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Constructions of Helicoidal Surfaces in Euclidean Space with Density

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Abstract: Our principal goal is to study the prescribed curvature problem in a manifold with density. In particular, we consider the Euclidean 3-space $\mathbb{R}^3$ with a positive density function e^{ϕ}, where $\phi = -x^2 - y^2$, $(x, y, z) \in \mathbb{R}^3$ and construct all the helicoidal surfaces in the space by solving the second-order non-linear ordinary differential equation with the weighted Gaussian curvature and the mean curvature functions. As a result, we give a classification of weighted minimal helicoidal surfaces as well as examples of helicoidal surfaces with some weighted Gaussian curvature and mean curvature functions in the space.

Keywords: manifold with density; weighted curvature; helicoidal surface

1. Introduction

Differential geometers have been of interest in studying surfaces of constant mean curvature and constant Gaussian curvature in space forms for a long time. As a generalization of surfaces with constant Gaussian curvature or mean curvature, Kenmotsu [1], who generalized an old result of Delaunay [2], constructed surfaces of revolution with the mean curvature as a smooth function. A helicoidal surface in the Euclidean 3-space $\mathbb{R}^3$ is defined as the orbit of a plane curve under a helicoidal motion. As for helicoidal surfaces in $\mathbb{R}^3$, the surfaces with prescribed mean or Gaussian curvature have been studied by Baikoussis and Koufogiorgos [3]. On the other hand, Beneki et al. [4] and Ji and Hou [5–7] extended it in a Minkowski space. Recently, in [8], Yoon et al. also constructed helicoidal surfaces in a Heisenberg group for such a case.

A density on a Riemannian manifold is a positive function Φ, weighting both volume and surface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the weighted volume and area are given by $dV = \Phi dV_0$ and $dA = \Phi dA_0$, respectively. Manifolds with densities (called also a weighted manifold) arise naturally in geometry as quotients of other Riemannian manifolds, in physics as spaces with different media, in probability as the famous Gauss space $\mathbb{G}^3$ with $\Phi = ce^{a^2r^2}$ for $a, c \in \mathbb{R}$ and $r^2 = x^2 + y^2 + z^2$. Also, it was instrumental in Perelman's proof of the Poincare conjecture [9].

By using the first variation of the weighted area, the mean curvature H_ϕ of a surface in the Euclidean 3-space $\mathbb{R}^3$ with density $\Phi = e^{\phi}$ can be defined. It is given by

$$H_\phi = H - \frac{1}{2}\langle \mathbf{N}, \bigtriangledown\phi \rangle, \tag{1}$$

where H and $\mathbf{N}$ are the mean curvature and the unit normal vector of a surface and $\bigtriangledown\phi$ is the gradient of ϕ, which is called the weighted mean curvature or the ϕ-mean curvature of a surface. The weighted mean curvature H_ϕ of a surface in $\mathbb{R}^3$ with density e^{ϕ} was introduced by Gromov [10] and it is a natural generalization of the mean curvature H of a surface. A surface with $H_\phi = 0$ is called a weighted minimal surface or a ϕ-minimal surface in $\mathbb{R}^3$.

Another curvature for a surface in the Euclidean 3-space is the Gaussian curvature. In [11], authors introduced a generalized Gaussian curvature of a surface in a manifold with density e^ϕ and it is defined by

$$G_\phi = G - \Delta\phi, \tag{2}$$

where G is the Riemannain Gaussian curvature of a surface and Δ is the Laplacian operator, which is called the weighted Gaussian curvature or the ϕ-Gaussian curvature of a surface. Also, they obtained a generalization of the Gauss–Bonnet formula for a smooth disc in a smooth surface with density e^ϕ.

For more details about manifolds with density and some relative topics, we refer readers to [12–17], etc. In particular, Hieu and Hoang [13] studied ruled surfaces and translation surfaces in $\mathbb{R}^3$ with density e^z and they classified the weighted minimal ruled surfaces and translation surfaces. Lopez [15] considered a linear density $e^{ax+by+cz}$, $a, b, c \in \mathbb{R}$, and he classified the weighted minimal translation surfaces and cyclic surfaces in a Euclidean 3-space $\mathbb{R}^3$. Also, Belarbi and Belkhelfa [18] investigated the properties of the weighted minimal graphs in $\mathbb{R}^3$ with a linear density.

