} h)\\ &= ((f \rhd_{I_{U,N}} h) \sqcup (g \rhd_{I_{U,N}} h)) \sqcap (g \rhd_{I_{U,N}} h).\end{aligned}$$

Since $f, g, h \in \mathscr{A} \subseteq \mathscr{F}_{CN}$, from Lemmas 2 and 3, we obtain that $(f \rhd_{I_{U,N}} h), (g \rhd_{I_{U,N}} h) \in \mathscr{F}_{CN}$. Again from Proposition 1, we can obtain that

$$((f \rhd_{I_{U,N}} h) \sqcup (g \rhd_{I_{U,N}} h)) \sqcap (g \rhd_{I_{U,N}} h) = (g \rhd_{I_{U,N}} h).$$

Thus, if $f \sqsubseteq g$, then $(g \rhd_{I_{U,N}} h) \sqcap (f \rhd_{I_{U,N}} h) = (g \rhd_{I_{U,N}} h)$ for any $h \in \mathscr{A}$, or $g \rhd_{I_{U,N}} h \sqsubseteq f \rhd_{I_{U,N}} h$, which means that $\rhd_{I_{U,N}}$ is decreasing in the first place with respect to the partial order $\sqsubseteq$. Remember that the partial orders $\sqsubseteq$ and $\preceq$ coincide in $\mathscr{F}_{CN}$. Then, $\rhd_{I_{U,N}}$ is decreasing in the first place with respect to the partial order $\preceq$ as well.

Similarly, if $f \preceq g$, then $h \rhd_{I_{U,N}} f \preceq h \rhd_{I_{U,N}} g$ for any $h \in \mathscr{A}$, namely, $\rhd_{I_{U,N}}$ is increasing in the second place with respect to the partial order $\preceq$, whence $\rhd_{I_{U,N}}$ is increasing in the second place with respect to the partial order $\sqsubseteq$.

To sum up, $\rhd_{I_{U,N}}$ is a type-2 fuzzy implication on $\mathbf{A}$. By simple computation, one can easily obtain (5). □

The following are some properties for type-2 fuzzy implications.

Theorem 5. *Let $\mathscr{A} \subseteq \mathscr{F}_{CN}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, $I_{U,N}$ be a (U,N)-implication derived from a disjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{I_{U,N}}$ be a type-2 fuzzy implication. Then, we have the following properties for $\rhd_{I_{U,N}}$:*

(i) $f \rhd_{I_{U,N}} \mathbf{e} = f \circ N$; *if* $N(e) = e$, *then* $\mathbf{e} \rhd_{I_{U,N}} f = f$.

(ii) $\mathbf{0} \rhd_{I_{U,N}} f = \mathbf{1}$; $(f \rhd_{I_{U,N}} \mathbf{0})(z) = \begin{cases} f^r(0) & z = 0,\\ f(0) & z = 1,\\ 0 & z \in (0,1).\end{cases}$

(iii) $f \rhd_{I_{U,N}} \mathbf{1} = \mathbf{1}$; $(\mathbf{1} \rhd_{I_{U,N}} f)(z) = \begin{cases} f^l(1) & z = 0,\\ f(1) & z = 1,\\ 0 & z \in (0,1).\end{cases}$

(iv) $f \rhd_{I_{U,N}} (g \vee h) = (f \rhd_{I_{U,N}} g) \vee (f \rhd_{I_{U,N}} h)$;
$(g \vee h) \rhd_{I_{U,N}} f = (g \rhd_{I_{U,N}} f) \vee (h \rhd_{I_{U,N}} f)$.

(v) $f \rhd_{I_{U,N}} (g \wedge h) \leq (f \rhd_{I_{U,N}} g) \wedge (f \rhd_{I_{U,N}} h)$;
$(g \wedge h) \rhd_{I_{U,N}} f \leq (g \rhd_{I_{U,N}} f) \wedge (h \rhd_{I_{U,N}} f)$.

