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Fractional Newton–Raphson Method Accelerated with Aitken's Method

A. Torres-Hernandez [1,*], F. Brambila-Paz [2,*], U. Iturrarán-Viveros [2], and R. Caballero-Cruz [1]

1 Department of Physics, Faculty of Science, UNAM, Mexico City 04510, Mexico; rcaballero@ciencias.unam.mx
2 Department of Mathematics, Faculty of Science, UNAM, Mexico City 04510, Mexico; ursula@ciencias.unam.mx
* Correspondence: anthony.torres@ciencias.unam.mx (A.T.-H.); fernando.brambila@ciencias.unam.mx (F.B.-P.)

Abstract: In the following paper, we present a way to accelerate the speed of convergence of the fractional Newton–Raphson (F N–R) method, which seems to have an order of convergence at least linearly for the case in which the order α of the derivative is different from one. A simplified way of constructing the Riemann–Liouville (R–L) fractional operators, fractional integral and fractional derivative is presented along with examples of its application on different functions. Furthermore, an introduction to Aitken's method is made and it is explained why it has the ability to accelerate the convergence of the iterative methods, in order to finally present the results that were obtained when implementing Aitken's method in the F N–R method, where it is shown that F N–R with Aitken's method converges faster than the simple F N–R.

Keywords: Newton–Raphson method; fractional calculus; fractional derivative; Aitken's method

MSC: 49M15; 90C53; 65H99; 26A33

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1. Fixed Point Method

A classical problem in mathematics, which is of common interest in physics and engineering, is finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \ : \ \|f(\xi)\| = 0\}, \tag{1}$$

where $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm. Although finding the zeros of a function may seem like a simple problem, in general, it involves solving an **algebraic equation system** as follows:

$$\begin{cases} [f]_1(x) = 0 \\ [f]_2(x) = 0 \\ \quad\vdots \\ [f]_n(x) = 0 \end{cases}, \tag{2}$$

where $[f]_k : \mathbb{R}^n \to \mathbb{R}$ denotes the k-th component of the function f. It should be noted that the system of Equation (2) may represent a **linear system** or a **nonlinear system**, and, in general, it is necessary to use numerical methods of the iterative type to solve it. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a function, and it is possible to build a sequence $\{x_i\}_{i=0}^{\infty}$ by defining the following iterative method:

$$x_{i+1} := \Phi(x_i), \tag{3}$$

if it is fulfilled that $x_i \to \xi \in \mathbb{R}^n$, and, if the function Φ is continuous around ξ, we obtain that

$$\xi = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \Phi(x_i) = \Phi\left(\lim_{i\to\infty} x_i\right) = \Phi(\xi), \tag{4}$$

the above result is the reason by which the method (3) is known as the **fixed point method**. Furthermore, the function Φ is called an **iteration function**. To understand the nature of the convergence of the iteration function Φ, the following definition is necessary [1]:

Definition 1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. The method (3) for determining $\xi \in \mathbb{R}^n$ is called* ***(locally) convergent****, if there exists $\delta > 0$ such that, for every initial value,*

$$x_0 \in B(\xi;\delta) := \{y \in \mathbb{R}^n \ : \ \|y - \xi\| < \delta\},$$

it is fulfilled that

$$\lim_{i\to\infty} \|x_i - \xi\| \to 0 \quad \Rightarrow \quad \lim_{i\to\infty} x_i = \xi. \tag{5}$$

If we have a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, for which we want to determine the set (1), in general, it is possible to write an iteration function Φ as follows [2]:

$$\Phi(x) = x - A(x)f(x),$$

where $A(x)$ is a matrix, which is given as follows:

$$A(x) := \Big([A]_{jk}(x)\Big) = \begin{pmatrix} [A]_{11}(x) & [A]_{12}(x) & \cdots & [A]_{1n}(x) \\ [A]_{21}(x) & [A]_{22}(x) & \cdots & [A]_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ [A]_{n1}(x) & [A]_{n2}(x) & \cdots & [A]_{nn}(x) \end{pmatrix}, \tag{6}$$

with $[A]_{jk}(x) : \mathbb{R}^n \to \mathbb{R}\ \forall j,k \leq n$. It is necessary to mention that the matrix $A(x)$ is determined according to the order of convergence desired.

