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Solvable Three-Dimensional Product-Type System of Difference Equations with Multipliers

Stevo Stević [1,2]

[1] Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia; sstevic@ptt.rs

[2] Operator Theory and Applications Research Group, Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract: The solvability of the following three-dimensional product-type system of difference equations

$$x_{n+1}=\alpha y_n^a z_{n-1}^b,\quad y_{n+1}=\beta z_n^c x_{n-1}^d,\quad z_{n+1}=\gamma x_n^f y_{n-1}^g,\quad n\in\mathbb{N}_0,$$

where $a,b,c,d,f,g\in\mathbb{Z}$, $\alpha,\beta,\gamma\in\mathbb{C}\setminus\{0\}$ and $x_{-i},y_{-i},z_{-i}\in\mathbb{C}\setminus\{0\}$, $i\in\{0,1\}$, is shown. This is the first three-dimensional system of the type with multipliers for which formulas are presented for their solutions in closed form in all the cases.

Keywords: solvable system of difference equations; three-dimensional system; product-type system

1. Introduction

There has been a huge interest on difference equations and systems of difference equations (see, for example, [1–33] and the references therein). The classical problem of solving difference equations, among other topics, has re-attracted some recent interest (see, for example, [3,20,23–28,30–33] and the references therein). The solvability problem is of interest, since it is expected that obtained formulas for solutions to some equations and systems can help in studying the long-term behavior of the solutions. In 2004, we presented a method for solving a generalization of a nonlinear difference equation for which some formulas for their solutions had been given without any explanation how they were obtained, which have had some impact in the interest, and the methods and ideas therein have been applied and developed later in a series of papers (see, for example, [3,20,30] and the references therein). The main feature of these papers is that the equations and systems therein have been solved by using some changes of variables that transform them into the linear solvable ones (for the classical theory of linear difference equations and systems, see, for example, [6,8–10]). In [1], a solvable equation is studied in another way. Motivated, essentially, by papers [11–13], several experts started investigating symmetric, cyclic and other closely related systems of difference equations, which we frequently call *close-to-symmetric/cyclic* (see, for example, [3,14–17,21–25,27–33] and the references therein). As it can be seen, many of the above quoted papers belong to both areas, that is, they deal with some solvable close-to-symmetric systems of difference equations.

In a series of papers, we have studied some equations and systems, which, for some values of parameters, are reduced to product-type ones (see, for example, [29], as well as [4,5,19] and the references therein). The product-type equations and systems have helped to some extent in the study of the original/general equations and systems.

If the constant in the system in [29] is taken to be zero, and only positive solutions are considered, then the following system is obtained:

$$x_{n+1}=y_n^p z_{n-1}^{-p},\quad y_{n+1}=z_n^p x_{n-1}^{-p},\quad z_{n+1}=x_n^p y_{n-1}^{-p},\quad n\in\mathbb{N}_0. \tag{1}$$

When the initial values are positive, then system (1) can be solved by using a well-known method (see, for example, the explanation in [24]). However, if they are complex, then we immediately see that there is a problem with multi-valuedness of functions

$$f_p(z) = z^p,$$

for non-integer p-s.

These facts have motivated us to study the solvability of product-type systems in the complex domain. The first paper in the area was [28] where we have studied a two-dimensional relative of system (1), while the solvability of the following generalization of system (1)

$$x_{n+1} = y_n^a z_{n-1}^{-b}, \quad y_{n+1} = z_n^c x_{n-1}^{-d}, \quad z_{n+1} = x_n^f y_{n-1}^{-g},$$

$n \in \mathbb{N}_0$, where $a, b, c, d, f, g \in \mathbb{Z}$ and $x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$, was studied in [24]. The next paper on the topic was [31], where we further developed our methods for solving product-type systems. In [26], we came across some product-type equations with multipliers, which motivated us to add some multipliers in the study of product-type systems, which had not been previously done in [24,28,31]. The first system with multipliers was studied in [23]. Having published [23], we got the idea of studying the solvability of product-type systems of the form

$$z_n = \alpha z_{n-k}^a w_{n-l}^b, \quad w_n = \beta w_{n-m}^c z_{n-s}^d, \quad n \in \mathbb{N}, \tag{2}$$

where $k, l, m, s \in \mathbb{N}$, $a, b, c, d \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}$. The study has been conducted in a series of papers (see [25,27,32,33], where some special cases of system (2) were studied in detail), with the aim to find all solvable systems of this type and to find closed-form formulas for their solutions. It should be pointed out that there are only finitely many solvable systems of the type, which is connected to the well known fact that there are polynomials of degree greater than or equal to five, which are not solvable by radicals. To the equations treated in [25,27,32,33] are essentially associated some polynomials of degree less than or equal to four, which helped in solving them. A detailed analysis of the structure of solutions to an equation whose associated polynomial is of the fourth order can be found in [27].

Since the study of solvable two-dimensional product-type systems is about finishing, it is a natural problem to find all three-dimensional product-type systems with multipliers, which are solvable in closed form. Here, we start considering the problem by investigating the solvability of the following three-dimensional close-to-cyclic product-type system of difference equations:

$$x_{n+1} = \alpha y_n^a z_{n-1}^b, \quad y_{n+1} = \beta z_n^c x_{n-1}^d, \quad z_{n+1} = \gamma x_n^f y_{n-1}^g, \tag{3}$$

for $n \in \mathbb{N}_0$, where $a, b, c, d, f, g \in \mathbb{Z}$, $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ and $x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$. As far as we know, this is the first paper devoted to investigating of a three-dimensional product-type system of difference equations with multipliers in the complex domain. Our main results extend several of the results on the solvability in the literature.

Note that the case when one of the multipliers α, β, γ or initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1\}$ is zero is excluded from the investigation, since, in the case, trivial or not well-defined solutions are obtained.

2. Main Result

This section presents our main result. Before we do this, we give a list of first several members of sequences x_n, y_n and z_n defined by (3), which will be used in the proof of the result. We have

$$\begin{aligned}
&x_1 = \alpha y_0^a z_{-1}^b, \quad x_2 = \alpha\beta^a x_{-1}^{ad} z_0^{ac+b}, \quad x_3 = \alpha\beta^a\gamma^{ac+b} y_{-1}^{acg+bg} x_0^{acf+ad+bf},\\
&x_4 = \alpha^{1+acf+ad+bf}\beta^a\gamma^{ac+b} z_{-1}^{abcf+abd+b^2f} y_0^{a^2cf+acg+a^2d+abf+bg},\\
&y_1 = \beta z_0^c x_{-1}^d, \quad y_2 = \beta\gamma^c y_{-1}^{cg} x_0^{fc+d}, \quad y_3 = \beta\gamma^c\alpha^{cf+d} z_{-1}^{bcf+bd} y_0^{acf+cg+ad},\\
&y_4 = \alpha^{cf+d}\beta^{1+acf+cg+ad}\gamma^c x_{-1}^{acdf+cdg+ad^2} z_0^{ac^2f+bcf+c^2g+acd+bd},\\
&z_1 = \gamma x_0^f y_{-1}^g, \quad z_2 = \gamma\alpha^f z_{-1}^{bf} y_0^{af+g}, \quad z_3 = \gamma\alpha^f\beta^{af+g} x_{-1}^{adf+dg} z_0^{acf+bf+cg},\\
&z_4 = \alpha^f\beta^{af+g}\gamma^{1+acf+bf+cg} y_{-1}^{acfg+bgf+cg^2} x_0^{acf^2+adf+bf^2+cgf+dg}.
\end{aligned} \tag{4}$$

Theorem 1. *Assume that $a, b, c, d, f, g \in \mathbb{Z}$, $\alpha, \beta, \gamma, x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$. Then, system (3) is solvable.*

Proof of Theorem 1. From (3) and since $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ and $x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$, it easily follows that $x_n y_n z_n \neq 0$ for every $n \geq -1$.

We have

$$x_{n+1} = \alpha y_n^a z_{n-1}^b = \alpha(\beta z_{n-1}^c x_{n-2}^d)^a z_{n-1}^b = \alpha\beta^a z_{n-1}^{ac+b} x_{n-2}^{ad} \tag{5}$$

$$= \alpha\beta^a(\gamma x_{n-2}^f y_{n-3}^g)^{ac+b} x_{n-2}^{ad} = \alpha\beta^a\gamma^{ac+b} x_{n-2}^{acf+bf+ad} y_{n-3}^{acg+bg}, \tag{6}$$

$$y_{n+1} = \beta z_n^c x_{n-1}^d = \beta(\gamma x_{n-1}^f y_{n-2}^g)^c x_{n-1}^d = \beta\gamma^c x_{n-1}^{cf+d} y_{n-2}^{cg} \tag{7}$$

$$= \beta\gamma^c(\alpha y_{n-2}^a z_{n-3}^b)^{cf+d} y_{n-2}^{cg} = \beta\gamma^c\alpha^{cf+d} y_{n-2}^{acf+ad+cg} z_{n-3}^{bcf+bd}, \tag{8}$$

$$z_{n+1} = \gamma x_n^f y_{n-1}^g = \gamma(\alpha y_{n-1}^a z_{n-2}^b)^f y_{n-1}^g = \gamma\alpha^f y_{n-1}^{af+g} z_{n-2}^{bf} \tag{9}$$

$$= \gamma\alpha^f(\beta z_{n-2}^c x_{n-3}^d)^{af+g} z_{n-2}^{bf} = \gamma\alpha^f\beta^{af+g} z_{n-2}^{acf+cg+bf} x_{n-3}^{adf+dg}, \tag{10}$$

for $n \geq 2$.