(vi) $f_1 \rhd_{I_{U,N}} (f_2 \rhd_{I_{U,N}} f_3) = (f_1 \rhd_{U_c} f_2) \rhd_{I_{U,N}} f_3$, *where U_c is a conjunctive uninorm given by $U_c(x,y) = N(U(N(x), N(y)))$ (namely, U_c is a representable uninorm dual with U with respect to N).*

Proof. (i) and (v) can be easily obtained.

(vi)

$$(f_1 \rhd_{I_{U,N}} (f_2 \rhd_{I_{U,N}} f_3))(z) = \bigvee_{U(U(p,q),\, y)=z} ((f_1 \circ N)(p) \wedge (f_2 \circ N)(q) \wedge f_3(y))$$

$$((f_1 \rhd_{U_c} f_2) \rhd_{I_{U,N}} f_3)(z) = \bigvee_{U(N\circ U_c(N(p),N(q),\, y)=z} ((f_1 \circ N)(p) \wedge (f_2 \circ N)(q) \wedge f_3(y))$$

Since $U_c(p,q) = N \circ U(N(p), N(q))$, then $f_1 \rhd_{I_{U,N}} (f_2 \rhd_{I_{U,N}} f_3) = (f_1 \rhd_{U_c} f_2) \rhd_{I_{U,N}} f_3$. □

Just like (U,N)-implications, we can obtain the following facts about (U,N)-coimplications.

Lemma 5. *Let $\mathscr{A} \subseteq \mathscr{F}$, $J_{U,N}$ be a (U,N)-coimplication derived from a conjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{J_{U,N}}$ be the extended operation of $J_{U,N}$. For any $f, g \in \mathscr{A}$, if $f, g \in \mathscr{F}_N$, then $f \rhd_{J_{U,N}} g$ is normal.*

Lemma 6. *Let $\mathscr{A} \subseteq \mathscr{F}$, $J_{U,N}$ be a (U,N)-coimplication derived from a conjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{J_{U,N}}$ be the extended operation of $J_{U,N}$. For any $f, g \in \mathscr{A}$, if $f, g \in \mathscr{F}_C$, then $f \rhd_{J_{U,N}} g \in \mathscr{F}_C$.*

Lemma 7. *Let $\mathscr{A} \subseteq \mathscr{F}$, $J_{U,N}$ be a (U,N)-coimplication derived from a conjunctive representable uninorm U and a strong fuzzy negation N, and $\rhd_{J_{U,N}}$ be the extended operation of $I_{U,N}$. Then,*

$$(f \sqcup g) \rhd_{J_{U,N}} h = (f \rhd_{J_{U,N}} h) \sqcap (g \rhd_{J_{U,N}} h)$$

or

$$h \rhd_{J_{U,N}} (f \sqcap g) = (h \rhd_{J_{U,N}} f) \sqcap (h \rhd_{J_{U,N}} g)$$

for any $f, g \in \mathscr{A}$ if and only if h is convex on $\mathscr{I}$.

Theorem 6. *Let $\mathscr{A} \subseteq \mathscr{F}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, $J_{U,N}$ be a (U,N)-coimplication derived from a conjunctive representable uninorm U and a strong fuzzy negation N. If $\mathscr{A} \subseteq \mathscr{F}_{CN}$, then $\rhd_{J_{U,N}}$ is a type-2 fuzzy coimplication on $\mathbf{A}$. In addition,*

$$(f \rhd_{J_{U,N}} g)(z) = \begin{cases} (g(0) \wedge f^{LR}) \vee (f(1) \wedge g^{LR}) & z = 0, \\ (f(0) \wedge g^{r}(0)) \vee (g(1) \wedge f^{l}(1)) & z = 1, \\ \bigvee_{x\in(0,1)} \Big(f(x) \wedge \big(g \circ h^{-1}(h(z) - h(N(x)))\big)\Big) \text{ or} & \\ \bigvee_{y\in(0,1)} \Big(g(y) \wedge \big(f \circ N \circ h^{-1}(h(z) - h(y))\big)\Big) & \text{otherwise,} \end{cases} \tag{6}$$

where h is an additive generator of U.