Order of Convergence

Before continuing, it is necessary to define the order of convergence of an iteration function Φ [1]:

Definition 2. *Let $\Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function with a fixed point $\xi \in \Omega$. Then, the method (3) is called* ***(locally) convergent of (at least) order p*** *($p \geq 1$), if there exist $\delta > 0$ and C a non-negative constant, with $C < 1$ if $p = 1$, such that, for any initial value $x_0 \in B(\xi;\delta)$, it is fulfilled that*

$$\|x_{k+1} - \xi\| \leq C\|x_k - \xi\|^p, \quad k = 0,1,2,\cdots, \tag{7}$$

where C is called a convergence factor.

The order of convergence is usually related to the speed at which the sequence generated by (3) converges. For the particular case $p = 1$, it is said that the method (3) has an **order of convergence (at least) linear**, and, for the case $p = 2$, it is said that the method (3) has an **order of convergence (at least) quadratic**. The following theorem allows for characterizing the order of convergence of an iteration function Φ with its derivatives [1–4]. Before continuing, we need to consider the following multi-index notation. Let $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$, if $\gamma \in \mathbb{N}_0^n$, then

$$\begin{cases} \gamma! := \prod_{k=1}^{n} [\gamma]_k! \\ |\gamma| := \sum_{k=1}^{n} [\gamma]_k \\ x^\gamma := \prod_{k=1}^{n} [x]_k^{[\gamma]_k} \\ \dfrac{\partial^\gamma}{\partial x^\gamma} := \dfrac{\partial^{|\gamma|}}{\partial [x]_1^{[\gamma]_1} \partial [x]_2^{[\gamma]_2} \cdots \partial [x]_n^{[\gamma]_n}} \end{cases}. \tag{8}$$

Theorem 1. *Let $\Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function with a fixed point $\xi \in \Omega$. Assuming that Φ is p-times differentiable in ξ for some $p \in \mathbb{N}$, and, furthermore,*

$$\begin{cases} \dfrac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma} = 0, \ \forall k \geq 1 \text{ and } \forall |\gamma| < p, & \text{if } p \geq 2 \\ \left\| \Phi^{(1)}(\xi) \right\| < 1, & \text{if } p = 1 \end{cases}, \tag{9}$$

where $\Phi^{(1)}$ denotes the ***Jacobian matrix*** *of the function Φ, then Φ is (locally) convergent of (at least) order p.*

Proof. The proof may be found in the Appendix A.1. □

The following corollary follows from the previous theorem:

Corollary 1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_i \to \xi$, and, if the following condition is true*

$$\lim_{x \to \xi} \left\| \Phi^{(1)}(x) \right\| \neq 0, \tag{10}$$

then Φ has an order of convergence (at least) linear in $B(\xi; \delta)$.

2. Newton–Raphson Method

We begin this section by considering the following proposition [4,5]:

Proposition 1. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a value $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function as follows:*

$$\Phi(x) = x - A(x) f(x), \tag{11}$$

with $A(x)$ a matrix. If the following condition is fulfilled:

$$\lim_{x \to \xi} A(x) = \left(f^{(1)}(\xi) \right)^{-1}, \tag{12}$$

where $f^{(1)}$ denotes the Jacobian matrix of the function f, which is defined as follows [6]

$$f^{(1)}(x) := \left([f]_{jk}^{(1)}(x) \right) = \left(\partial_k [f]_j(x) \right), \tag{13}$$

where

$$[f]_{jk}^{(1)}(x) = \partial_k [f]_j(x) := \frac{\partial}{\partial [x]_k} [f]_j(x), \quad 1 \leq j, k \leq n,$$

then the iteration function Φ, fulfills a necessary (but not sufficient) condition to be (locally) convergent of order (at least) quadratic in $B(\xi;\delta)$.

