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On Characterizations of Directional Derivatives and Subdifferentials of Fuzzy Functions

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Abstract: In this paper, based on a partial order, we study the characterizations of directional derivatives and the subdifferential of fuzzy function. At the same time, we also discuss the relation between the directional derivative and the subdifferential.

Keywords: fuzzy function; fuzzy number; directional derivative; subdifferential

1. Introduction

In 1965, Zadeh [1] introduced concepts and operations with respect to fuzzy sets, and many authors contributed to the development of fuzzy set theory and applications. Later, Zadeh proposed the fuzzy number [2–4] and put forward the theory of the fuzzy numerical function together with Chang [5]. These theories and those associated with optimization theory have been extensively studied in some fields, such as economics, engineering, the stock market, greenhouse gas emissions and management science [6–14].

In order to solve complex optimization problems in real life, various optimization algorithms have been presented in [9,10,15–23]. Jia et al. [9] presented a new algorithm for solving the optimization problem based on the stock exchange. Afterwards, in [10,15,16], a multiple genetic algorithm and multi-objective differential evolution were used to solve multiple optimization problems efficiently. Moreover, Wah et al. [17] applied a genetic algorithm to optimize flow rectification efficiency of the diffuser element based on the valveless diaphragm micropump application. Precup et al. [18] applied the grey wolf optimizer algorithm to deal with the fuzzy optimization problem. In [19], in order to solve the meta-heuristics optimization problem quickly, a bio-inspired optimization algorithm based on fuzzy logic was proposed. Peraza et al. [20] presented a new algorithm that can solve the complex optimization problems based on uncertainty management. They [22] introduced a fuzzy harmony search algorithm with fuzzy logic and this algorithm was utilized to solve the fuzzy optimization problem. Amador et al. [23], presented a new optimization algorithm based on the fuzzy logic system.

It is well known that convexity plays a key role in fuzzy optimization theory. Therefore, the properties of convexity of fuzzy function and related problems are attached a wide range of research [24–30]. Subdifferentials are very important tools in convex fuzzy optimization theory. Based on a variety of different backgrounds, the derivative and differential of fuzzy function have been widely discussed. Goetschel et al. [31,32] defined the derivative of fuzzy function, which is a generalized derivative of the set-valued function. Afterwards, Buckley et al. [33,34] defined the derivatives of fuzzy function using left- and right-hand functions of its α-level sets and established sufficient conditions for the existence of fuzzy derivatives. Subsequently, Wang et al. [29] proposed the new concepts of directional derivative, differential and subdifferential of fuzzy function from $\mathbb{R}^m$ to E^1, and discussed the characterizations of directional derivative and differential of fuzzy function by using the directional derivative and the differential of two crisp functions that are determined.

In this paper, we investigate several characterizations of directional derivative of fuzzy function about the direction d, based on a partial order and introduce the concept of the subdifferential of fuzzy function.

2. Preliminaries

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, that is,

$$\mathcal{K}_C = \{[a_L, a_R] | a_L, a_R \in \mathbb{R} \text{ and } a_L \le a_R\}.$$

Given two intervals $A = [a_L, a_R]$ and $B = [b_L, b_R]$, the distance between A and B is defined by

$$H(A, B) = \max\{|a_L - b_L|, |a_R - b_R|\}.$$

Then, $(\mathcal{K}_C, H)$ is a complete metric space [35].

Definition 1. *In reference [27], suppose that $E^1 = \{u | u : \mathbb{R} \to [0, 1]\}$ satisfies the following conditions:*

(1) u is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(2) u is upper semicontinuous;
(3) u is convex, that is,

$$u(\lambda x + (1 - \lambda)y) \ge min\{u(x), u(y)\}$$

for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$;
(4) $[u]^0 = \overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Any $u \in E^1$, is called a fuzzy number. The α-level set of fuzzy number u is a closed and bounded interval $[u_L(\alpha), u_R(\alpha)]$, where $u_L(\alpha)$ denotes the left-hand end point of $[u]^\alpha$ and $u_R(\alpha)$ denotes the right-hand end point of $[u]^\alpha$ [36].

For $u, v \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by,

$$(u + v)(x) = \sup_{s+t=x} \min\{u(s), v(t)\},$$

$$(ku)(x) = \begin{cases} u(k^{-1}x), & k \ne 0, \\ 0, & k = 0. \end{cases}$$

It is well known that for $u, v \in E^1$, $k \in \mathbb{R}$, then $u + v$, $ku \in E^1$, $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[ku]^\alpha = k[u]^\alpha$.