Theorem 7. *Let $\mathscr{A} \subseteq \mathscr{F}_{CN}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, $J_{U,N}$ be a (U,N)-coimplication derived from a conjunctive representable uninorm U and a strong fuzzy negation N and $\rhd_{J_{U,N}}$ be a type-2 fuzzy coimplication on $\mathbf{A}$. Then we have the following facts.*

(i) $f \rhd_{J_{U,N}} \mathbf{e} = f \circ N$; *if* $N(e) = e$, *then* $\mathbf{e} \rhd_{J_{U,N}} f = f$.

(ii) $f \rhd_{J_{U,N}} \mathbf{0} = \mathbf{0}$; $(\mathbf{0} \rhd_{J_{U,N}} f)(z) = \begin{cases} f^{r}(0) & z = 1, \\ f(0) & z = 0, \\ 0 & z \in (0,1). \end{cases}$

(iii) $\mathbf{1} \rhd_{J_{U,N}} f = \mathbf{0}$; $(f \rhd_{J_{U,N}} \mathbf{1})(z) = \begin{cases} f^{l}(1) & z = 1, \\ f(1) & z = 0, \\ 0 & z \in (0,1). \end{cases}$

(iv) $f \rhd_{J_{U,N}} (g \vee h) = (f \rhd_{J_{U,N}} g) \vee (f \rhd_{J_{U,N}} h)$;
$(g \vee h) \rhd_{J_{U,N}} f = (g \rhd_{J_{U,N}} f) \vee (h \rhd_{J_{U,N}} f)$.

(v) $f \rhd_{J_{U,N}} (g \wedge h) \leq (f \rhd_{J_{U,N}} g) \wedge (f \rhd_{J_{U,N}} h)$;
$(g \wedge h) \rhd_{J_{U,N}} f \leq (g \rhd_{J_{U,N}} f) \wedge (h \rhd_{J_{U,N}} f)$.

(vi) $f_1 \rhd_{J_{U,N}} (f_2 \rhd_{J_{U,N}} f_3) = (f_1 \rhd_{U_d} f_2) \rhd_{J_{U,N}} f_3$*, where* U_d *is a disjunctive uninorm given by* $U_d(x,y) = N\big(U(N(x), N(y))\big)$ *(namely,* U_d *is a representable uninorm dual with* U *with respect to* N*).*

5. Extended RU-Implications (RU-Coimplications) and Their Properties

Definition 10. *(i) A function* $I_U : \mathscr{I}^2 \to \mathscr{I}$ *is called an RU-operation if there exists a uninorm* U *such that*

$$I_U(x,y) = \sup\{z \in [0,1] | U(x,z) \le y\},\ x, y \in \mathscr{I}.$$

(ii) A function $J_U : \mathscr{I}^2 \to \mathscr{I}$ *is called an RU-cooperation if there exists a uninorm* U *such that*

$$J_U(x,y) = \inf\{z \in [0,1] | U(x,z) \ge y\},\ x, y \in \mathscr{I}.$$

The authors [35] have proved that I_U is a fuzzy implication if and only if $U(0,z) = 0$ for any $z \in [0,1)$. Since for any representable uninorm U, whether it is disjunctive or conjunctive, it always holds that $U(0,z) = 0$ for any $z \in [0,1)$. Then, we can get RU-implications from any representable uninorm. The authors in Ref. [35] also have proved that if U is a representable uninorm with an additive generator h, then I_U is given by

$$I_U(x,y) = \begin{cases} h^{-1}(h(y) - h(x)) & (x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\}, \\ 1 & (x,y) = (0,0) \text{ or } (1,1). \end{cases} \quad (7)$$