Proof. The proof may be found in Appendix A.2. □

The following fixed point method may be obtained from the previous proposition:

$$x_{i+1} := \Phi(x_i) = x_i - \left(f^{(1)}(x_i)\right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots, \tag{14}$$

which is known as the **Newton–Raphson method**, also known as Newton's method [7]. Given the condition (12), it could be wrongly considered that the Newton–Raphson method always has an order of convergence (at least) quadratic, but as mentioned in the **Proposition 1**, the form of the iteration function (14) is not sufficient to guarantee this order of convergence. This occurs because, even if the condition (12) is fulfilled, the order of convergence becomes conditioned by the way in which the function f is constituted, for example for the one variable case, if the function f has a root ξ, with a certain algebraic multiplicity $m \geq 2$, that is,

$$f(x) = (x-\xi)^m g(x), \quad g(\xi) \neq 0,$$

the Newton–Raphson method presents an order of convergence at least linear [1], the aforementioned may be observed in the following proposition:

Proposition 2. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then, the iteration function Φ of the Newton–Raphson method, given by (14), fulfills the following condition:*

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!} |x_i - \xi|^p, \tag{15}$$

where

$$p = \begin{cases} 1, & \text{if } f(x) = (x-\xi)^m g(x) \\ 2, & \text{if } f^{(1)}(\xi) \neq 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \\ 3, & \text{if } f^{(1)}(\xi) \neq 0,\ f^{(2)}(\xi) = 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \\ 4, & \text{if } f^{(1)}(\xi) \neq 0,\ f^{(2)}(\xi) = 0,\ f^{(3)}(\xi) = 0 \text{ and } f(x) \neq (x-\xi)^m g(x) \end{cases}, \tag{16}$$

with $g(\xi) \neq 0$ and $m \geq 2$.

Proof. The proof may be found in Appendix A.3. □

The previous proposition is important because, when the N–R method is implemented in a function f, the zeros of the function are assumed to be unknown, and their algebraic multiplicities $m \geq 2$, in case they exist, are also unknown. With the above in mind, the following corollary is obtained, which is derived from **Proposition 1**, **Proposition 2**, and **Corollary 1**.

Corollary 2. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a zero $\xi \in \Omega$. If there exists at least a value $k > 0$, and a function $g_k : \mathbb{R}^n \to \mathbb{R}$, such that*

$$[f]_{j_k}(x) = [(x-\xi)]_k^m g_k(x), \quad g_k(\xi) \neq 0,$$

for some value j_k, with

$$1 \leq j_k, k \leq n \quad \text{and} \quad m \geq 2,$$

then the Jacobian matrix of the iteration function Φ *of the N–R method, given by* (14)*, fulfills that all entries in its k-th column are nonzero at the value* ξ*, that is,*

$$[\Phi]_{jk}^{(1)}(\xi) \neq 0, \quad \forall j > 0,$$

Consequently, the N–R method has an order of convergence (at least) linear.

3. Fractional Calculus

The fractional calculus is a mathematical analysis branch whose applications have been increasing since the end of the 20th century and beginnings of the 21st century [8–10], the fractional calculus arose around 1695 due to Leibniz's notation for the derivatives of integer order

$$f^{(n)}(x) := \frac{d^n}{dx^n} f(x), \quad n \in \mathbb{N}.$$

Thanks to this notation, L'Hopital could ask in a letter to Leibniz about the interpretation of taking $n = 1/2$ in a derivative, since at that moment Leibniz could not give a physical or geometrical interpretation to this question, he simply answered L'Hopital in a letter, "... is an apparent paradox of which, one day, useful consequences will be drawn" [11]. The name of fractional calculus comes from a historical question since, in this branch of mathematical analysis, the derivatives and integrals of a certain order α are studied, with $\alpha \in \mathbb{R}$ or $\mathbb{C}$.