From Equations (5), (7) and (9), we get, respectively,

$$z_{n-1}^{ac+b} = \frac{x_{n+1}}{\alpha\beta^a x_{n-2}^{ad}}, \tag{11}$$

$$x_{n-1}^{cf+d} = \frac{y_{n+1}}{\beta\gamma^c y_{n-2}^{cg}}, \tag{12}$$

$$y_{n-1}^{af+g} = \frac{z_{n+1}}{\gamma\alpha^f z_{n-2}^{bf}}, \tag{13}$$

for $n \in \mathbb{N}$.

Taking Equation (10) to the $ac+b$-th power and using (11),

$$\frac{x_{n+3}}{\alpha\beta^a x_n^{ad}} = (\gamma\alpha^f\beta^{af+g})^{ac+b}\left(\frac{x_n}{\alpha\beta^a x_{n-3}^{ad}}\right)^{acf+cg+bf} x_{n-3}^{(adf+dg)(ac+b)},$$

is obtained, that is,

$$x_{n+3} = \alpha^{1-cg}\beta^{bg+a}\gamma^{ac+b} x_n^{acf+bf+ad+cg} x_{n-3}^{bdg}, \quad n \geq 2. \tag{14}$$

Taking Equation (6) to the $cf+d$-th power and using (12),

$$\frac{y_{n+3}}{\beta\gamma^c y_n^{cg}} = (\alpha\beta^a\gamma^{ac+b})^{cf+d}\left(\frac{y_n}{\beta\gamma^c y_{n-3}^{cg}}\right)^{acf+bf+ad} y_{n-3}^{(acg+bg)(cf+d)},$$

is obtained, that is,

$$y_{n+3} = \alpha^{cf+d}\beta^{1-bf}\gamma^{bd+c}y_n^{acf+bf+ad+cg}y_{n-3}^{bdg}, \quad n \geq 2. \tag{15}$$

Taking Equation (8) to the $af+g$-th power and using (13) is obtained

$$\frac{z_{n+3}}{\gamma\alpha^f z_n^{bf}} = (\beta\gamma^c\alpha^{cf+d})^{af+g}\left(\frac{z_n}{\gamma\alpha^f z_{n-3}^{bf}}\right)^{acf+ad+cg} z_{n-3}^{(bcf+bd)(af+g)},$$

that is,

$$z_{n+3} = \alpha^{dg+f}\beta^{af+g}\gamma^{1-ad}z_n^{acf+bf+ad+cg}z_{n-3}^{bdg}, \quad n \geq 2. \tag{16}$$

Let $\delta_1 := \alpha^{1-cg}\beta^{bg+a}\gamma^{ac+b}$, $\delta_2 := \alpha^{cf+d}\beta^{1-bf}\gamma^{bd+c}$, $\delta_3 := \alpha^{dg+f}\beta^{af+g}\gamma^{1-ad}$,

$$a_1^{(i)} = acf + bf + ad + cg, \quad b_1^{(i)} = bdg, \quad u_1^{(i)} = 1, \tag{17}$$

for $i = -1, 0, 1$. Then, Equations (14)–(16) can be written in the form

$$x_{3m+i} = \delta_1^{u_1^{(i)}} x_{3(m-1)+i}^{a_1^{(i)}} x_{3(m-2)+i}^{b_1^{(i)}}, \tag{18}$$

$$y_{3m+i} = \delta_2^{u_1^{(i)}} y_{3(m-1)+i}^{a_1^{(i)}} y_{3(m-2)+i}^{b_1^{(i)}}, \tag{19}$$

$$z_{3m+i} = \delta_3^{u_1^{(i)}} z_{3(m-1)+i}^{a_1^{(i)}} z_{3(m-2)+i}^{b_1^{(i)}}, \tag{20}$$

for $m \geq 2$ and $i = -1, 0, 1$.

By using Equation (18) with $m \to m-1$, into Equation (18), we have

$$\begin{aligned} x_{3m+i} &= \delta_1^{u_1^{(i)}} x_{3(m-1)+i}^{a_1^{(i)}} x_{3(m-2)+i}^{b_1^{(i)}} \\ &= \delta_1^{u_1^{(i)}} (\delta_1 x_{3(m-2)+i}^{a_1^{(i)}} x_{3(m-3)+i}^{b_1^{(i)}})^{a_1^{(i)}} x_{3(m-2)+i}^{b_1^{(i)}} \\ &= \delta_1^{u_1^{(i)}+a_1^{(i)}} x_{3(m-2)+i}^{a_1^{(i)}a_1^{(i)}+b_1^{(i)}} x_{3(m-3)+i}^{b_1^{(i)}a_1^{(i)}} \\ &= \delta_1^{u_2^{(i)}} x_{3(m-2)+i}^{a_2^{(i)}} x_{3(m-3)+i}^{b_2^{(i)}}, \end{aligned} \tag{21}$$

for $m \geq 3$ and $i = -1, 0, 1$, where

$$a_2^{(i)} := a_1^{(i)}a_1^{(i)} + b_1^{(i)}, \quad b_2^{(i)} := b_1^{(i)}a_1^{(i)}, \quad u_2^{(i)} := u_1^{(i)} + a_1^{(i)}. \tag{22}$$

Assume that, for some $k \geq 2$, we have proved that

$$x_{3m+i} = \delta_1^{u_k^{(i)}} x_{3(m-k)+i}^{a_k^{(i)}} x_{3(m-k-1)+i}^{b_k^{(i)}}, \tag{23}$$

for $m \geq k+1$ and $i = -1, 0, 1$, where

$$a_k^{(i)} = a_1^{(i)}a_{k-1}^{(i)} + b_{k-1}^{(i)}, \quad b_k^{(i)} = b_1^{(i)}a_{k-1}^{(i)}, \quad u_k^{(i)} = u_{k-1}^{(i)} + a_{k-1}^{(i)}. \tag{24}$$

Then, by using Equation (18) with $m \to m-k$ into Equation (23), we get

$$\begin{aligned}
x_{3m+i} &= \delta_1^{u_k^{(i)}}\left(\delta_1 x_{3(m-k-1)+i}^{a_1^{(i)}} x_{3(m-k-2)+i}^{b_1^{(i)}}\right)^{a_k^{(i)}} x_{3(m-k-1)+i}^{b_k^{(i)}} \\
&= \delta_1^{u_k^{(i)}+a_k^{(i)}} x_{3(m-k-1)+i}^{a_1^{(i)}a_k^{(i)}+b_k^{(i)}} x_{3(m-k-2)+i}^{b_1^{(i)}a_k^{(i)}} \\
&= \delta_1^{u_{k+1}^{(i)}} x_{3(m-k-1)+i}^{a_{k+1}^{(i)}} x_{3(m-k-2)+i}^{b_{k+1}^{(i)}},
\end{aligned} \tag{25}$$

for $m \geq k+2$ and $i = -1, 0, 1$, where

$$a_{k+1}^{(i)} := a_1^{(i)} a_k^{(i)} + b_k^{(i)}, \quad b_{k+1}^{(i)} := b_1^{(i)} a_k^{(i)}, \quad u_{k+1}^{(i)} := u_k^{(i)} + a_k^{(i)}. \tag{26}$$

Relations (21), (22), (25), (26) along with the induction show that (23) and (24) hold for every k and m such that $2 \leq k \leq m-1$ and each $i = -1, 0, 1$.

From the first two equations in (26), it follows that

$$a_{k+1}^{(i)} - (acf + bf + ad + cg)a_k^{(i)} - bdga_{k-1}^{(i)} = 0, \quad k \geq 2. \tag{27}$$

The third equalities in (17) and (24) imply

$$u_k^{(i)} = 1 + \sum_{j=1}^{k-1} a_j^{(i)}, \tag{28}$$

for $k \in \mathbb{N}$ and $i = -1, 0, 1$.

Similarly is obtained

$$y_{3m+i} = \delta_2^{u_k^{(i)}} y_{3(m-k)+i}^{a_k^{(i)}} y_{3(m-k-1)+i}^{b_k^{(i)}}, \tag{29}$$

$$z_{3m+i} = \delta_3^{u_k^{(i)}} z_{3(m-k)+i}^{a_k^{(i)}} z_{3(m-k-1)+i}^{b_k^{(i)}}, \tag{30}$$

for $m \geq k+1$ and $i = -1, 0, 1$.