Similarly, it can be proved that that J_U is a (RU)-coimplications if and only if $U(1,z) = 1$ for any $z \in (0,1]$. For any representable uninorm U, whether it is disjunctive or conjunctive, it always holds that $U(1,z) = 1$ for any $z \in (0,1]$. Thus, we can get RU-coimplications from any representable uninorm. By simple computation, we can obtain that

$$J_U(x,y) = \begin{cases} h^{-1}(h(y) - h(x)) & (x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\}, \\ 0 & (x,y) = (0,0) \text{ or } (1,1). \end{cases} \quad (8)$$

Lemma 8. *Let* U *be a representable uninorm with an additive generator* h *and* I_U *be its RU-implication. Then,* $I_U(1,y) = 0$ *for* $y \ne 1$*,* $I_U(x,0) = 0$ *for* $x \ne 0$*,* $I_U(e,y) = 0{=}y$ *for* $y \in [0,1]$*,* $I_U(0,y) = 1 = I_U(x,1)$ *for* $x, y \in [0,1]$*,* $I_U(x,y) \in (0,1)$ *for* $(x,y) \in (0,1)^2$ *and* I_U *is continuous if and only if* $(x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$*.*

Proof. It is easy to prove it. □

Lemma 9. *Let* $\mathscr{A} \subseteq \mathscr{F}$*,* I_U *be an RU-implication derived from a representable uninorm* U *and* $\rhd_{I_U}$ *be its extended operation. For any* $f, g \in \mathscr{A}$*, if* $f, g \in \mathscr{F}_N$*, then* $f \rhd_{I_U} g$ *is normal.*

Proof. It is similar to Lemma 2. □

Lemma 10. *Let* $\mathscr{A} \subseteq \mathscr{F}$*,* U *be a representable uninorm,* I_U *be a RU-implication derived from* U *and* $\rhd_{I_U}$ *be its extended operation. For any* $f, g \in \mathscr{A}$*, if* $f, g \in \mathscr{F}_C$*, then* $f \rhd_{I_U} g \in \mathscr{F}_C$*.*

Proof. It is similar to Lemma 3. □

Lemma 11. *Let* $\mathscr{A} \subseteq \mathscr{F}$*,* U *be a representable uninorm,* I_U *be a RU-implication derived from* U *and* $\rhd_{I_U}$ *be its extended operation. Then,*

$$(f \sqcap g) \rhd_{I_U} h = (f \rhd_{I_U} h) \sqcup (g \rhd_{I_U} h)$$

or

$$h \rhd_{I_U} (f \sqcup g) = (h \rhd_{I_U} f) \sqcup (h \rhd_{I_U} g)$$

for any $f, g \in \mathscr{A}$ *if and only if* h *is convex on* $\mathscr{I}$*.*

Proof. We only prove the first distributive equation. The second equation can be similarly proved.

Let
$$(I) = ((f \sqcap g) \rhd_{I_U} h)(z) = \bigvee_{I_U(p \wedge q, y) = z} (f(p) \wedge g(q) \wedge h(y))$$
and
$$(II) = ((f \rhd_{I_U} h) \sqcup (g \rhd_{I_U} h))(z) = \bigvee_{I_U(p,s) \vee I_U(q,t) = z} (f(p) \wedge g(q) \wedge h(s) \wedge h(t))$$

($\Rightarrow$) Let $f = \mathbf{e}$, $g(q) = \begin{cases} 1, & q \geq e, \\ 0, & \text{otherwise.} \end{cases}$

Then, for any $z \in (0, 1)$,
$$(I) = \bigvee_{I_U(e,y) = z} (h(y)) = h(z)$$
and
$$\begin{aligned} (II) &= \bigvee_{I_U(e,s) \vee I_U(q,t) = z,\ q \geq e} (h(s) \wedge h(t)) \\ &= \bigvee_{s \vee I_U(q,t) = z,\ q \geq e} (h(s) \wedge h(t)) \\ &\geq \bigvee_{I_U(q,t) = z,\ s \leq z,\ q \geq e} (h(s) \wedge h(t)) \\ &= h^L(z) \wedge \Big(\bigvee_{I_U(q,t) = z,\ q \geq e} (h(t)) \Big). \end{aligned}$$