For $x = (x_1, x_2, ..., x_m)$, $y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$, it is said that $x \ge y$ if and only if $x_i \ge y_i$ $(i = 1, 2, ..., m)$. It is said that $x > y$ if and only if $x \ge y$ and $x \ne y$.

Definition 2. *In reference [29], for $u, v \in E^1$, then,*

(1) $u \preceq v$ if and only if $[u]^\alpha = [u_L(\alpha), u_R(\alpha)] \le [v]^\alpha = [u_L(\alpha), u_R(\alpha)]$ for each $\alpha \in [0, 1]$, where $[u]^\alpha \le [v]^\alpha$ if and only if $u_L(\alpha) \le v_L(\alpha)$ and $u_R(\alpha) \le v_R(\alpha)$.
(2) $u \prec v$ if and only if $u \preceq v$ and there exists $\alpha_0 \in [0, 1]$ such that $u_L(\alpha_0) < v_L(\alpha_0)$ or $u_R(\alpha_0) < v_R(\alpha_0)$.
(3) if either $u \preceq v$ or $v \preceq u$, then u and v are comparable. Otherwise, u and v are non-comparable.

Definition 3. *In reference [37], if any $u, v \in E^1$, there exists $w \in E^1$ such that $u = v + w$, then the standard Hukuhara difference (H-difference) of u and v is defined by $u \tilde{-} v = w$.*

Definition 4. *In reference [37], (Fuzzy function) let D be a convex set of $\mathbb{R}$ and $F : D \to E^1$ be a fuzzy function. The α-level set of F at $x \in D$, which is a closed and bounded interval, can be denoted by $[F(x)]^\alpha = [F_L(x, \alpha), F_R(x, \alpha)]$. Thus, F can be understood by the two functions $F_L(x, \alpha)$ and $F_R(x, \alpha)$, which are functions* *from $D \times [0,1]$ to the set of real numbers $\mathbb{R}$, $F_L(x,\alpha)$ is a bounded increasing function of α and $F_R(x,\alpha)$ is a bounded decreasing function of α. Moreover, $F_L(x,\alpha) \leq F_R(x,\alpha)$ for each $\alpha \in [0,1]$.*

Definition 5. *[35] For $u, v \in E^1$, the d_∞-distance is defined by the Hausdorff metric as,*

$$\begin{aligned} d_\infty(u,v) &= \sup_{\alpha\in[0,1]} H([u]^\alpha, [v]^\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\{|u_L(\alpha) - v_L(\alpha)|, |u_R(\alpha) - v_R(\alpha)|\}. \end{aligned}$$

Definition 6. *Let $X = (a,b)$ and let $F : X \to E^1$ be a fuzzy function and $\{F_n(x)\} : X \to E^1$, $n \in N$ be a sequence of fuzzy function. If, for any $\varepsilon > 0$, there exists a positive integer $M = M(\varepsilon) \in N$ such that,*

$$D(F_n(x), F(x)) < \varepsilon$$

for any $x \in X$, all $n \geq M$. Then the sequence $\{F_n(x)\}$ is convergent to $F(x)$.

Definition 7. *In reference [34], let $F : X \to E^1$ be a fuzzy function. Assume that the partial derivatives of $F_L(x,\alpha), F_R(x,\alpha)$ with respect to $x \in \mathbb{R}$ for each $\alpha \in [0,1]$ exist. The partial derivatives of $F_L(x,\alpha)$ and $F_R(x,\alpha)$ are denoted by $F_L'(x,\alpha)$ and $F_R'(x,\alpha)$, respectively. Let $\Gamma(x,\alpha) = [F_L'(x,\alpha), F_R'(x,\alpha)]$ for $x \in \mathbb{R}$, $\alpha \in [0,1]$. $\Gamma(x,\alpha)$ defines the α-level set of fuzzy interval for $x \in \mathbb{R}$. Then F is S-differentiable and is written as,*

$$[\frac{dF(x)}{dx}]^\alpha = \Gamma(x,\alpha) = [F_L'(x,\alpha), F_R'(x,\alpha)]$$

for $x \in \mathbb{R}$, $\alpha \in [0,1]$.

3. Directional Derivative of the Fuzzy Function

Inspired by [29], we discuss some relations among the gradient and directional derivative of fuzzy function. Moreover, several characterizations of the directional derivative of fuzzy function about the direction d are investigated, based on the partial order $\preceq$.