By taking $k = m-1$ in Equations (23), (29) and (30), using the second relation in (24), and the fact that due to (17) and (24), sequences $(a_k^{(i)})_{k\in\mathbb{N}}$, as well as $(b_k^{(i)})_{k\in\mathbb{N}}$ and $(u_k^{(i)})_{k\in\mathbb{N}}$, are the same for $i \in \{-1, 0, 1\}$, and denoting them by a_k, b_k, u_k, we get

$$x_{3m+i} = \delta_1^{u_{m-1}} x_{3+i}^{a_{m-1}} x_i^{b_1 a_{m-2}}, \tag{31}$$

$$y_{3m+i} = \delta_2^{u_{m-1}} y_{3+i}^{a_{m-1}} y_i^{b_1 a_{m-2}}, \tag{32}$$

$$z_{3m+i} = \delta_3^{u_{m-1}} z_{3+i}^{a_{m-1}} z_i^{b_1 a_{m-2}}, \tag{33}$$

for $m \geq 3$ and $i = -1, 0, 1$.

Employing (4) and (24), in (31)–(33), we have

$$\begin{aligned}
x_{3m-1} &= \delta_1^{u_{m-1}} x_2^{a_{m-1}} x_{-1}^{b_1 a_{m-2}} = \left(\alpha^{1-cg}\beta^{bg+a}\gamma^{ac+b}\right)^{u_{m-1}} \left(\alpha\beta^a x_{-1}^{ad} z_0^{ac+b}\right)^{a_{m-1}} x_{-1}^{b_1 a_{m-2}} \\
&= \alpha^{u_m - cgu_{m-1}} \beta^{bgu_{m-1}+au_m} \gamma^{(ac+b)u_{m-1}} x_{-1}^{ada_{m-1}+bdga_{m-2}} z_0^{(ac+b)a_{m-1}},
\end{aligned} \tag{34}$$

$$\begin{aligned}
x_{3m} &= \delta_1^{u_{m-1}} x_3^{a_{m-1}} x_0^{b_1 a_{m-2}} \\
&= \left(\alpha^{1-cg}\beta^{bg+a}\gamma^{ac+b}\right)^{u_{m-1}} \left(\alpha\beta^a\gamma^{ac+b} y_{-1}^{acg+bg} x_0^{acf+ad+bf}\right)^{a_{m-1}} x_0^{b_1 a_{m-2}} \\
&= \alpha^{u_m - cgu_{m-1}} \beta^{bgu_{m-1}+au_m} \gamma^{(ac+b)u_m} y_{-1}^{(ac+b)ga_{m-1}} x_0^{a_m - cga_{m-1}},
\end{aligned} \tag{35}$$

$$
\begin{aligned}
x_{3m+1} &= \delta_1^{u_{m-1}} x_4^{a_{m-1}} x_1^{b_1 a_{m-2}} = (\alpha^{1-cg}\beta^{bg+a}\gamma^{ac+b})^{u_{m-1}} (\alpha y_0^a z_{-1}^b)^{b_1 a_{m-2}} \\
&\quad \times (\alpha^{1+acf+ad+bf}\beta^a\gamma^{ac+b} z_{-1}^{abcf+abd+b^2f} y_0^{a^2cf+acg+a^2d+abf+bg})^{a_{m-1}} \\
&= \alpha^{u_{m+1}-cgu_m}\beta^{bgu_{m-1}+au_m}\gamma^{(ac+b)u_m} z_{-1}^{b(a_m-cga_{m-1})} y_0^{aa_m+bga_{m-1}}, && (36) \\
y_{3m-1} &= \delta_2^{u_{m-1}} y_2^{a_{m-1}} y_{-1}^{b_1 a_{m-2}} = (\alpha^{cf+d}\beta^{1-bf}\gamma^{bd+c})^{u_{m-1}} (\beta\gamma^c y_{-1}^{cg} x_0^{fc+d})^{a_{m-1}} y_{-1}^{b_1 a_{m-2}} \\
&= \alpha^{(cf+d)u_{m-1}}\beta^{u_m-bfu_{m-1}}\gamma^{bdu_{m-1}+cu_m} y_{-1}^{cga_{m-1}+bdga_{m-2}} x_0^{(fc+d)a_{m-1}}, && (37) \\
y_{3m} &= \delta_2^{u_{m-1}} y_3^{a_{m-1}} y_0^{b_1 a_{m-2}} \\
&= (\alpha^{cf+d}\beta^{1-bf}\gamma^{bd+c})^{u_{m-1}} (\beta\gamma^c\alpha^{cf+d} z_{-1}^{bcf+bd} y_0^{acf+cg+ad})^{a_{m-1}} y_0^{b_1 a_{m-2}} \\
&= \alpha^{(cf+d)u_m}\beta^{u_m-bfu_{m-1}}\gamma^{bdu_{m-1}+cu_m} z_{-1}^{b(cf+d)a_{m-1}} y_0^{a_m-bfa_{m-1}}, && (38) \\
y_{3m+1} &= \delta_2^{u_{m-1}} y_4^{a_{m-1}} y_1^{b_1 a_{m-2}} \\
&= (\alpha^{cf+d}\beta^{1+acf+cg+ad}\gamma^c x_{-1}^{acdf+cdg+ad^2} z_0^{ac^2f+bcf+c^2g+acd+bd})^{a_{m-1}} \\
&\quad \times (\alpha^{cf+d}\beta^{1-bf}\gamma^{bd+c})^{u_{m-1}} (\beta z_0^c x_{-1}^d)^{b_1 a_{m-2}} \\
&= \alpha^{(cf+d)u_m}\beta^{u_{m+1}-bfu_m}\gamma^{bdu_{m-1}+cu_m} x_{-1}^{d(a_m-bfa_{m-1})} z_0^{ca_m+bda_{m-1}}, && (39) \\
z_{3m-1} &= \delta_3^{u_{m-1}} z_2^{a_{m-1}} z_{-1}^{b_1 a_{m-2}} = (\alpha^{dg+f}\beta^{af+g}\gamma^{1-ad})^{u_{m-1}} (\gamma\alpha^f z_{-1}^{bf} y_0^{af+g})^{a_{m-1}} z_{-1}^{b_1 a_{m-2}} \\
&= \alpha^{dgu_{m-1}+fu_m}\beta^{(af+g)u_{m-1}}\gamma^{u_m-adu_{m-1}} z_{-1}^{bfa_{m-1}+bdga_{m-2}} y_0^{(af+g)a_{m-1}}, && (40) \\
z_{3m} &= \delta_3^{u_{m-1}} z_3^{a_{m-1}} z_0^{b_1 a_{m-2}} \\
&= (\alpha^{dg+f}\beta^{af+g}\gamma^{1-ad})^{u_{m-1}} (\gamma\alpha^f\beta^{af+g} x_{-1}^{adf+dg} z_0^{acf+bf+cg})^{a_{m-1}} z_0^{b_1 a_{m-2}} \\
&= \alpha^{dgu_{m-1}+fu_m}\beta^{(af+g)u_m}\gamma^{u_m-adu_{m-1}} x_{-1}^{d(af+g)a_{m-1}} z_0^{a_m-ada_{m-1}}, && (41) \\
z_{3m+1} &= \delta_3^{u_{m-1}} z_4^{a_{m-1}} z_1^{b_1 a_{m-2}} \\
&= (\alpha^f\beta^{af+g}\gamma^{1+acf+bf+cg} y_{-1}^{acfg+bgf+cg^2} x_0^{acf^2+adf+bf^2+cgf+dg})^{a_{m-1}} \\
&\quad \times (\alpha^{dg+f}\beta^{af+g}\gamma^{1-ad})^{u_{m-1}} (\gamma x_0^f y_{-1}^g)^{b_1 a_{m-2}} \\
&= \alpha^{dgu_{m-1}+fu_m}\beta^{(af+g)u_m}\gamma^{u_{m+1}-adu_m} y_{-1}^{g(a_m-ada_{m-1})} x_0^{fa_m+dga_{m-1}}, && (42)
\end{aligned}
$$

for $m \geq 3$.

Case $b = 0$. In this case, we have

$$a_{n+1} = (acf + ad + cg)a_n, \quad n \geq 2,$$

from which along with (17) and (24), it follows that

$$a_n = (acf + ad + cg)^n, \tag{43}$$

for $n \in \mathbb{N}$.

From (28) and (43), it follows that

$$u_n = \sum_{j=0}^{n-1} (acf + ad + cg)^j, \quad n \in \mathbb{N},$$

and consequently

$$u_n = \frac{(acf + ad + cg)^n - 1}{acf + ad + cg - 1}, \quad n \in \mathbb{N}, \tag{44}$$

when $acf + ad + cg \neq 1$, while

$$u_n = n, \quad n \in \mathbb{N}, \tag{45}$$

when $acf + ad + cg = 1$.