Just as the proof of Proposition 2, we can similarly prove that $\bigvee_{I_U(q,t)=z,\ q \geq e} (h(t)) = \bigvee_{t \geq z} (h(t)) = h^R(z)$. Thus, $(II) = h^L(z) \wedge h^R(z)$ and hence $h(z) = h^L(z) \wedge h^R(z)$ for any $z \in (0, 1)$. Again, $h(z) = h^L(z) \wedge h^R(z)$ always holds for $z = 0$ or $z = 1$. Consequently, $h(z) = h^L(z) \wedge h^R(z)$ for any $z \in \mathscr{I}$. That is to say, h is convex on $\mathscr{I}$.

($\Leftarrow$) If $z = 1$, then $(I) = (II)$ always holds. In fact,
$$\begin{aligned} (I) &= \bigvee_{I_U(p \wedge q, y) = 1} (f(p) \wedge g(q) \wedge h(y)) \\ &= \bigvee_{p \wedge q = 0 \text{ or } y = 1} (f(p) \wedge g(q) \wedge h(y)) \\ &= \Big(\bigvee_{p = 0,\ q \geq 0,\ y \geq 0} (f(p) \wedge g(q) \wedge h(y)) \Big) \\ &\bigvee \Big(\bigvee_{q = 0,\ p \geq 0,\ y \geq 0} (f(p) \wedge g(q) \wedge h(y)) \Big) \\ &\bigvee \Big(\bigvee_{p \geq 0,\ q \geq 0,\ y = 1} (f(p) \wedge g(q) \wedge h(y)) \Big) \\ &= (f(0) \wedge g^{LR} \wedge h^{LR}) \vee (g(0) \wedge f^{LR} \wedge h^{LR}) \vee (h(1) \wedge g^{LR} \wedge f^{LR}), \end{aligned}$$

and

$$\begin{aligned}
(II) &= \bigvee_{I_U(p,s)\vee I_U(q,t)=1} (f(p)\wedge g(q)\wedge h(s)\wedge h(t)) \\
&= (\bigvee_{I_U(p,s)=1,\ I_U(q,t)\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&\bigvee (\bigvee_{I_U(q,t)=1,\ I_U(p,s)\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&= (\bigvee_{p=0,\ s,q,t\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&\bigvee (\bigvee_{s=1,\ p,q,t\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&\bigvee (\bigvee_{q=0,\ p,s,t\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&\bigvee (\bigvee_{t=1,\ p,q,s\geq 0} (f(p)\wedge g(q)\wedge h(s)\wedge h(t))) \\
&= (f(0)\wedge h^{LR}\wedge g^{LR}\wedge h^{LR})\vee(f^{LR}\wedge h^{LR}\wedge g^{LR}\wedge h(1)) \\
&\vee (g(0)\wedge h^{LR}\wedge f^{LR}\wedge h^{LR})\vee(h(1)\wedge f^{LR}\wedge g^{LR}\wedge h^{LR}) \\
&= (f(0)\wedge g^{LR}\wedge h^{LR})\vee(g(0)\wedge f^{LR}\wedge h^{LR})\vee(h(1)\wedge g^{LR}\wedge f^{LR}).
\end{aligned}$$

Hence, $(I) = (II)$ for $z = 1$.

If $z \in [0,1)$, then it is obvious that $(I) \leq (II)$. Now, we will prove $(II) \leq (I)$ for $z \in [0,1)$. Let $I_U(p,s) \vee I_U(q,t) = z$ in (II).

(i) Suppose $I_U(p,s) = I_U(q,t) = z$. Then, let $y = s \wedge t$. Thus, $I_U(p \wedge q, y) = z$ and $f(p) \wedge g(q) \wedge h(y) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

(ii) Suppose $I_U(p,s) = z > I_U(q,t)$.