Definition 8. *In reference [36], (Gradient of a fuzzy function) let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function. For $x \in D$ and $\frac{\partial}{\partial x_i}$ $(i = 1,2,...,m)$ stand for the partial differentiation with respect to the ith variable x_i. For each $\alpha \in [0,1]$, $F_L(x,\alpha)$ and $F_R(x,\alpha)$ have continuous partial derivatives so that $\frac{\partial F_L(x,\alpha)}{\partial x_i}$ and $\frac{\partial F_R(x,\alpha)}{\partial x_i}$ are continuous about x. Define*

$$[\frac{\partial F(x)}{\partial x_i}]^\alpha = [\frac{\partial F_L(x,\alpha)}{\partial x_i}, \frac{\partial F_R(x,\alpha)}{\partial x_i}] \tag{1}$$

for each $i = 1,2,...,m$, $\alpha \in [0,1]$. If for each $i = 1,2,...,m$, (1) defines the α-level set of fuzzy number, then F is S-differential at x. The gradient of the fuzzy function $F(x)$ at x, denoted by $\overset{\sim}{\nabla} F(x)$, is defined as:

$$\overset{\sim}{\nabla} F(x) = \left(\frac{\partial F(x)}{\partial x_1}, \frac{\partial F(x)}{\partial x_2}, ..., \frac{\partial F(x)}{\partial x_m}\right).$$

Remark 1. *For the gradient of fuzzy function, we use the symbol $\overset{\sim}{\nabla}$, whereas for the gradient of a real valued function, we use the symbol ∇.*

Definition 9. *In reference [38], let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function.*

(1) F is convex on D if

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda F(x_1) + (1-\lambda)F(x_2) \tag{2}$$

for any $x_1, x_2 \in D$ *and each* $\lambda \in [0, 1]$.

(2) *F is strictly convex on D if*

$$F(\lambda x_1 + (1-\lambda)x_2) \prec \lambda F(x_1) + (1-\lambda)F(x_2) \tag{3}$$

for any $x_1, x_2 \in D$ *with* $x_1 \neq x_2$ *and each* $\lambda \in (0, 1)$.

Theorem 1. *In reference [36], let D be a convex set of* $\mathbb{R}^m$ *and* $F : D \to E^1$ *be a fuzzy function, F is convex on D, if and only if for each* $\alpha \in [0, 1]$, $F_L(x, \alpha)$ *and* $F_R(x, \alpha)$ *are convex on D, that is, for each* $\lambda \in [0, 1]$, x_1, $x_2 \in D$, *and each* $\alpha \in [0, 1]$,

$$F_L((\lambda x_1 + (1-\lambda)x_2), \alpha) \leq \lambda F_L(x_1, \alpha) + (1-\lambda)F_L(x_2, \alpha) \tag{4}$$

and

$$F_R((\lambda x_1 + (1-\lambda)x_2), \alpha) \leq \lambda F_R(x_1, \alpha) + (1-\lambda)F_R(x_2, \alpha). \tag{5}$$

Theorem 2. *Let D be a convex set of* $\mathbb{R}^m$ *and* $F : D \to E^1$ *be a S-differentiable fuzzy function. Then F is a convex fuzzy function on D if and only if, for any* $x_1, x_2 \in D$ *with* $(x_1 > x_2)$ *such that*

$$\tilde{\nabla} F(x_2)^T (x_1 - x_2) \preceq F(x_1) \tilde{-} F(x_2). \tag{6}$$

Proof. F is a convex fuzzy function on D. According to Definition 9, we obtain that:

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda F(x_1) + (1-\lambda)F(x_2) \tag{7}$$

for any $x_1, x_2 \in D$ with $x_1 > x_2$, and each $\lambda \in [0, 1]$. By Theorem 1, for each $\lambda \in [0, 1]$ we have that

$$F_L((\lambda x_1 + (1-\lambda)x_2), \alpha) \leq \lambda F_L(x_1, \alpha) + (1-\lambda)F_L(x_2, \alpha) \tag{8}$$

and

$$F_R((\lambda x_1 + (1-\lambda)x_2), \alpha) \leq \lambda F_R(x_1, \alpha) + (1-\lambda)F_R(x_2, \alpha). \tag{9}$$

Now combining (8) and (9) imply that

$$\frac{F_L(x_2 + \lambda(x_1 - x_2), \alpha) - F_L(x_2, \alpha)}{\lambda} \leq F_L(x_1, \alpha) - F_L(x_2, \alpha) \tag{10}$$

and

$$\frac{F_R(x_2 + \lambda(x_1 - x_2), \alpha) - F_R(x_2, \alpha)}{\lambda} \leq F_R(x_1, \alpha) - F_R(x_2, \alpha). \tag{11}$$