In this article, we focus on a class of helicoidal surfaces in the Euclidean 3-space $\mathbb{R}^3$ with density e^ϕ, where $\phi(p) = -x^2 - y^2$, $p = (x, y, z) \in \mathbb{R}^3$. In particular, we construct all helicoidal surfaces in the space, in terms of the weighted Gaussian curvature and mean curvature, as smooth functions.

2. Preliminaries

We consider a regular plane curve $\gamma(u) = (g(u), 0, f(u))$ with $g(u) > 0$ in the xz-plane which is defined on a open interval $I \subset \mathbb{R}$. A surface M in the Euclidean 3-space $\mathbb{R}^3$ defined by

$$X(u,v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g(u) \\ 0 \\ f(u) \end{pmatrix} + h \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{3}$$

where h is a constant, is said to be the helicoidal surface with axis Oz, a pitch h and the profile curve γ. That is, M can be parametrized by

$$X(u,v) = (g(u)\cos v, g(u)\sin v, f(u) + hv).$$

We assume, without loss of generality, $\gamma(u) = (u, 0, f(u))$ is the profile curve in the xz-plane defined on any open interval I of positive real numbers. Then, the helicoidal surface M in $\mathbb{R}^3$ is given by

$$X(u,v) = (u\cos v, u\sin v, f(u) + hv), \tag{4}$$

where f is a smooth function defined on I.

By a direct computation, the Gaussian curvature G and the mean curvature H of the surface are given by

$$G = \frac{1}{D^2}\left[u^3 f'(u) f''(u) - h^2\right],$$
$$H = \frac{1}{2D^{\frac{3}{2}}}\left[(u^2 + h^2)u f''(u) + u^2 f'^3(u) + (u^2 + 2h^2) f'(u)\right],$$

where $D = (1 + f'^2(u))u^2 + h^2 > 0$. On the other hand, the unit normal vector $\mathbf{N}$ of the surface is

$$\mathbf{N} = \frac{1}{\sqrt{D}}\left(h\sin v - uf'(u)\cos v, -uf'(u)\sin v - h\cos v, u\right).$$

Suppose that M is the surface in $\mathbb{R}^3$ with density e^{ϕ}, where $\phi = -x^2 - y^2$. Then, in this case, the weighted mean curvature H_ϕ and the weighted Gaussian curvature G_ϕ can be expressed as

$$H_\phi = \frac{1}{2D^{\frac{3}{2}}} \left[(u^2 + h^2) u f''(u) + (u^2 - 2u^4) f'^3(u) + (u^2 + 2h^2 - 2u^4 - 2h^2u^2) f'(u) \right] \tag{5}$$

and

$$G_\phi = \frac{1}{D^2} \left(u^3 f'(u) f''(u) - h^2 \right) - 4, \tag{6}$$

respectively.

3. Main Theorems and Examples

In this section, we construct helicoidal surfaces with prescribed weighted Gaussian curvature and weighted mean curvature in the Euclidean 3-space $\mathbb{R}^3$ with density $e^{-x^2-y^2}$, where $(x, y, z) \in \mathbb{R}^3$.

3.1. *The Solution of Equation (5)*

Equation (5) is a second-order nonlinear ordinary differential equation. To solve it, we put

$$A = \frac{f'(u)}{\sqrt{D}}. \tag{7}$$

Then, Equation (5) can be expressed in the form:

$$H_\phi = uA' + (2 - 2u^2)A,$$

equivalently,

$$A' + \left(\frac{2}{u} - 2u \right) A = \frac{1}{u} H_\phi. \tag{8}$$

It is a first-order linear ordinary differential equation with respect to A and its general solution is given by

$$A = \frac{e^{u^2}}{u^2} \left(\int u e^{-u^2} H_\phi du + c_1 \right), \tag{9}$$

where $c_1 \in \mathbb{R}$. On the other hand, Equations (7) and (9) imply

$$\left[u^2 - e^{2u^2} \left(\int u e^{-u^2} H_\phi du + c_1 \right)^2 \right] f'^2(u) = \frac{u^2 + h^2}{u^2} e^{2u^2} \left(\int u e^{-u^2} H_\phi du + c_1 \right)^2. \tag{10}$$

Since

$$u^2 - e^{2u^2} \left(\int u e^{-u^2} H_\phi du + c_1 \right)^2 = \frac{1}{D} (u^4 + u^2 h^2) > 0,$$

thus the general solution of Equation (10) becomes

$$f(u) = \pm \int \frac{e^{u^2} \sqrt{u^2 + h^2} (\int u e^{-u^2} H_\phi du + c_1)}{u \left(u^2 - e^{2u^2} (\int u e^{-u^2} H_\phi du + c_1)^2 \right)^{\frac{1}{2}}} du + c_2, \tag{11}$$

where c_2 is constant.