In this case, if $I_U(q,s) \leq z = I_U(p,s)$, then $q \geq p$. Let $y = s$ and then $I_U(p \wedge q, y) = I_U(p,s) = z$ and $f(p) \wedge g(q) \wedge h(y) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

If $I_U(q,s) > z$, then $I_U(q,t) < z < I_U(q,s)$. It can be proved that $q \neq 0$ and 1. Otherwise, if $q = 0$, then $I_U(q,t) = I_U(q,s) = 1$, which implies $1 = I_U(q,t) < z < I_U(q,s) = 1$—a contradiction; if $q = 1$ and $z \in (0,1)$, then from $I_U(1,t) < z < I_U(1,s)$, we have that $t < 1$ and $s = 1$ and hence $I_U(p,s) = I_U(p,1) = 1$, which contradicts $I_U(p,s) = z \in (0,1)$; if $q = 1$ and $z = 0$, then the inequality $I_U(1,t) < z < I_U(1,s)$ can not hold. Thus, $q \in (0,1)$. Because $I_U(q,\cdot)$ is continuous for $q \in (0,1)$, then there exists some $c \in (s,t)$ such that $I_U(q,c) = z$. Again because h is convex, then $h(c) \geq h(s) \wedge h(t)$ and consequently $f(p) \wedge g(q) \wedge h(c) \geq f(p) \wedge g(q) \wedge h(s) \wedge h(t)$.

(iii) Suppose $I_U(q,t) = z > I_U(p,s)$. It is similar to (ii).

From the above, we know that if h is convex on $[0,1]$, then $(II) \leq (I)$ for any $z \in \mathscr{I}$. □

Theorem 8. *Let $\mathscr{A} \subseteq \mathscr{F}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, U be a representable uninorm, I_U be a RU-implication derived from U and $\rhd_{I_U}$ be its extended operation. If $\mathscr{A} \subseteq \mathscr{F}_{CN}$, then $\rhd_{I_U}$ is a type-2 fuzzy implication. In addition,*

$$(f \rhd_{I_U} g)(z) = \begin{cases} (f(1)\wedge g^l(1))\vee(f^r(0)\wedge g(0)) & z=0, \\ (f(0)\wedge g^{LR})\vee(g(1)\wedge f^{LR}) & z=1, \\ \bigvee_{x\in(0,1)} \Big(f(x)\wedge\big(g\circ h^{-1}(h(z)+h(x))\big)\Big) \ or & \\ \bigvee_{y\in(0,1)} \Big(g(y)\wedge\big(f\circ h^{-1}(h(y)-h(z))\big)\Big) & otherwise, \end{cases} \tag{9}$$

where h is an additive generator of U.

Proof. The proof is similar to Theorem 4. □

Theorem 9. *Let $\mathscr{A} \subseteq \mathscr{F}_{CN}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, U be a representable uninorm, I_U be a RU-implication derived from U, and $\rhd_{I_U}$ be a type-2 fuzzy implication on $\mathbf{A}$. Then, we have the following facts:*

(i) $\mathbf{e} \rhd_{I_U} g = g$, where e is the neutral element of uninorm U.

(ii) $\mathbf{0} \rhd_{I_U} g = \mathbf{1}$; $(f \rhd_{I_U} \mathbf{0})(z) = \begin{cases} f^r(0) & z = 0 \\ f(0) & z = 1, \\ 0 & \textit{otherwise.} \end{cases}$

(iii) $f \rhd_{I_U} \mathbf{1} = \mathbf{1}$; $(\mathbf{1} \rhd_{I_U} g)(z) = \begin{cases} g^l(1) & z = 0 \\ g(1) & z = 1, \\ 0 & \textit{otherwise.} \end{cases}$

(iv) If U_c is a conjunctive uninorm given by $U_c(x,y) = N \circ U(N(x), N(y))$, i.e., U_c is dual with U w.r.t N, then $f_1 \rhd_{I_U} (f_2 \rhd_{I_U} f_3) = (f_1 \rhd_{U_c} f_2) \rhd_{I_U} f_3$.