Currently, the fractional calculus does not have a unified definition of what is considered a fractional derivative because one of the conditions required to consider an expression as a fractional derivative is to recover the results of conventional calculus when the order $\alpha \to n$, with $n \in \mathbb{N}$ [12]; among the most common definitions of fractional derivatives are the Riemann–Liouville (R–L) fractional derivative and the Caputo fractional derivative [13–15], the latter is usually the most studied since the Caputo fractional derivative allows us a physical interpretation to problems with initial conditions; this derivative fulfills the property of the classical calculus that the derivative of a constant is null regardless of the order α of the derivative; however, this does not occur with the R–L fractional derivative, and this characteristic can be used to solve nonlinear systems [4,16,17].

Unlike the Caputo fractional derivative, the R–L fractional derivative does not allow for a physical interpretation to the problems with an initial condition because its use induces fractional initial conditions; however, the fact that this derivative does not cancel the constants for α, with $\alpha \notin \mathbb{N}$, allows for obtaining a "spectrum" of the behavior of the constants for different orders of the derivative, which is not possible with conventional calculus. It is worth mentioning that, depending on the function f, the results of Riemann–Liouville and Caputo fractional derivatives can sometimes be rewritten in terms of Mittag–Leffler functions or hypergeometric functions [18,19].

3.1. Construction of the Riemann–Liouville Fractional Derivative

We begin with some definitions and standard properties for those readers who have not had previous contact with fractional calculus. The R–L fractional derivative is constructed in a simplified way, taking into account that the integral operator is defined for a locally integrable function f, that is, $f \in L^1_{loc}(a, \infty)$, then

$$_aI_x f(x) := \int_a^x f(t)dt,$$

applying two times the integral operator

$$_aI_x^2 f(x) = \int_a^x \left(\int_a^{x_1} f(t)dt \right) dx_1 = \int_a^x \left({_aI_{x_1}} f(x_1) \right) dx_1,$$

doing an integration by parts, taking $u = {}_aI_{x_1}f(x_1)$ and $dv = dx_1$, as a consequence

$$\begin{aligned} {}_aI_x^2f(x) =& x_1{}_aI_{x_1}f(x_1)\Big|_a^x - \int_a^x x_1f(x_1)dx_1 \\ =& x_aI_xf(x) - {}_aI_x(xf(x)) \\ =& \int_a^x (x-t)f(t)dt, \end{aligned} \tag{17}$$

repeating the previous process, applying the integral operator three times

$${}_aI_x^3f(x) = \int_a^x \left({}_aI_{x_1}^2f(x_1)\right)dx_1,$$

doing an integration by parts, taking $u = {}_aI_{x_1}^2f(x_1)$ and $dv = dx_1$, then

$$\begin{aligned} {}_aI_x^3f(x) =& x_1{}_aI_{x_1}^2f(x_1)\Big|_a^x - \int_a^x \left(x_1{}_aI_{x_1}f(x_1)\right)dx_1 \\ =& x_aI_x^2f(x) - {}_aI_x(x_aI_xf(x)) \\ =& \int_a^x (x-t)_aI_tf(t)dt, \end{aligned}$$

doing again an integration by parts, taking $u = {}_aI_tf(t)$ and $dv = (x-t)dt$, as a consequence

$$\begin{aligned} {}_aI_x^3f(x) =& -\frac{1}{2}(x-t)^2{}_aI_tf(t)\Big|_a^x + \frac{1}{2}\int_a^x (x-t)^2f(t)dt \\ =& \frac{1}{2}\int_a^x (x-t)^2f(t)dt. \end{aligned} \tag{18}$$

Repeating the previous process, applying n times the integral operator and doing $n-1$ integrations by parts, it is possible to obtain the following expression of the n-th iterated integral [13]

$${}_aI_x^nf(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt, \tag{19}$$

to make a generalization of the previous expression, it is enough to take into account the relationship between the Gamma function and the factorial function, $\Gamma(n) = (n-1)!$, and doing $n \to \alpha \in \mathbb{R}$, the expression for the (right) R–L fractional integral is obtained [13]