Taking limits for $\lambda \to 0^+$, we get

$$\nabla F_L(x_2, \alpha)^T (x_1 - x_2) \leq F_L(x_1, \alpha) - F_L(x_2, \alpha)$$

and

$$\nabla F_R(x_2, \alpha)^T (x_1 - x_2) \leq F_R(x_1, \alpha) - F_R(x_2, \alpha).$$

That is,

$$\tilde{\nabla} F(x_2)^T (x_1 - x_2) \preceq F(x_1) \tilde{-} F(x_2). \tag{12}$$

Conversely, since F is a S-differentiable fuzzy function, and there exist $x, y \in D$ with $x > y$ such that

$$\tilde{\nabla} F(y)^T (x - y) \preceq F(x) \tilde{-} F(y).$$

For any $x_1, x_2 \in D$ and each $\lambda \in [0,1]$. Suppose that $x = x_1$ and $y = (1-\lambda)x_1 + \lambda x_2$. It follows that

$$\tilde{\nabla} F(y)^T (x_1 - y) \preceq F(x_1) \tilde{-} F(y). \tag{13}$$

That is,

$$\nabla F_L(y,\alpha)^T (x_1 - y) \leq F_L(x_1,\alpha) - F_L(y,\alpha) \tag{14}$$

and

$$\nabla F_R(y,\alpha)^T (x_1 - y) \leq F_R(x_1,\alpha) - F_R(y,\alpha). \tag{15}$$

Let $x = x_2$ and $y = (1-\lambda)x_1 + \lambda x_2$, we get

$$\tilde{\nabla} F(y)^T (x_2 - y) \preceq F(x_2) \tilde{-} F(y). \tag{16}$$

That is,

$$\nabla F_L(y,\alpha)^T (x_2 - y) \leq F_L(x_2,\alpha) - F_L(y,\alpha) \tag{17}$$

and

$$\nabla F_R(y,\alpha)^T (x_2 - y) \leq F_R(x_2,\alpha) - F_R(y,\alpha). \tag{18}$$

Now combining (14) $\times$ $(1-\lambda)$ and (17) $\times$ λ, we have

$$\nabla F_L(y,\alpha)^T ((1-\lambda)x_1 + \lambda x_2 - y) \leq (1-\lambda)F_L(x_1,\alpha) + \lambda F_L(x_2,\alpha) - F_L(y,\alpha). \tag{19}$$

Similarly,

$$\nabla F_R(y,\alpha)^T ((1-\lambda)x_1 + \lambda x_2 - y) \leq (1-\lambda)F_R(x_1,\alpha) + \lambda F_R(x_2,\alpha) - F_R(y,\alpha). \tag{20}$$

The equations (19) and (20) imply

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda F(x_1) + (1-\lambda)F(x_2).$$

Therefore, F is a convex fuzzy function on D. □

Theorem 3. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a S-differentiable fuzzy function. Then F is a strictly convex fuzzy function on D if and only if, for any $x_1, x_2 \in D$ with $(x_1 > x_2)$ such that*

$$\tilde{\nabla} F(x_2)^T (x_1 - x_2) \prec F(x_1) \tilde{-} F(x_2). \tag{21}$$

Proof. The proof is similar to the proof of Theorem 2. □

Theorem 4. *In reference [39], let D be a convex set of $\mathbb{R}^m$ and $f : D \to (-\infty, +\infty]$ be a convex real valued function. For $x \in D$, let $d \in \mathbb{R}^m$ such that $x + \lambda d \in D$ for any $\lambda > 0$ and sufficiently small. If $h(\lambda) : (0, +\infty) \to (-\infty, +\infty]$ is defined by*

$$h(\lambda) = \frac{f(x + \lambda d) - f(x)}{\lambda},$$

then $h(\lambda)$ is a nondecreasing function. Moreover, if f is differential at x, then

$$\lim_{\lambda \to 0^+} h(\lambda) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda d) - f(x)}{\lambda} = \nabla f(x)^T d.$$

Definition 10. *In reference [36], (directional derivative of a fuzzy function) Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function. For $x \in D$, let $d \in \mathbb{R}^m$ such that $x + \lambda d$ for any $\lambda > 0$ and sufficiently small.*