On the other hand, from (34)–(42), we get

$$\begin{aligned}
x_{3m-1} &= \alpha^{u_m - cgu_{m-1}} \beta^{au_m} \gamma^{acu_{m-1}} x_{-1}^{ada_{m-1}} z_0^{aca_{m-1}}, &(46)\\
x_{3m} &= \alpha^{u_m - cgu_{m-1}} \beta^{au_m} \gamma^{acu_m} y_{-1}^{acga_{m-1}} x_0^{a_m - cga_{m-1}}, &(47)\\
x_{3m+1} &= \alpha^{u_{m+1} - cgu_m} \beta^{au_m} \gamma^{acu_m} y_0^{aa_m}, &(48)\\
y_{3m-1} &= \alpha^{(cf+d)u_{m-1}} \beta^{u_m} \gamma^{cu_m} y_{-1}^{cga_{m-1}} x_0^{(fc+d)a_{m-1}}, &(49)\\
y_{3m} &= \alpha^{(cf+d)u_m} \beta^{u_m} \gamma^{cu_m} y_0^{a_m}, &(50)\\
y_{3m+1} &= \alpha^{(cf+d)u_m} \beta^{u_{m+1}} \gamma^{cu_m} x_{-1}^{da_m} z_0^{ca_m}, &(51)\\
z_{3m-1} &= \alpha^{dgu_{m-1} + fu_m} \beta^{(af+g)u_{m-1}} \gamma^{u_m - adu_{m-1}} y_0^{(af+g)a_{m-1}}, &(52)\\
z_{3m} &= \alpha^{dgu_{m-1} + fu_m} \beta^{(af+g)u_m} \gamma^{u_m - adu_{m-1}} x_{-1}^{d(af+g)a_{m-1}} z_0^{a_m - ada_{m-1}}, &(53)\\
z_{3m+1} &= \alpha^{dgu_{m-1} + fu_m} \beta^{(af+g)u_m} \gamma^{u_{m+1} - adu_m} y_{-1}^{g(a_m - ada_{m-1})} x_0^{fa_m + dga_{m-1}}, &(54)
\end{aligned}$$

for $m \geq 3$.

Subcase $acf + ad + cg \neq 1$. Employing (43) and (44) in (46)–(54),

$$\begin{aligned}
x_{3m-1} &= \alpha^{\frac{a(cf+d)(acf+ad+cg)^{m-1}+cg-1}{acf+ad+cg-1}} \beta^{a\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \gamma^{ac\frac{(acf+ad+cg)^{m-1}-1}{acf+ad+cg-1}}\\
&\quad \times x_{-1}^{ad(acf+ad+cg)^{m-1}} z_0^{ac(acf+ad+cg)^{m-1}}, &(55)\\
x_{3m} &= \alpha^{\frac{a(cf+d)(acf+ad+cg)^{m-1}+cg-1}{acf+ad+cg-1}} \beta^{a\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \gamma^{ac\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}}\\
&\quad \times y_{-1}^{acg(acf+ad+cg)^{m-1}} x_0^{a(cf+d)(acf+ad+cg)^{m-1}}, &(56)\\
x_{3m+1} &= \alpha^{\frac{a(cf+d)(acf+ad+cg)^m+cg-1}{acf+ad+cg-1}} \beta^{a\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \gamma^{ac\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}}\\
&\quad \times y_0^{a(acf+ad+cg)^m}, &(57)\\
y_{3m-1} &= \alpha^{(cf+d)\frac{(acf+ad+cg)^{m-1}-1}{acf+ad+cg-1}} \beta^{\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \gamma^{c\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}}\\
&\quad \times y_{-1}^{cg(acf+ad+cg)^{m-1}} x_0^{(fc+d)(acf+ad+cg)^{m-1}}, &(58)\\
y_{3m} &= \alpha^{(cf+d)\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \beta^{\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \gamma^{c\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} y_0^{(acf+ad+cg)^m}, &(59)\\
y_{3m+1} &= \alpha^{(cf+d)\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}} \beta^{\frac{(acf+ad+cg)^{m+1}-1}{acf+ad+cg-1}} \gamma^{c\frac{(acf+ad+cg)^m-1}{acf+ad+cg-1}}\\
&\quad \times x_{-1}^{d(acf+ad+cg)^m} z_0^{c(acf+ad+cg)^m}, &(60)\\
z_{3m-1} &= \alpha^{\frac{(dg+acf^2+adf+cgf)(acf+ad+cg)^{m-1}-dg-f}{acf+ad+cg-1}} \beta^{(af+g)\frac{(acf+ad+cg)^{m-1}-1}{acf+ad+cg-1}}\\
&\quad \times \gamma^{\frac{c(af+g)(acf+ad+cg)^{m-1}+ad-1}{acf+ad+cg-1}} y_0^{(af+g)(acf+ad+cg)^{m-1}}, &(61)
\end{aligned}$$

$$\begin{aligned} z_{3m} &= \alpha^{\frac{(dg+acf^2+adf+cgf)(acf+ad+cg)^{m-1}-dg-f}{acf+ad+cg-1}} \beta^{(af+g)\frac{(acf+ad+cg)^{m}-1}{acf+ad+cg-1}} \\ &\quad \times \gamma^{\frac{c(af+g)(acf+ad+cg)^{m-1}+ad-1}{acf+ad+cg-1}} x_{-1}^{d(af+g)(acf+ad+cg)^{m-1}} \\ &\quad \times z_0^{c(af+g)(acf+ad+cg)^{m-1}}, \end{aligned} \tag{62}$$

$$\begin{aligned} z_{3m+1} &= \alpha^{\frac{(dg+acf^2+adf+cgf)(acf+ad+cg)^{m-1}-dg-f}{acf+ad+cg-1}} \beta^{(af+g)\frac{(acf+ad+cg)^{m}-1}{acf+ad+cg-1}} \\ &\quad \times \gamma^{\frac{c(af+g)(acf+ad+cg)^{m}+ad-1}{acf+ad+cg-1}} y_{-1}^{cg(af+g)(acf+ad+cg)^{m-1}} \\ &\quad \times x_0^{(acf^2+adf+cgf+dg)(acf+ad+cg)^{m-1}}, \end{aligned} \tag{63}$$

is obtained for $m \geq 3$.

Subcase $acf + ad + cg = 1$. Employing (43) and (45) in (46)–(54),

$$x_{3m-1} = \alpha^{(1-cg)m+cg} \beta^{am} \gamma^{ac(m-1)} x_{-1}^{ad} z_0^{ac}, \tag{64}$$
$$x_{3m} = \alpha^{(1-cg)m+cg} \beta^{am} \gamma^{acm} y_{-1}^{acg} x_0^{1-cg}, \tag{65}$$
$$x_{3m+1} = \alpha^{(1-cg)m+1} \beta^{am} \gamma^{acm} y_0^{a}, \tag{66}$$
$$y_{3m-1} = \alpha^{(cf+d)(m-1)} \beta^{m} \gamma^{cm} y_{-1}^{cg} x_0^{fc+d}, \tag{67}$$
$$y_{3m} = \alpha^{(cf+d)m} \beta^{m} \gamma^{cm} y_0, \tag{68}$$
$$y_{3m+1} = \alpha^{(cf+d)m} \beta^{m+1} \gamma^{cm} x_{-1}^{d} z_0^{c}, \tag{69}$$
$$z_{3m-1} = \alpha^{(dg+f)m-dg} \beta^{(af+g)(m-1)} \gamma^{(1-ad)m+ad} y_0^{af+g}, \tag{70}$$
$$z_{3m} = \alpha^{(dg+f)m-dg} \beta^{(af+g)m} \gamma^{(1-ad)m+ad} x_{-1}^{d(af+g)} z_0^{1-ad}, \tag{71}$$
$$z_{3m+1} = \alpha^{(dg+f)m-dg} \beta^{(af+g)m} \gamma^{(1-ad)m+1} y_{-1}^{g(1-ad)} x_0^{f+dg}, \tag{72}$$

is obtained for $m \geq 3$.

Case $d = 0$. In this case, we have

$$a_{n+1} = (acf + bf + cg)a_n, \quad n \geq 2,$$

from which, along with (17) and (24), it follows that

$$a_n = (acf + bf + cg)^n, \tag{73}$$

for $n \in \mathbb{N}$.

From (28) and (73), it follows that

$$u_n = \sum_{j=0}^{n-1} (acf + bf + cg)^j, \quad n \in \mathbb{N},$$

and consequently

$$u_n = \frac{(acf + bf + cg)^n - 1}{acf + bf + cg - 1}, \quad n \in \mathbb{N}, \tag{74}$$

when $acf + bf + cg \neq 1$, while

$$u_n = n, \quad n \in \mathbb{N}, \tag{75}$$

when $acf + bf + cg = 1$.