Proof. It is easy to prove it. □

For extended RU-coimplications that are derived from representable uninorms, we can similarly obtain the following results.

Theorem 10. *Let $\mathscr{A} \subseteq \mathscr{F}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, U be a representable uninorm, J_U be a RU-coimplication derived from U and $\rhd_{J_U}$ be its extended operation. If $\mathscr{A} \subseteq \mathscr{F}_{CN}$, then $\rhd_{J_U}$ is a type-2 fuzzy coimplication on $\mathbf{A}$. In addition,*

$$(f \rhd_{J_U} g)(z) = \begin{cases} (f(0) \wedge g^{LR}) \vee (f^{LR} \wedge g(0)) & z = 0, \\ (f(0) \wedge g^r(0)) \vee (g(1) \wedge f^l(1)) & z = 1, \\ \bigvee\limits_{x \in (0,1)} \Big(f(x) \wedge \big(g \circ h^{-1}(h(z) + h(x))\big)\Big) \textit{ or} & \\ \bigvee\limits_{y \in (0,1)} \Big(g(y) \wedge \big(f \circ h^{-1}(h(y) - h(z))\big)\Big) & \textit{otherwise,} \end{cases} \tag{10}$$

where h is an additive generator of U.

Theorem 11. *Let $\mathscr{A} \subseteq \mathscr{F}_{CN}$, $\mathbf{A} = (\mathscr{A}, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$, U be a representable uninorm, J_U be a RU-coimplication derived from U and $\rhd_{J_U}$ be a type-2 fuzzy coimplication. Then, the following hold:*

(i) $\mathbf{e} \rhd_{J_U} f = f$, where e is the neutral element of uninorm U.

(ii) $f \rhd_{J_U} \mathbf{0} = \mathbf{0}$; $(\mathbf{0} \rhd_{J_U} f)(z) = \begin{cases} f^r(0) & z = 1 \\ f(0) & z = 0, \\ 0 & \textit{otherwise.} \end{cases}$

(iii) $\mathbf{1} \rhd_{J_U} f = \mathbf{0}$; $(f \rhd_{J_U} \mathbf{1})(z) = \begin{cases} f^l(1) & z = 1 \\ f(1) & z = 0, \\ 0 & \textit{otherwise.} \end{cases}$

(iv) If U_d is a disjunctive uninorm given by $U_d(x,y) = N \circ U(N(x), N(y))$, then $f_1 \rhd_{J_U} (f_2 \rhd_{J_U} f_3) = (f_1 \rhd_{U_c} f_2) \rhd_{J_U} f_3$.

6. Conclusions

Uninorms and fuzzy implications are important operations in type-1 fuzzy sets. In this work, by Zadeh's extension principle, we extended uninorms and fuzzy implications (coimplications) to type-2 fuzzy sets and defined type-2 uninorms and fuzzy implications (coimplications). We focused on discussing in which algebras of fuzzy truth values extended representable uninorms and its fuzzy implications (coimplications) are type-2 uninorms and fuzzy implications (coimplications), respectively. First, extended representable uninorms were discussed. According to the distributive equation of extended conjunctive representable uninorms over type-2 meet, we had the sufficient and necessary conditions under which extended conjunctive representable uninorms are type-2 uninorms. Similar results were obtained for extended disjunctive representable uninorms. As for extended fuzzy implications, including extended (U,N)-implications and RU-implications, which are derived from representable uninorms, we proved that, in the algebra of convex and normal fuzzy truth values, they are type-2 fuzzy implications. Similarly, we obtained results for extended (U,N)-coimplications and RU-coimplications.

Since Wang and Hu [29] proposed the concept of generated extended fuzzy implications, in future work, we will also study generalized extended uninorms.

Acknowledgments: This work was supported by the National Natural Science Foundations of China (No.11501281).

Author Contributions: Aifang Xie is the sole author who read and approved the final manuscript.

Conflicts of Interest: Because Aifang Xie is the sole author, there does not exist competing interests with others.

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