Conversely, let h be a given non-zero real constant and $H_\phi(u)$ be a real-valued smooth function defined on an open interval $I \subset (0, +\infty)$. Then, for any $u_0 \in I$, there exist an open subinterval I' of u_0 $(I' \subset I)$ and an open interval J of $\mathbb{R}$ containing

$$c_1' = -\left(\int ue^{-u^2} H_\phi du\right)(u_0)$$

such that the function

$$F(u, c_1) = u^2 - e^{2u^2}\left(\int ue^{-u^2} H_\phi du + c_1\right)^2 > 0$$

for any $(u, c_1) \in I' \times J$. In fact, because $F(u_0, c_1') = u_0^2 > 0$, by the continuity of F, it is positive in a subset of $I' \times J \subset \mathbb{R}^2$. Therefore, for any $(u, c_1) \in I' \times J$, $c_2 \in \mathbb{R}$, $h \in \mathbb{R}$ and any given smooth function H_ϕ, we can define the two-parameter family of curves

$$\gamma(u, H_\phi, h, c_1, c_2) = \left(u,\ 0,\ \pm \int \frac{e^{u^2}\sqrt{u^2+h^2}(\int ue^{-u^2} H_\phi du + c_1)}{u\left(u^2 - e^{2u^2}(\int ue^{-u^2} H_\phi du + c_1)^2\right)^{\frac{1}{2}}} du + c_2\right).$$

Applying the one-parameter subgroup Φ_t^h on these curves, we can obtain a two-parameter family of helicoidal surfaces with the weighted mean curvature H_ϕ.

Theorem 1. *Let $\gamma(u) = (u, 0, f(u))$ be a profile curve of the helicoidal surface Equation (4) in the Euclidean 3-space with density $e^{-x^2-y^2}$ of which the weighted mean curvature at the point $(u, 0, f(u))$ is given by $H_\phi(u)$. Then, for some constants c_1, c_2 and h, there exists the two-parameter family of helicoidal surfaces generated by plane curves*

$$\gamma(u, H_\phi(u), h, c_1, c_2) = \left(u,\ 0,\ \pm \int \frac{e^{u^2}\sqrt{u^2+h^2}(\int ue^{-u^2} H_\phi du + c_1)}{u\left(u^2 - e^{2u^2}(\int ue^{-u^2} H_\phi du + c_1)^2\right)^{\frac{1}{2}}} du + c_2\right).$$

Conversely, let $H_\phi(u)$ be a smooth function. Then, we can construct the two-parameter family of curves $\gamma(u, H_\phi(u), c_1, c_2)$ and so it is the two-parameter family of helicoidal surfaces with the weighted mean curvature $H_\phi(u)$ and a pitch h.

Corollary 1. *Let M be a weighted minimal helicoidal surface in the Euclidean 3-space with density $e^{-x^2-y^2}$. Then, M is an open part of either a helicoid or a surface parameterized by*

$$X(u, v) = (u \cos v,\ u \sin v, f(u) + hv),$$

where

$$f(u) = \pm \int \frac{c_1 e^{u^2}\sqrt{u^2+h^2}}{u\sqrt{u^2 - c_1^2 e^{2u^2}}} du + c_2 \tag{12}$$

for some constants c_1 and c_2.

Proof. If f is a constant function, it is a trivial solution for $H_\phi = 0$. It follows that a helicoidal surface is a helicoid. Otherwise, we can easily compute Equation (11), in such case f is given by (12). □

Example 1. *We consider a helicoidal surface with the constant weighted mean curvature*

$$H_\phi(u) = -2$$

and the pitch $h = 1$ in the Euclidean 3-space with density $e^{-x^2-y^2}$. Then, by Equation (11), we can calculate the profile curve $\gamma(u)$; from this, the parametrization of the surface is given by (see Figure 1)

$$X(u,v) = \left(u\cos v, u\sin v, \frac{1}{2}\ln\left(2u^2 + 2\sqrt{u^4-1}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{u^4-1}}\right) + v \right).$$

Example 2. *Consider a helicoidal surface with the weighted mean curvature*

$$H_\phi(u) = \frac{1}{\sqrt{2}u}(1-2u^2)$$

and the pitch $h = 1$ in the Euclidean 3-space with density $e^{-x^2-y^2}$. By a direct computation with the help of Equation (11), we can compute the profile curve $\gamma(u)$, which implies that the parametrization of the surface is expressed in the form (see Figure 2):

$$X(u,v) = \left(u\cos v, u\sin v, \sqrt{u^2+1} - \tan^{-1}\left(\frac{1}{\sqrt{u^2+1}}\right) + v \right).$$