On the other hand, from (34)–(42), we get

$$x_{3m-1} = \alpha^{u_m - cgu_{m-1}} \beta^{bgu_{m-1}+au_m} \gamma^{(ac+b)u_{m-1}} z_0^{(ac+b)a_{m-1}}, \tag{76}$$

$$x_{3m} = \alpha^{u_m - cgu_{m-1}} \beta^{bgu_{m-1}+au_m} \gamma^{(ac+b)u_m} y_{-1}^{(ac+b)ga_{m-1}} x_0^{a_m - cga_{m-1}}, \tag{77}$$

$$x_{3m+1} = \alpha^{u_{m+1} - cgu_m} \beta^{bgu_{m-1}+au_m} \gamma^{(ac+b)u_m} z_{-1}^{b(a_m - cga_{m-1})} y_0^{aa_m + bga_{m-1}}, \tag{78}$$

$$y_{3m-1} = \alpha^{cfu_{m-1}} \beta^{u_m - bfu_{m-1}} \gamma^{cu_m} y_{-1}^{cga_{m-1}} x_0^{fca_{m-1}}, \tag{79}$$

$$y_{3m} = \alpha^{cfu_m} \beta^{u_m - bfu_{m-1}} \gamma^{cu_m} z_{-1}^{bcfa_{m-1}} y_0^{a_m - bfa_{m-1}}, \tag{80}$$

$$y_{3m+1} = \alpha^{cfu_m} \beta^{u_{m+1} - bfu_m} \gamma^{cu_m} z_0^{ca_m}, \tag{81}$$

$$z_{3m-1} = \alpha^{fu_m} \beta^{(af+g)u_{m-1}} \gamma^{u_m} z_{-1}^{bfa_{m-1}} y_0^{(af+g)a_{m-1}}, \tag{82}$$

$$z_{3m} = \alpha^{fu_m} \beta^{(af+g)u_m} \gamma^{u_m} z_0^{a_m}, \tag{83}$$

$$z_{3m+1} = \alpha^{fu_m} \beta^{(af+g)u_m} \gamma^{u_{m+1}} y_{-1}^{ga_m} x_0^{fa_m}, \tag{84}$$

for $m \geq 3$.

We have to consider two subcases.

Subcase $acf + bf + cg \neq 1$. Employing (73) and (74) in (76)–(84),

$$x_{3m-1} = \alpha^{\frac{(ac+b)f(acf+bf+cg)^{m-1}+cg-1}{acf+bf+cg-1}} \beta^{\frac{(a^2cf+abf+acg+bg)(acf+bf+cg)^{m-1}-a-bg}{acf+bf+cg-1}} \times \gamma^{(ac+b)\frac{(acf+bf+cg)^{m-1}-1}{acf+bf+cg-1}} z_0^{(ac+b)(acf+bf+cg)^{m-1}}, \tag{85}$$

$$x_{3m} = \alpha^{\frac{(ac+b)f(acf+bf+cg)^{m-1}+cg-1}{acf+bf+cg-1}} \beta^{\frac{(a^2cf+abf+acg+bg)(acf+bf+cg)^{m-1}-a-bg}{acf+bf+cg-1}} \times \gamma^{(ac+b)\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} y_{-1}^{(ac+b)g(acf+bf+cg)^{m-1}} \times x_0^{(ac+b)f(acf+bf+cg)^{m-1}}, \tag{86}$$

$$x_{3m+1} = \alpha^{\frac{(ac+b)f(acf+bf+cg)^{m}+cg-1}{acf+bf+cg-1}} \beta^{\frac{(a^2cf+abf+acg+bg)(acf+bf+cg)^{m-1}-a-bg}{acf+bf+cg-1}} \times \gamma^{(ac+b)\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} z_{-1}^{bf(ac+b)(acf+bf+cg)^{m-1}} \times y_0^{(a^2cf+abf+acg+bg)(acf+bf+cg)^{m-1}}, \tag{87}$$

$$y_{3m-1} = \alpha^{cf\frac{(acf+bf+cg)^{m-1}-1}{acf+bf+cg-1}} \beta^{\frac{c(af+g)(acf+bf+cg)^{m-1}+bf-1}{acf+bf+cg-1}} \gamma^{c\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \times y_{-1}^{cg(acf+bf+cg)^{m-1}} x_0^{fc(acf+bf+cg)^{m-1}}, \tag{88}$$

$$y_{3m} = \alpha^{cf\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \beta^{\frac{c(af+g)(acf+bf+cg)^{m-1}+bf-1}{acf+bf+cg-1}} \gamma^{c\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \times z_{-1}^{bcf(acf+bf+cg)^{m-1}} y_0^{c(af+g)(acf+bf+cg)^{m-1}}, \tag{89}$$

$$y_{3m+1} = \alpha^{cf\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \beta^{\frac{c(af+g)(acf+bf+cg)^{m}+bf-1}{acf+bf+cg-1}} \gamma^{c\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \times z_0^{c(acf+bf+cg)^{m}}, \tag{90}$$

$$z_{3m-1} = \alpha^{f\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \beta^{(af+g)\frac{(acf+bf+cg)^{m-1}-1}{acf+bf+cg-1}} \gamma^{\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \times z_{-1}^{bf(acf+bf+cg)^{m-1}} y_0^{(af+g)(acf+bf+cg)^{m-1}}, \tag{91}$$

$$z_{3m} = \alpha^{f\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \beta^{(af+g)\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \gamma^{\frac{(acf+bf+cg)^{m}-1}{acf+bf+cg-1}} \times z_0^{(acf+bf+cg)^{m}}, \tag{92}$$

$$z_{3m+1} = \alpha^{f\frac{(acf+bf+cg)^m-1}{acf+bf+cg-1}}\beta^{(af+g)\frac{(acf+bf+cg)^m-1}{acf+bf+cg-1}}\gamma^{\frac{(acf+bf+cg)^{m+1}-1}{acf+bf+cg-1}}\times y_{-1}^{g(acf+bf+cg)^m}x_0^{f(acf+bf+cg)^m}, \quad (93)$$

is obtained for $m \geq 3$.

Subcase $acf + bf + cg = 1$. Employing (73) and (75) in (76)–(84),

$$x_{3m-1} = \alpha^{(1-cg)m+cg}\beta^{(bg+a)m-bg}\gamma^{(ac+b)(m-1)}z_0^{ac+b}, \quad (94)$$
$$x_{3m} = \alpha^{(1-cg)m+cg}\beta^{(bg+a)m-bg}\gamma^{(ac+b)m}y_{-1}^{(ac+b)g}x_0^{1-cg}, \quad (95)$$
$$x_{3m+1} = \alpha^{(1-cg)m+1}\beta^{(bg+a)m-bg}\gamma^{(ac+b)m}z_{-1}^{b(1-cg)}y_0^{a+bg}, \quad (96)$$
$$y_{3m-1} = \alpha^{cf(m-1)}\beta^{(1-bf)m+bf}\gamma^{cm}y_{-1}^{cg}x_0^{fc}, \quad (97)$$
$$y_{3m} = \alpha^{cfm}\beta^{(1-bf)m+bf}\gamma^{cm}z_{-1}^{bcf}y_0^{1-bf}, \quad (98)$$
$$y_{3m+1} = \alpha^{cfm}\beta^{(1-bf)m+1}\gamma^{cm}z_0^{c}, \quad (99)$$
$$z_{3m-1} = \alpha^{fm}\beta^{(af+g)(m-1)}\gamma^{m}z_{-1}^{bf}y_0^{af+g}, \quad (100)$$
$$z_{3m} = \alpha^{fm}\beta^{(af+g)m}\gamma^{m}z_0, \quad (101)$$
$$z_{3m+1} = \alpha^{fm}\beta^{(af+g)m}\gamma^{m+1}y_{-1}^{g}x_0^{f}, \quad (102)$$

is obtained for $m \geq 3$.

Case $g = 0$. In this case, we have

$$a_{n+1} = (acf + ad + bf)a_n, \quad n \geq 2,$$

from which along with (17) and (24), it follows that

$$a_n = (acf + ad + bf)^n, \quad n \in \mathbb{N}. \quad (103)$$

From (28) and (103), it follows that

$$u_n = \sum_{j=0}^{n-1}(acf + ad + bf)^j, \quad n \in \mathbb{N},$$

and consequently

$$u_n = \frac{(acf + ad + bf)^n - 1}{acf + ad + bf - 1}, \quad n \in \mathbb{N}, \quad (104)$$

when $acf + ad + bf \neq 1$, while

$$u_n = n, \quad n \in \mathbb{N}, \quad (105)$$

when $acf + ad + bf = 1$.

On the other hand, from (34)–(42), we get

$$x_{3m-1} = \alpha^{u_m}\beta^{au_m}\gamma^{(ac+b)u_{m-1}}x_{-1}^{ada_{m-1}}z_0^{(ac+b)a_{m-1}}, \quad (106)$$
$$x_{3m} = \alpha^{u_m}\beta^{au_m}\gamma^{(ac+b)u_m}x_0^{a_m}, \quad (107)$$
$$x_{3m+1} = \alpha^{u_{m+1}}\beta^{au_m}\gamma^{(ac+b)u_m}z_{-1}^{ba_m}y_0^{aa_m}, \quad (108)$$

$$
\begin{aligned}
y_{3m-1} &= \alpha^{(cf+d)u_{m-1}}\beta^{u_m-bfu_{m-1}}\gamma^{bdu_{m-1}+cu_m}x_0^{(fc+d)a_{m-1}}, && (109)\\
y_{3m} &= \alpha^{(cf+d)u_m}\beta^{u_m-bfu_{m-1}}\gamma^{bdu_{m-1}+cu_m}z_{-1}^{b(cf+d)a_{m-1}}y_0^{a_m-bfa_{m-1}}, && (110)\\
y_{3m+1} &= \alpha^{(cf+d)u_m}\beta^{u_{m+1}-bfu_m}\gamma^{bdu_{m-1}+cu_m}x_{-1}^{d(a_m-bfa_{m-1})}z_0^{ca_m+bda_{m-1}}, && (111)\\
z_{3m-1} &= \alpha^{fu_m}\beta^{afu_{m-1}}\gamma^{u_m-adu_{m-1}}z_{-1}^{bfa_{m-1}}y_0^{afa_{m-1}}, && (112)\\
z_{3m} &= \alpha^{fu_m}\beta^{afu_m}\gamma^{u_m-adu_{m-1}}x_{-1}^{adfa_{m-1}}z_0^{a_m-ada_{m-1}}, && (113)\\
z_{3m+1} &= \alpha^{fu_m}\beta^{afu_m}\gamma^{u_{m+1}-adu_m}x_0^{fa_m}, && (114)
\end{aligned}
$$

for $m \geq 3$.