The directional derivative of F at x along the vector d (if it exists) is a fuzzy number denoted by $F^{'}(x,d)$ whose α-level set is defined as:

$$\left[F^{'}(x,d)\right]^{\alpha} = \left[F_{L}^{'}((x,d),\alpha), F_{R}^{'}((x,d),\alpha)\right],$$

where

$$F_{L}^{'}((x,d),\alpha) = \lim_{\lambda \to 0^{+}} \frac{F_{L}(x+\lambda d,\alpha) - F_{L}(x,\alpha)}{\lambda}$$

and

$$F_{R}^{'}((x,d),\alpha) = \lim_{\lambda \to 0^{+}} \frac{F_{R}(x+\lambda d,\alpha) - F_{R}(x,\alpha)}{\lambda}.$$

Theorem 5. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a convex and S-differentiable fuzzy function. For $x \in D$, let $d \in \mathbb{R}^m$, $d = (d_1, d_2, ..., d_m)$, $d_i > 0, i = 1, 2, ..., m$, for $m \in N$. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x,d)$. Then*

$$F^{'}(x,d) \preceq F(x+d) \tilde{-} F(x).$$

Proof. Since $F : D \to E^1$ is a convex and S-differentiable fuzzy function. From Theorem 2, for $x \in D$, $d \in \mathbb{R}^m$, we obtain

$$\tilde{\nabla} F(x)^T d \preceq F(x+d) \tilde{-} F(x). \tag{22}$$

That is,

$$\nabla F_L(x,\alpha)^T d \leq F_L(x+d,\alpha) - F_L(x,\alpha) \tag{23}$$

and

$$\nabla F_R(x,\alpha)^T d \leq F_R(x+d,\alpha) - F_R(x,\alpha). \tag{24}$$

Since F is a convex fuzzy function. From Definition 10 and Theorem 4, we conclude that

$$F_L'((x,d),\alpha) = \lim_{\lambda \to 0^{+}} \frac{F_L(x+\lambda d,\alpha) - F_L(x,\alpha)}{\lambda} = \nabla F_L(x,\alpha)^T d \tag{25}$$

and

$$F_R'((x,d),\alpha) = \lim_{\lambda \to 0^{+}} \frac{F_R(x+\lambda d,\alpha) - F_R(x,\alpha)}{\lambda} = \nabla F_R(x,\alpha)^T d \tag{26}$$

Now combining (23), (24), (25) and (26), we have

$$F_L'((x,d),\alpha) = \nabla F_L(x,\alpha)^T d \leq F_L(x+d,\alpha) - F_L(x,\alpha) \tag{27}$$

and

$$F_R'((x,d),\alpha) = \nabla F_R(x,\alpha)^T d \leq F_R(x+d,\alpha) - F_R(x,\alpha). \tag{28}$$

The Equations (27) and (28) imply $F^{'}(x,d) \preceq F(x+d) \tilde{-} F(x)$. □

Theorem 6. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a S-differentiable fuzzy function. For $x \in D$, let $d \in \mathbb{R}^m$ such that $x + \lambda d \in D$ for any $\lambda > 0$ and sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x,d)$. Then $F'(x,d)$ is a strictly positive homogeneous fuzzy function.*

Proof. Since the directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x,d)$. By Definition 10, we have,

$$F_{L}^{'}((x,d),\alpha) = \lim_{\lambda \to 0^{+}} \frac{F_{L}(x+\lambda d,\alpha) - F_{L}(x,\alpha)}{\lambda}$$

and

$$F_R'((x,d),\alpha) = \lim_{\lambda \to 0^+} \frac{F_R(x+\lambda d,\alpha) - F_R(x,\alpha)}{\lambda}.$$

For any $t > 0$ and let $\tau = t\lambda$, we obtain

$$\begin{aligned} F_L'((x,td),\alpha) &= \lim_{\lambda \to 0^+} \frac{F_L(x+\lambda td,\alpha)-F_L(x,\alpha)}{\lambda} \\ &= \lim_{\tau \to 0^+} \frac{[F_L(x+\tau d,\alpha)-F_L(x,\alpha)]t}{\tau} \\ &= tF_L'((x,d),\alpha) \end{aligned} \tag{29}$$

and

$$\begin{aligned} F_R'((x,td),\alpha) &= \lim_{\lambda \to 0^+} \frac{F_R(x+\lambda td,\alpha)-F_R(x,\alpha)}{\lambda} \\ &= \lim_{\tau \to 0^+} \frac{[F_R(x+\tau d,\alpha)-F_R(x,\alpha)]t}{\tau} \\ &= tF_R'((x,d),\alpha). \end{aligned} \tag{30}$$

Now combining (29) and (30), we have that

$$\begin{aligned} \left[F'(x,td)\right]^\alpha &= [F_L'((x,td),\alpha), F_R'((x,td),\alpha)] \\ &= t[F_L'((x,d),\alpha), F_R'((x,d),\alpha)] \\ &= t[F'(x,d]^\alpha. \end{aligned}$$