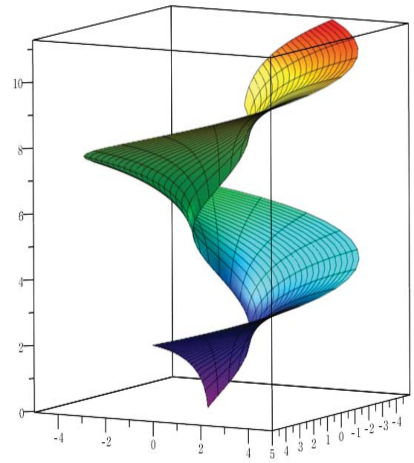

Figure 1. A helicoidal surface with $H_\phi(u) = -2$.

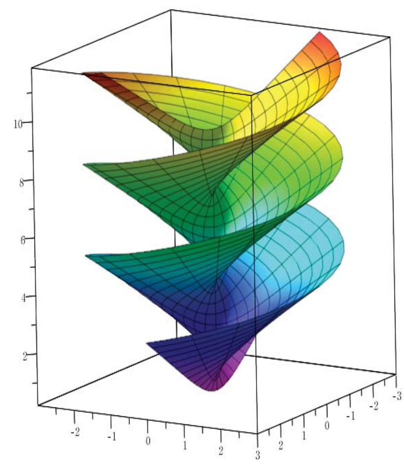

Figure 2. A helicoidal surface with $H_\phi(u) = \frac{1}{\sqrt{2}u}(1-2u^2)$.

3.2. *The Solution of Equation (6)*

To solve the second-order nonlinear ordinary differential Equation (6), we put

$$B = \frac{u^2 f'^2(u) + h^2}{D}. \tag{13}$$

Then, the weighted Gaussian curvature G_ϕ can be rewritten as:

$$G_\phi = \frac{1}{2u} B' - 4,$$

that is,

$$B' = 2uG_\phi + 8u. \tag{14}$$

The solution of the last equation is

$$B = 4u^2 + \int 2uG_\phi du + c_1 \tag{15}$$

for some constant c_1. Combining Equations (13) and (15), one gets

$$u^2 \left(1 - 4u^2 - \int 2uG_\phi du - c_1\right) f'^2(u) = (u^2 + h^2) \left(4u^2 + \int 2uG_\phi du + c_1\right) - h^2. \tag{16}$$

Since

$$1 - 4u^2 - \int 2uG_\phi du - c_1 = \frac{u^2}{D} > 0,$$

the general solution of Equation (16) is given by

$$f(u) = \pm \int \frac{1}{u} \left[\frac{(u^2 + h^2)\left(4u^2 + \int 2uG_\phi du + c_1\right) - h^2}{1 - 4u^2 - \int 2uG_\phi du - c_1} \right]^{\frac{1}{2}} du + c_2, \tag{17}$$

where $c_2 \in \mathbb{R}$.

Conversely, let h be a given real number and G_ϕ be a smooth function defined on an open interval $I \subset (0, +\infty)$. Let

$$F(u, c_1) = 1 - 4u^2 - \int 2uG_\phi du - c_1$$

be a function defined on $I \times \mathbb{R} \subset \mathbb{R}^2$. For any $u_0 \in I$, denote

$$c_1' = -\left(4u^2 + \int 2uG_\phi du\right)(u_0).$$

Thus, we can find an open subinterval $I' \subset I$ containing u_0 and an open interval J of $\mathbb{R}$ containing c_1' such that the function $F(u, c_1)$ is positive for any $(u, c_1) \in I' \times J$. In fact, because $F(u_0, c_1') = 1$, by the continuity of F, it is positive in a subset of $I' \times J \subset \mathbb{R}^2$. Theeqrefore, for any $(u, c_1) \in I' \times J$, $h \in \mathbb{R}$, $c_2 \in \mathbb{R}$ and given the smooth function G_ϕ, we can define the two-parameter family of curves

$$\gamma(u, G_\phi, h, c_1, c_2) = \left(u, 0, \pm \int \frac{1}{u} \left[\frac{(u^2 + h^2)\left(4u^2 + \int 2uG_\phi du + c_1\right) - h^2}{1 - 4u^2 - \int 2uG_\phi du - c_1} \right]^{\frac{1}{2}} du + c_2\right).$$

Consequently, we get a two-parameter family of helicoidal surfaces with the weighted Gaussian curvature $G_\phi(u)$, $u \in I'$ and we have the following theorem.