Subcase $acf + ad + bf \neq 1$. Employing (103) and (104) in (106)–(114),

$$
\begin{aligned}
x_{3m-1} &= \alpha^{\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{a\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\gamma^{(ac+b)\frac{(acf+ad+bf)^{m-1}-1}{acf+ad+bf-1}}\\
&\quad\times x_{-1}^{ad(acf+ad+bf)^{m-1}}z_0^{(ac+b)(acf+ad+bf)^{m-1}}, && (115)\\
x_{3m} &= \alpha^{\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{a\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\gamma^{(ac+b)\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\\
&\quad\times x_0^{(acf+ad+bf)^m}, && (116)\\
x_{3m+1} &= \alpha^{\frac{(acf+ad+bf)^{m+1}-1}{acf+ad+bf-1}}\beta^{a\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\gamma^{(ac+b)\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\\
&\quad\times z_{-1}^{b(acf+ad+bf)^m}y_0^{a(acf+ad+bf)^m}, && (117)\\
y_{3m-1} &= \alpha^{(cf+d)\frac{(acf+ad+bf)^{m-1}-1}{acf+ad+bf-1}}\beta^{\frac{a(cf+d)(acf+ad+bf)^{m-1}+bf-1}{acf+ad+bf-1}}\\
&\quad\times \gamma^{\frac{(ac^2f+acd+bcf+bd)(acf+ad+bf)^{m-1}-bd-c}{acf+ad+bf-1}}x_0^{(fc+d)(acf+ad+bf)^{m-1}}, && (118)\\
y_{3m} &= \alpha^{(cf+d)\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{\frac{a(cf+d)(acf+ad+bf)^{m-1}+bf-1}{acf+ad+bf-1}}\\
&\quad\times \gamma^{\frac{(ac^2f+acd+bcf+bd)(acf+ad+bf)^{m-1}-bd-c}{acf+ad+bf-1}}z_{-1}^{b(fc+d)(acf+ad+bf)^{m-1}}\\
&\quad\times y_0^{a(cf+d)(acf+ad+bf)^{m-1}}, && (119)\\
y_{3m+1} &= \alpha^{(cf+d)\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{\frac{a(cf+d)(acf+ad+bf)^m+bf-1}{acf+ad+bf-1}}\\
&\quad\times \gamma^{\frac{(ac^2f+acd+bcf+bd)(acf+ad+bf)^{m-1}-bd-c}{acf+ad+bf-1}}x_{-1}^{ad(cf+d)(acf+ad+bf)^{m-1}}\\
&\quad\times z_0^{(ac^2f+acd+bcf+bd)(acf+ad+bf)^{m-1}}, && (120)\\
z_{3m-1} &= \alpha^{f\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{af\frac{(acf+ad+bf)^{m-1}-1}{acf+ad+bf-1}}\gamma^{\frac{f(ac+b)(acf+ad+bf)^{m-1}+ad-1}{acf+ad+bf-1}}\\
&\quad\times z_{-1}^{bf(acf+ad+bf)^{m-1}}y_0^{af(acf+ad+bf)^{m-1}}, && (121)\\
z_{3m} &= \alpha^{f\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{af\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\gamma^{\frac{f(ac+b)(acf+ad+bf)^{m-1}+ad-1}{acf+ad+bf-1}}\\
&\quad\times x_{-1}^{adf(acf+ad+bf)^{m-1}}z_0^{f(ac+b)(acf+ad+bf)^{m-1}}, && (122)\\
z_{3m+1} &= \alpha^{f\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\beta^{af\frac{(acf+ad+bf)^m-1}{acf+ad+bf-1}}\gamma^{\frac{f(ac+b)(acf+ad+bf)^m+ad-1}{acf+ad+bf-1}}\\
&\quad\times x_0^{f(acf+ad+bf)^m}, && (123)
\end{aligned}
$$

is obtained for $m \geq 3$.

Subcase $acf + ad + bf = 1$. Employing (103) and (105) in (106)–(114),

$$x_{3m-1} = \alpha^m \beta^{am} \gamma^{(ac+b)(m-1)} x_{-1}^{ad} z_0^{ac+b}, \tag{124}$$
$$x_{3m} = \alpha^m \beta^{am} \gamma^{(ac+b)m} x_0, \tag{125}$$
$$x_{3m+1} = \alpha^{m+1} \beta^{am} \gamma^{(ac+b)m} z_{-1}^b y_0^a, \tag{126}$$
$$y_{3m-1} = \alpha^{(cf+d)(m-1)} \beta^{(1-bf)m+bf} \gamma^{(bd+c)m-bd} x_0^{fc+d}, \tag{127}$$
$$y_{3m} = \alpha^{(cf+d)m} \beta^{(1-bf)m+bf} \gamma^{(bd+c)m-bd} z_{-1}^{b(cf+d)} y_0^{1-bf}, \tag{128}$$
$$y_{3m+1} = \alpha^{(cf+d)m} \beta^{(1-bf)m+1} \gamma^{(bd+c)m-bd} x_{-1}^{d(1-bf)} z_0^{c+bd}, \tag{129}$$
$$z_{3m-1} = \alpha^{fm} \beta^{af(m-1)} \gamma^{(1-ad)m+ad} z_{-1}^{bf} y_0^{af}, \tag{130}$$
$$z_{3m} = \alpha^{fm} \beta^{afm} \gamma^{(1-ad)m+ad} x_{-1}^{adf} z_0^{1-ad}, \tag{131}$$
$$z_{3m+1} = \alpha^{fm} \beta^{afm} \gamma^{(1-ad)m+1} x_0^f, \tag{132}$$

is obtained for $m \geq 3$.

Case $bdg \neq 0$. Let $\lambda_{1,2}$ be the roots of the characteristic polynomial

$$P(\lambda) = \lambda^2 - (acf + ad + bf + cg)\lambda - bdg,$$

associated with the equation

$$u_{n+2} - (acf + ad + bf + cg)u_{n+1} - bdgu_n = 0, \quad n \in \mathbb{N}. \tag{133}$$

It is known that general solution of Equation (133) has the following form

$$u_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n, \quad n \in \mathbb{N},$$

if $(acf + ad + bf + cg)^2 \neq -4bdg$, where

$$\lambda_{1,2} = \frac{acf + ad + bf + cg \pm \sqrt{(acf + ad + bf + cg)^2 + 4bdg}}{2},$$

and α_1 and α_2 are arbitrary constants, while in the case $(acf + ad + bf + cg)^2 = -4bdg$, the general solution has the following form

$$u_n = (\beta_1 n + \beta_2)\lambda_1^n, \quad n \in \mathbb{N},$$

where $\lambda_1 = (acf + ad + bf + cg)/2$, and β_1, β_2 are arbitrary constants.

Since $bdg \neq 0$, from (133), we have

$$u_n = \frac{u_{n+2} - (acf + ad + bf + cg)u_{n+1}}{bdg},$$

from which it follows that u_n can be calculated also for every $n \leq 0$. Hence, it is easily seen that, for a_n,

$$a_{-1} = 0, \quad a_0 = 1, \tag{134}$$

holds, from which it follows that

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{135}$$

when $(acf+ad+bf+cg)^2 \neq -4bdg$, and consequently

$$u_n = 1 + \sum_{j=1}^{n-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{(\lambda_2 - 1)\lambda_1^{n+1} - (\lambda_1 - 1)\lambda_2^{n+1} + \lambda_1 - \lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)(\lambda_2 - 1)}. \tag{136}$$