Therefore, $F'(x,d)$ is a strictly positive homogeneous fuzzy function. □

Theorem 7. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a convex and S-differential fuzzy function. For $x \in D$, let $d \in \mathbb{R}^m$ such that $x + \lambda d \in D$ for any $\lambda > 0$ sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F'(x,d)$. Then $F'(x,d)$ is a convex fuzzy function about the direction d.*

Proof. For any $\lambda_1, \lambda_2 \in (0,1)$, any $d^1, d^2 \in \mathbb{R}^m$, let $\lambda_1 = 1 - \lambda_2$. By Definition 10 and Theorem 1, we get that

$$\begin{aligned} F_L'((x,\lambda_1 d^1 + \lambda_2 d^2),\alpha) &= \lim_{\lambda \to 0^+} \frac{F_L\left(x+\lambda(\lambda_1 d^1+\lambda_2 d^2),\alpha\right)-F_L(x,\alpha)}{\lambda} \\ &= \lim_{\lambda \to 0^+} \frac{F_L\left(\lambda_1(x+\lambda d^1)+\lambda_2(x+\lambda d^2),\alpha\right)-F_L(x,\alpha)}{\lambda} \\ &\leq \lim_{\lambda \to 0^+} \frac{\lambda_1[F_L\left(x+\lambda d^1,\alpha\right)-F_L(x,\alpha)]}{\lambda} + \lim_{\lambda \to 0^+} \frac{\lambda_2[F_L\left(x+\lambda d^2,\alpha\right)-F_L(x,\alpha)]}{\lambda} \\ &= \lambda_1 F_L'\left(\left(x,d^1\right),\alpha\right) + \lambda_2 F_L'\left(\left(x,d^2\right),\alpha\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} F_R'((x,\lambda_1 d^1 + \lambda_2 d^2),\alpha) &= \lim_{\lambda \to 0^+} \frac{F_R\left(x+\lambda(\lambda_1 d^1+\lambda_2 d^2),\alpha\right)-F_R(x,\alpha)}{\lambda} \\ &= \lim_{\lambda \to 0^+} \frac{F_R\left(\lambda_1(x+\lambda d^1)+\lambda_2(x+\lambda d^2),\alpha\right)-F_R(x,\alpha)}{\lambda} \\ &\leq \lim_{\lambda \to 0^+} \frac{\lambda_1[F_R\left(x+\lambda d^1,\alpha\right)-F_R(x,\alpha)]}{\lambda} + \lim_{\lambda \to 0^+} \frac{\lambda_2[F_R\left(x+\lambda d^2,\alpha\right)-F_R(x,\alpha)]}{\lambda} \\ &= \lambda_1 F_R'\left(\left(x,d^1\right),\alpha\right) + \lambda_2 F_R'\left(\left(x,d^2\right),\alpha\right). \end{aligned} \tag{32}$$

Hence, by Theorem 1, $F'(x,d)$ is a convex fuzzy function about the direction d. □

Theorem 8. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a convex and S-differential fuzzy function. For $x \in D$, let $d \in \mathbb{R}^m$ such that $x + \lambda d \in D$ for any $\lambda > 0$ sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F'(x,d)$. Then $F'(x,d)$ is subadditive about the direction d.*

Proof. F is a S-differential fuzzy function on D, and the directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x,d)$. For arbitrary $d^1, d^2 \in \mathbb{R}^m$. By Theorem 7, we have that

$$F'(x, \frac{1}{2}d^1 + \frac{1}{2}d^2) \preceq \frac{1}{2}F'(x, d^1) + \frac{1}{2}F'(x, d^2). \tag{33}$$

By Theorem 6, we obtain that

$$F'(x, \frac{1}{2}d^1 + \frac{1}{2}d^2) = \frac{1}{2}F'(x, d^1 + d^2). \tag{34}$$

Now combining (33) and (34), it follows that

$$F'(x, d^1 + d^2) \preceq F'(x, d^1) + F'(x, d^2).$$

Therefore, $F'(x,d)$ is subadditive about the direction d. □

Definition 11. *In reference [40], let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function. Let $\bar{x} \in D$, if there exists $\delta_0 > 0$ and no $x \in U(\bar{x}, \delta_0) \bigcap D$ such that $F(x) \preceq F(\bar{x})$, then $\bar{x}$ is a local minimum solution of $F(x)$.*