Theorem 2. *Let $\gamma(u) = (u, 0, f(u))$ be a profile curve of the helicoidal surface (4) in the Euclidean 3-space with density $e^{-x^2-y^2}$ of which the weighted Gaussian curvature at the point $(u, 0, f(u))$ is given by $G_\phi(u)$. Then, for some constants c_1, c_2 and h, there exists the two-parameter family of the helicoidal surface generated by plane curves*

$$\gamma(u, G_\phi, h, c_1, c_2) = \left(u, 0, \pm \int \frac{1}{u} \left[\frac{(u^2+h^2)\left(4u^2 + \int 2uG_\phi du + c_1\right) - h^2}{1 - 4u^2 - \int 2uG_\phi du - c_1}\right]^{\frac{1}{2}} du + c_2\right).$$

Conversely, let $G_\phi(u)$ be a smooth function. Then, for any $u_0 \in I$, we can construct the two-parameter family of curves $\gamma(u, G_\phi(u), h, c_1, c_2)$, $u \in I' \subset I$ and so it is the two-parameter family of helicoidal surfaces with the weighted Gaussian curvature $G_\phi(u)$, $u \in I'$.

Example 3. *We consider a helicoidal surface in the Euclidean 3-space with density $e^{-x^2-y^2}$ with a negative weighted Gaussian curvature*

$$G_\phi(u) = -\frac{1}{(2u^2+1)^2} - 4.$$

In such a case, an integration of Equation (17) implies $f(u) = u$ for $h = 1$, $c_1 = 0$ and $c_2 = 0$. It follows that the helicoidal surface is parametrized by (see Figure 3)

$$X(u, v) = (u\cos v, u\sin v, u + v).$$

Example 4. *Consider a helicoidal surface in the Euclidean 3-space with density $e^{-x^2-y^2}$ with a weighted Gaussian curvature*

$$G_\phi(u) = \frac{2u^2-1}{(u^4-u^2-1)^2} - 4.$$

Then, from Equation (17), we have $f(u) = \sin^{-1} u$ for $h = 1$, $c_1 = 0$ and $c_2 = 0$ and, in this case, a parametrization of the helicoidal surface is given by (see Figure 4)

$$X(u, v) = (u\cos v, u\sin v, \sin^{-1} u + v).$$

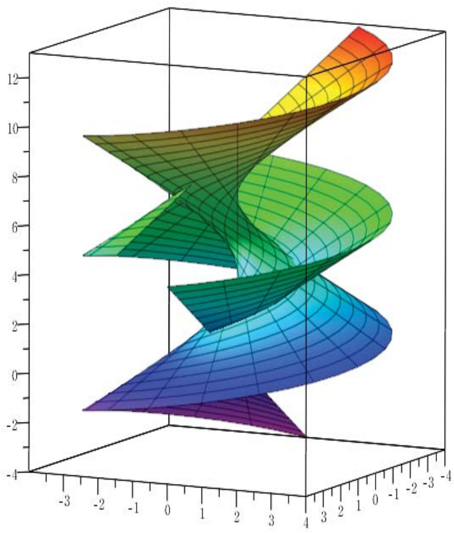

Figure 3. A helicoidal surface with $G_\phi(u) = -\frac{1}{(2u^2+1)^2} - 4$.

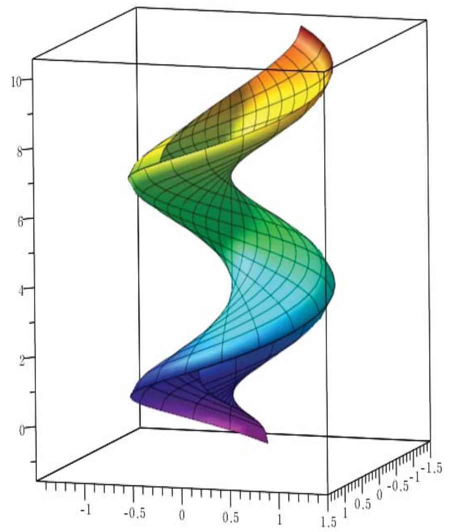

Figure 4. A helicoidal surface with $G_\phi(u) = \frac{2u^2-1}{(u^4-u^2-1)^2} - 4$.

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