By using (135) and (136) in (34)–(42), and after some calculations, we get

$$\begin{aligned} x_{3m-1} = {} & \alpha^{\frac{(\lambda_2-1)(\lambda_1-cg)\lambda_1^m-(\lambda_1-1)(\lambda_2-cg)\lambda_2^m+(\lambda_1-\lambda_2)(1-cg)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \beta^{\frac{(\lambda_2-1)(a\lambda_1+bg)\lambda_1^m-(\lambda_1-1)(a\lambda_2+bg)\lambda_2^m+(\lambda_1-\lambda_2)(bg+a)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \gamma^{(ac+b)\frac{(\lambda_2-1)\lambda_1^m-(\lambda_1-1)\lambda_2^m+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times x_{-1}^{d\frac{(a\lambda_1+bg)\lambda_1^{m-1}-(a\lambda_2+bg)\lambda_2^{m-1}}{\lambda_1-\lambda_2}} z_0^{(ac+b)\frac{\lambda_1^m-\lambda_2^m}{\lambda_1-\lambda_2}}, \end{aligned} \tag{137}$$

$$\begin{aligned} x_{3m} = {} & \alpha^{\frac{(\lambda_2-1)(\lambda_1-cg)\lambda_1^m-(\lambda_1-1)(\lambda_2-cg)\lambda_2^m+(\lambda_1-\lambda_2)(1-cg)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \beta^{\frac{(\lambda_2-1)(a\lambda_1+bg)\lambda_1^m-(\lambda_1-1)(a\lambda_2+bg)\lambda_2^m+(\lambda_1-\lambda_2)(bg+a)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \gamma^{(ac+b)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times y_{-1}^{(ac+b)g\frac{\lambda_1^m-\lambda_2^m}{\lambda_1-\lambda_2}} x_0^{\frac{(\lambda_1-cg)\lambda_1^m-(\lambda_2-cg)\lambda_2^m}{\lambda_1-\lambda_2}}, \end{aligned} \tag{138}$$

$$\begin{aligned} x_{3m+1} = {} & \alpha^{\frac{(\lambda_2-1)(\lambda_1-cg)\lambda_1^{m+1}-(\lambda_1-1)(\lambda_2-cg)\lambda_2^{m+1}+(\lambda_1-\lambda_2)(1-cg)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \beta^{\frac{(\lambda_2-1)(a\lambda_1+bg)\lambda_1^m-(\lambda_1-1)(a\lambda_2+bg)\lambda_2^m+(\lambda_1-\lambda_2)(bg+a)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \gamma^{(ac+b)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} z_{-1}^{b\frac{(\lambda_1-cg)\lambda_1^m-(\lambda_2-cg)\lambda_2^m}{\lambda_1-\lambda_2}} \\ & \times y_0^{\frac{(a\lambda_1+bg)\lambda_1^m-(a\lambda_2+bg)\lambda_2^m}{\lambda_1-\lambda_2}}, \end{aligned} \tag{139}$$

$$\begin{aligned} y_{3m-1} = {} & \alpha^{(cf+d)\frac{(\lambda_2-1)\lambda_1^m-(\lambda_1-1)\lambda_2^m+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \beta^{\frac{(\lambda_2-1)(\lambda_1-bf)\lambda_1^m-(\lambda_1-1)(\lambda_2-bf)\lambda_2^m+(\lambda_1-\lambda_2)(1-bf)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \gamma^{\frac{(\lambda_2-1)(c\lambda_1+bd)\lambda_1^m-(\lambda_1-1)(c\lambda_2+bd)\lambda_2^m+(\lambda_1-\lambda_2)(bd+c)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times y_{-1}^{g\frac{(c\lambda_1+bd)\lambda_1^{m-1}-(c\lambda_2+bd)\lambda_2^{m-1}}{\lambda_1-\lambda_2}} x_0^{(fc+d)\frac{\lambda_1^m-\lambda_2^m}{\lambda_1-\lambda_2}}, \end{aligned} \tag{140}$$

$$\begin{aligned} y_{3m} = {} & \alpha^{(cf+d)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \beta^{\frac{(\lambda_2-1)(\lambda_1-bf)\lambda_1^m-(\lambda_1-1)(\lambda_2-bf)\lambda_2^m+(\lambda_1-\lambda_2)(1-bf)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times \gamma^{\frac{(\lambda_2-1)(c\lambda_1+bd)\lambda_1^m-(\lambda_1-1)(c\lambda_2+bd)\lambda_2^m+(\lambda_1-\lambda_2)(bd+c)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ & \times z_{-1}^{b(cf+d)\frac{\lambda_1^m-\lambda_2^m}{\lambda_1-\lambda_2}} y_0^{\frac{(\lambda_1-bf)\lambda_1^m-(\lambda_2-bf)\lambda_2^m}{\lambda_1-\lambda_2}}, \end{aligned} \tag{141}$$

$$\begin{aligned} y_{3m+1} &= \alpha^{(cf+d)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \beta^{\frac{(\lambda_2-1)(\lambda_1-bf)\lambda_1^{m+1}-(\lambda_1-1)(\lambda_2-bf)\lambda_2^{m+1}+(\lambda_1-\lambda_2)(1-bf)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \gamma^{\frac{(\lambda_2-1)(c\lambda_1+bd)\lambda_1^{m}-(\lambda_1-1)(c\lambda_2+bd)\lambda_2^{m}+(\lambda_1-\lambda_2)(bd+c)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times x_{-1}^{d\frac{(\lambda_1-bf)\lambda_1^{m}-(\lambda_2-bf)\lambda_2^{m}}{\lambda_1-\lambda_2}} z_0^{\frac{(c\lambda_1+bd)\lambda_1^{m}-(c\lambda_2+bd)\lambda_2^{m}}{\lambda_1-\lambda_2}}, \end{aligned} \tag{142}$$

$$\begin{aligned} z_{3m-1} &= \alpha^{\frac{(\lambda_2-1)(f\lambda_1+dg)\lambda_1^{m}-(\lambda_1-1)(f\lambda_2+dg)\lambda_2^{m}+(\lambda_1-\lambda_2)(dg+f)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \beta^{(af+g)\frac{(\lambda_2-1)\lambda_1^{m}-(\lambda_1-1)\lambda_2^{m}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \gamma^{\frac{(\lambda_2-1)(\lambda_1-ad)\lambda_1^{m}-(\lambda_1-1)(\lambda_2-ad)\lambda_2^{m}+(\lambda_1-\lambda_2)(1-ad)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times z_{-1}^{b\frac{(f\lambda_1+dg)\lambda_1^{m-1}-(f\lambda_2+dg)\lambda_2^{m-1}}{\lambda_1-\lambda_2}} y_0^{(af+g)\frac{\lambda_1^{m}-\lambda_2^{m}}{\lambda_1-\lambda_2}}, \end{aligned} \tag{143}$$

$$\begin{aligned} z_{3m} &= \alpha^{\frac{(\lambda_2-1)(f\lambda_1+dg)\lambda_1^{m}-(\lambda_1-1)(f\lambda_2+dg)\lambda_2^{m}+(\lambda_1-\lambda_2)(dg+f)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \beta^{(af+g)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \gamma^{\frac{(\lambda_2-1)(\lambda_1-ad)\lambda_1^{m}-(\lambda_1-1)(\lambda_2-ad)\lambda_2^{m}+(\lambda_1-\lambda_2)(1-ad)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times x_{-1}^{d(af+g)\frac{\lambda_1^{m}-\lambda_2^{m}}{\lambda_1-\lambda_2}} z_0^{\frac{(\lambda_1-ad)\lambda_1^{m}-(\lambda_2-ad)\lambda_2^{m}}{\lambda_1-\lambda_2}}, \end{aligned} \tag{144}$$

$$\begin{aligned} z_{3m+1} &= \alpha^{\frac{(\lambda_2-1)(f\lambda_1+dg)\lambda_1^{m}-(\lambda_1-1)(f\lambda_2+dg)\lambda_2^{m}+(\lambda_1-\lambda_2)(dg+f)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \beta^{(af+g)\frac{(\lambda_2-1)\lambda_1^{m+1}-(\lambda_1-1)\lambda_2^{m+1}+\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times \gamma^{\frac{(\lambda_2-1)(\lambda_1-ad)\lambda_1^{m+1}-(\lambda_1-1)(\lambda_2-ad)\lambda_2^{m+1}+(\lambda_1-\lambda_2)(1-ad)}{(\lambda_1-\lambda_2)(\lambda_1-1)(\lambda_2-1)}} \\ &\quad\times y_{-1}^{g\frac{(\lambda_1-ad)\lambda_1^{m}-(\lambda_2-ad)\lambda_2^{m}}{\lambda_1-\lambda_2}} x_0^{\frac{(f\lambda_1+dg)\lambda_1^{m}-(f\lambda_2+dg)\lambda_2^{m}}{\lambda_1-\lambda_2}}, \end{aligned} \tag{145}$$

for $m \in \mathbb{N}_0$.