Theorem 9. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function. For $\bar{x} \in D$, let $d \in \mathbb{R}^m$ such that $\bar{x} + \lambda d \in D$ for any $\lambda > 0$ sufficiently small. The directional derivative of F at $\bar{x}$ along the vector d is a fuzzy number denoted by $F^{'}(\bar{x}, d)$ if $0 \in$ inter $[F^{'}(\bar{x}, d)]^0$, where inter A denotes the interior of the set A. Then, $\bar{x}$ is a local minimum solution of $F(x)$.*

Proof. Suppose that $\bar{x}$ is not a local minimum solution of $F(x)$. Hence, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ and any $\delta > 0$ such that $x_n = \bar{x} + \lambda d \in U(\bar{x}, \delta) \bigcap D$ ($|\lambda d| < \delta$) and

$$F(\bar{x} + \lambda d) = F(x_n) \preceq F(\bar{x})$$

for all $n \in N$. That is,

$$F_L(\bar{x} + \lambda d, \alpha) \leq F_L(\bar{x}, \alpha)$$

and

$$F_R(\bar{x} + \lambda d, \alpha) \leq F_R(\bar{x}, \alpha).$$

for each $\alpha \in [0, 1]$. From Definition 10, we conclude that

$$\begin{aligned} F_L^{'}((\bar{x}, d), \alpha) &= \lim_{\lambda \to 0^+} \tfrac{F_L(\bar{x}+\lambda d, \alpha) - F_L(\bar{x}, \alpha)}{\lambda} \\ &\leq 0 \end{aligned} \tag{35}$$

and

$$\begin{aligned} F_R^{'}((\bar{x}, d), \alpha) &= \lim_{\lambda \to 0^+} \tfrac{F_R(\bar{x}+\lambda d, \alpha) - F_R(\bar{x}, \alpha)}{\lambda} \\ &\leq 0. \end{aligned} \tag{36}$$

for each $\alpha \in [0, 1]$. Hence, we have $0 \notin$ inter $[F^{'}(\bar{x}, d)]^0$. This is a contradiction with the hypothesis. Then $\bar{x}$ is a local minimum solution of $F(x)$. □

4. Subdifferential of Fuzzy Function

Definition 12. *(Subdifferential of a fuzzy function) Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy number-valued function. For $\bar{x} \in D$, let $d \in \mathbb{R}^m$ such that $\bar{x} + \lambda d \in D$ for any $\lambda > 0$ and sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x, d)$.*

(1) *A fuzzy function $l(d) : \mathbb{R}^m \to E^1$ with*

$$l(d) \preceq F'(\bar{x}, d) \quad for\ all\ d \in \mathbb{R}^m.$$

Then $l(d)$ is called a subgradient of F at $\bar{x}$.

(2) *Define the set*

$$\partial F(\bar{x}) = \{l(d)|l(d) \preceq F'(\bar{x}, d) \quad for\ all\ d \in \mathbb{R}^m\}$$

The set $\partial F(\bar{x})$ of all subgradients of F at $\bar{x}$ is called the subdifferential of F at $\bar{x}$.

Now, we present some basic properties of subdifferential of fuzzy function.

Theorem 10. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a S-differential fuzzy function. For $\bar{x} \in D$, let $d \in \mathbb{R}^m$ such that $\bar{x} + \lambda d \in D$ for any $\lambda > 0$ and is sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x, d)$. Then, $\partial F(\bar{x})$ is convex.*

Proof. Take any $l^1(d), l^2(d) \in \partial F(\bar{x})$ and $\lambda \in [0, 1]$. By Definition 12, it follows that

$$l^1(d) \preceq F'(\bar{x}, d) \tag{37}$$

and

$$l^2(d) \preceq F'(\bar{x}, d) \tag{37}$$

that is,

$$l_L^1(d, \alpha) \leq F_L'((\bar{x}, d), \alpha) \tag{39}$$

$$l_R^1(d, \alpha) \leq F_R'((\bar{x}, d), \alpha), \tag{40}$$

and

$$l_L^2(d, \alpha) \leq F_L'((\bar{x}, d), \alpha) \tag{41}$$

$$l_R^2(d, \alpha) \leq F_R'((\bar{x}, d), \alpha). \tag{42}$$

Now combining $\lambda \times (39)$ and $(1 - \lambda) \times (41)$, we have that

$$\begin{aligned} \lambda l_L^1(d, \alpha) + (1 - \lambda) l_L^2(d, \alpha) &\leq \lambda F_L'((\bar{x}, d), \alpha) + (1 - \lambda) F_L'((\bar{x}, d), \alpha) \\ &= F_L'((\bar{x}, d), \alpha). \end{aligned} \tag{43}$$