If $(acf + ad + bf + cg)^2 = -4bdg$, then

$$\lambda_1 = \lambda_2 = \frac{acf + ad + bf + cg}{2}.$$

Then, from (134), it follows that

$$a_n = (n+1)\lambda_1^n, \tag{146}$$

from which it follows that

$$u_n = 1 + \sum_{j=1}^{n-1} (j+1)\lambda_1^j = \frac{1-(n+1)\lambda_1^n + n\lambda_1^{n+1}}{(1-\lambda_1)^2}. \tag{147}$$

By using (146) and (147) in (34)–(42), and, after some calculations, we get

$$\begin{aligned}
x_{3m-1} &= \alpha^{\frac{1-cg+cgm\lambda_1^{m-1}-(m(1+cg)+1-cg)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{a+bg-bgm\lambda_1^{m-1}+(m(bg-a)-bg-a)\lambda_1^m+am\lambda_1^{m+1}}{(1-\lambda_1)^2}} \gamma^{(ac+b)\frac{1-m\lambda_1^{m-1}+(m-1)\lambda_1^m}{(1-\lambda_1)^2}} \qquad (148)\\
&\times x_{-1}^{d((a\lambda_1+bg)m-bg)\lambda_1^{m-2}} z_0^{(ac+b)m\lambda_1^{m-1}},\\
x_{3m} &= \alpha^{\frac{1-cg+cgm\lambda_1^{m-1}-(m(1+cg)+1-cg)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{a+bg-bgm\lambda_1^{m-1}+(m(bg-a)-bg-a)\lambda_1^m+am\lambda_1^{m+1}}{(1-\lambda_1)^2}} \gamma^{(ac+b)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \qquad (149)\\
&\times y_{-1}^{g(ac+b)m\lambda_1^{m-1}} x_0^{(m(\lambda_1-cg)+\lambda_1)\lambda_1^{m-1}},\\
x_{3m+1} &= \alpha^{\frac{1-cg+cg(m+1)\lambda_1^m-(m(1+cg)+2)\lambda_1^{m+1}+(m+1)\lambda_1^{m+2}}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{a+bg-bgm\lambda_1^{m-1}+(m(bg-a)-bg-a)\lambda_1^m+am\lambda_1^{m+1}}{(1-\lambda_1)^2}} \gamma^{(ac+b)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times z_{-1}^{b(m(\lambda_1-cg)+\lambda_1)\lambda_1^{m-1}} y_0^{((a\lambda_1+bg)m+a\lambda_1)\lambda_1^{m-1}}, \qquad (150)\\
y_{3m-1} &= \alpha^{(cf+d)\frac{1-m\lambda_1^{m-1}+(m-1)\lambda_1^m}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{1-bf+bfm\lambda_1^{m-1}-(m(1+bf)+1-bf)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \gamma^{\frac{c+bd-bdm\lambda_1^{m-1}+(m(bd-c)-bd-c)\lambda_1^m+cm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times y_{-1}^{g((c\lambda_1+bd)m-bd)\lambda_1^{m-2}} x_0^{(fc+d)m\lambda_1^{m-1}}, \qquad (151)\\
y_{3m} &= \alpha^{(cf+d)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{1-bf+bfm\lambda_1^{m-1}-(m(1+bf)+1-bf)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \gamma^{\frac{c+bd-bdm\lambda_1^{m-1}+(m(bd-c)-bd-c)\lambda_1^m+cm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times z_{-1}^{b(cf+d)m\lambda_1^{m-1}} y_0^{((\lambda_1-bf)m+\lambda_1)\lambda_1^{m-1}}, \qquad (152)\\
y_{3m+1} &= \alpha^{(cf+d)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \beta^{\frac{1-bf+bf(m+1)\lambda_1^m-(m(1+bf)+2)\lambda_1^{m+1}+(m+1)\lambda_1^{m+2}}{(1-\lambda_1)^2}} \\
&\times \gamma^{\frac{c+bd-bdm\lambda_1^{m-1}+(m(bd-c)-bd-c)\lambda_1^m+cm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times x_{-1}^{d((\lambda_1-bf)m+\lambda_1)\lambda_1^{m-1}} z_0^{((c\lambda_1+bd)m+c\lambda_1)\lambda_1^{m-1}}, \qquad (153)\\
z_{3m-1} &= \alpha^{\frac{f+dg-dgm\lambda_1^{m-1}+(m(dg-f)-dg-f)\lambda_1^m+fm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times \beta^{(af+g)\frac{1-m\lambda_1^{m-1}+(m-1)\lambda_1^m}{(1-\lambda_1)^2}} \gamma^{\frac{1-ad+adm\lambda_1^{m-1}-(m(1+ad)+1-ad)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\times z_{-1}^{b((f\lambda_1+dg)m-dg)\lambda_1^{m-2}} y_0^{(af+g)m\lambda_1^{m-1}}, \qquad (154)
\end{aligned}$$

$$
\begin{aligned}
z_{3m} &= \alpha^{\frac{f+dg-dgm\lambda_1^{m-1}+(m(dg-f)-dg-f)\lambda_1^m+fm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\quad \times \beta^{(af+g)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \gamma^{\frac{1-ad+adm\lambda_1^{m-1}-(m(1+ad)+1-ad)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\quad \times x_{-1}^{d(af+g)m\lambda_1^{m-1}} z_0^{((\lambda_1-ad)m+\lambda_1)\lambda_1^{m-1}}, \qquad (155)\\
z_{3m+1} &= \alpha^{\frac{f+dg-dgm\lambda_1^{m-1}+(m(dg-f)-dg-f)\lambda_1^m+fm\lambda_1^{m+1}}{(1-\lambda_1)^2}} \\
&\quad \times \beta^{(af+g)\frac{1-(m+1)\lambda_1^m+m\lambda_1^{m+1}}{(1-\lambda_1)^2}} \gamma^{\frac{1-ad+ad(m+1)\lambda_1^m-(m(1+ad)+2)\lambda_1^{m+1}+(m+1)\lambda_1^{m+2}}{(1-\lambda_1)^2}} \\
&\quad \times y_{-1}^{g((\lambda_1-ad)m+\lambda_1)\lambda_1^{m-1}} x_0^{((f\lambda_1+dg)m+f\lambda_1)\lambda_1^{m-1}}, \qquad (156)
\end{aligned}
$$

for $m \in \mathbb{N}_0$.

It is time-consuming, but not difficult to see, that all formulas above really present closed-form formulas for solutions to system (3) in the corresponding cases, which was checked by the author by hand. □

Corollary 1. *Assume that $a, b, c, d, f, g \in \mathbb{Z}$, $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$, and $x_{-i}, y_{-i}, z_{-i}, \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$. Then, the following statements are true:*

- *(a) If $b = 0$ and $acf + ad + cg \neq 1$, then the general solution to system (3) is given by (55)–(63).*
- *(b) If $b = 0$ and $acf + ad + cg = 1$, then the general solution to system (3) is given by (64)–(72).*
- *(c) If $d = 0$ and $acf + bf + cg \neq 1$, then the general solution to system (3) is given by (85)–(93).*
- *(d) If $d = 0$ and $acf + bf + cg = 1$, then the general solution to system (3) is given by (94)–(102).*
- *(e) If $g = 0$ and $acf + ad + bf \neq 1$, then the general solution to system (3) is given by (115)–(123).*
- *(f) If $g = 0$ and $acf + ad + bf = 1$, then the general solution to system (3) is given by (124)–(132).*
- *(g) If $bdg \neq 0$ and $(acf + ad + bf + cg)^2 \neq -4bdg$, then the general solution to system (3) is given by (137)–(145).*
- *(h) If $bdg \neq 0$ and $(acf + ad + bf + cg)^2 = -4bdg$, then the general solution to system (3) is given by (148)–(156).*

Remark. Bearing in mind that system (3) is obtained from the one in [24] by adding some mulitipliers only, it is a natural question if (3) is its real generalization, that is, whether there are cases when the systems are not equivalent. If these two systems are equivalent, it is highly expected that the equivalence can be obtained by using scaling of dependent variables, that is, by using the transformation

$$(x_n, y_n, z_n) = (\lambda_1 \widehat{x}_n, \lambda_2 \widehat{y}_n, \lambda_3 \widehat{z}_n), \quad n \in \mathbb{N}_0, \qquad (157)$$

where λ_j, $j = \overline{1,3}$, are some nonzero numbers.

If we use (157) in (3), we get

$$\lambda_1 \widehat{x}_{n+1} = \alpha \lambda_2^a \lambda_3^b \widehat{y}_n^a \widehat{z}_{n-1}^b, \;\; \lambda_2 \widehat{y}_{n+1} = \beta \lambda_3^c \lambda_1^d \widehat{z}_n^c \widehat{x}_{n-1}^d, \;\; \lambda_3 \widehat{z}_{n+1} = \gamma \lambda_1^f \lambda_2^g \widehat{x}_n^f \widehat{y}_{n-1}^g,$$

from which it follows that, for the equivalence of the systems, it must hold

$$\lambda_1 \lambda_2^{-a} \lambda_3^{-b} = \alpha, \quad \lambda_1^{-d} \lambda_2 \lambda_3^{-c} = \beta, \quad \lambda_1^{-f} \lambda_2^{-g} \lambda_3 = \gamma. \qquad (158)$$

However, the nonlinear system (158) need not have solutions if the determinant

$$\Delta := \begin{vmatrix} 1 & -a & -b \\ -d & 1 & -c \\ -f & -g & 1 \end{vmatrix}$$

is equal to zero (if $\lambda_j > 0$, $j = \overline{1,3}$, and $\alpha, \beta, \gamma \in \mathbb{R}_+$, the statement is immediately obtained by transforming system (158) to a linear one, by taking the logarithm; the case when some of these parameters are complex is dealt with in another way). It is easy to see that the equation $\Delta = 0$ has many solutions in the set $\mathbb{Z}^6$. Hence, the systems, in general cases, are not equivalent.

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