Similarly, we obtain

$$\begin{aligned} \lambda l_R^1(d, \alpha) + (1 - \lambda) l_R^2(d, \alpha) &\leq \lambda F_R'((\bar{x}, d), \alpha) + (1 - \lambda) F_R'((\bar{x}, d), \alpha) \\ &= F_R'((\bar{x}, d), \alpha). \end{aligned} \tag{44}$$

The Equations (43) and (44) imply

$$\lambda \times l^1(d) + (1 - \lambda) l^2(d) \preceq F'(\bar{x}, d).$$

By Definition 12, we obtain

$$\lambda \times l^1(d) + (1 - \lambda) l^2(d) \in \partial F(\bar{x}).$$

Then $\partial F(\bar{x})$ is convex. □

Theorem 11. *Let D be a convex set of $\mathbb{R}^m$ and $F : D \to E^1$ be a fuzzy function. For $\bar{x} \in D$, let $d \in \mathbb{R}^m$ such that $\bar{x} + \lambda d \in D$ for any $\lambda > 0$ and sufficiently small. The directional derivative of F at x along the vector d is a fuzzy number denoted by $F^{'}(x, d)$. Then $\partial F(\bar{x})$ is closed.*

Proof. Take an arbitrary sequence of fuzzy functions $\{l_n(d)\}_{n\in N}$ of subgradient is convergent to fuzzy functions $l(d)$. By Definition 6, for any $\varepsilon > 0$, there exists a positive integer $M = M(\varepsilon) \in N$ such that

$$D(l_n(x), l(x)) < \varepsilon \tag{45}$$

for any $d \in \mathbb{R}^m$, all $n \geq M$. Hence, by Definition 5, we obtain that

$$\sup_{\alpha\in[0,1]} \max\{|l_{n.L}(d,\alpha) - l_L(d,\alpha)|, |l_{n.R}(d,\alpha) - l_R(d,\alpha)|\} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$, there exists a positive integer $M = M(\varepsilon) \in N$ such that $|l_{n.L}(d,\alpha) - l_L(d,\alpha)| < \varepsilon$ and $|l_{n.R}(d,\alpha) - l_R(d,\alpha)| < \varepsilon$ for any $d \in \mathbb{R}^m$, all $n \geq M$, and each $\alpha \in [0,1]$.

That is,

$$\lim_{n\to\infty} l_{n.L}(d,\alpha) = l_L(d,\alpha) \tag{46}$$

and

$$\lim_{n\to\infty} l_{n.R}(d,\alpha) = l_R(d,\alpha) \tag{47}$$

In view of Definition 12, we get

$$l_n(d) \preceq F'(\bar{x}, d).$$

That is,

$$l_{n.L}(d,\alpha) \leq F'_L((\bar{x}, d), \alpha) \tag{48}$$

and

$$l_{n.R}(d,\alpha) \leq F'_R((\bar{x}, d), \alpha). \tag{49}$$

Now combining (46), (47), (48) and (49), we obtain

$$l_L(d,\alpha) \leq F'_L((\bar{x}, d), \alpha)$$

and

$$l_R(d,\alpha) \leq F'_R((\bar{x}, d), \alpha).$$

That is,

$$l(d) \preceq F'(\bar{x}, d).$$

Thus, $l(d)$ is a subgradient. That is, the subdifferential $\partial F(\bar{x})$ is closed. □

5. Conclusions

We have investigated several characterizations of directional derivative of fuzzy function about the direction, based on the partial order. For example, we present strictly positive homogeneity, convexity and subadditivity of directional derivative of fuzzy functions. And we also propose the sufficient optimality condition for fuzzy optimization problems. Afterwards, we introduce the concept of the subdifferential of convex fuzzy function. And we present some basic characterizations of subdifferential of fuzzy function and application in the convex fuzzy programming. Thus, we will apply the subdifferentiablity of fuzzy function to deal with the multiobjective fuzzy optimization problem in the future. Constrained optimization problems involving fuzzy functions are an interesting field for future study. For example, finance represents a good field to implement models for sensitive analysis through fuzzy mathematics. Several authors are working hard to shape sources of uncertainty: prices, interest rates, volatilities, etc. (see Guerra et al. [41], Buckley [42]). Therefore, fuzzy optimization problems based on parameter uncertainty sources are a topic of interest in many applications. Inspired by [20,22], directional derivatives and subdifferentials of fuzzy functions will be extensively applied in some fields, such as economics, engineering, stock market greenhouse gas emission and interest rates.

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