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \tag{20}$$

taking into account that the differential operator $(D_x = d/dx)$ is the inverse operator to the left of the integral operator $({}_aI_x)$, that is,

$$D_x^n({}_aI_x^nf(x)) = \frac{d^n}{dx^n}({}_aI_x^nf(x)) = f(x),$$

we may consider extending the previous result analogously to the fractional calculus using the expression

$${}_aD_x^\alpha f(x) := {}_aI_x^{-\alpha}f(x).$$

Unfortunately, this would cause convergence problems because the Gamma function is not defined for $\alpha \in \mathbb{Z}_{\leq 0}$, to solve this problem, the above expression is rewritten as

$$
{}_aD_x^\alpha f(x) = {}_aI_x^{-\alpha} f(x) = \frac{d^n}{dx^n}\left({}_aI_x^n\left({}_aI_x^{-\alpha} f(x)\right)\right) = \frac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right),
$$

for the above expression to make sense, it is necessary to consider $n - \alpha \geq 0$, there are infinite ways that n may be taken to fulfill the above condition, but the most convenient way is to consider

$$
n = n(\alpha).
$$

Considering the above, we can define the (right) R–L fractional derivative as follows:

$$
{}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt, \quad n = \lceil \alpha \rceil, \tag{21}
$$

in such a way that the previous expression fulfills that

$$
\begin{aligned}
\lim_{\alpha \to 1} {}_aD_x^\alpha f(x) &= \lim_{\alpha \to 1} \frac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right) \\
&= \frac{d}{dx}\left({}_aI_x^0 f(x)\right) \\
&= \frac{d}{dx} f(x).
\end{aligned}
$$

Finally, it is possible to unify the R–L fractional operators, fractional integral, and fractional derivative, and define the **(right) Riemann–Liouville fractional derivative** as follows [13,14]:

$$
{}_aD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ \dfrac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right), & \text{if } \alpha \geq 0 \end{cases}, \tag{22}
$$

where $n = \lceil \alpha \rceil$.

Examples of the Riemann–Liouville Fractional Derivative

Before continuing, it is necessary to define the Beta function and the incomplete Beta function [19], which are defined as follows:

$$
B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad B_r(p,q) := \int_0^r t^{p-1}(1-t)^{q-1}dt, \tag{23}
$$

where p and q are positive values. Considering the following proposition:

Proposition 3. *Let f be a function, with*

$$
f(x) = (x-c)^\mu, \quad \mu > -1, \quad c \in \mathbb{R},
$$

then, for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the Riemann–Liouville fractional derivative of the above function may be written as

$$
{}_aD_x^\alpha f(x) = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-c)^{\mu-\alpha} G_{-\alpha}\left(\dfrac{a-c}{x-c}, \mu+1\right), & \text{if } \alpha < 0 \\ \displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)}(x-c)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1\right), & \text{if } \alpha \geq 0 \end{cases}, \tag{24}
$$

where

$$
G_\alpha\left(\frac{a-c}{x-c}, \mu+1\right) := 1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,\alpha)}{B(\mu+1,\alpha)}. \tag{25}
$$

Proof. The proof may be found in Appendix A.4. □

From the previous proposition, we can note that the Riemann–Liouville fractional derivative presents an explicit dependence of the value $n = \lceil \alpha \rceil$. However, there exists a particular case in which this dependence disappears, as shown in the following proposition:

Proposition 4. *Let f be a function, with*

$$f(x) = (x-a)^{\mu}, \quad \mu > -1, \quad a \in \mathbb{R},$$

then for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the Riemann–Liouville fractional derivative of the above function may be written in general form as

$$_aD_x^{\alpha}(x-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha}. \tag{26}$$

Proof. The proof may be found in Appendix A.5. □

From the previous proposition, the following corollary is obtained

Corollary 3. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function, with $f \in L^1_{loc}(a,\infty)$. Assuming furthermore that $f \in C^{\infty}(a,\infty)$, such that f may be written in terms of its Taylor series around the point $x = a$, that is,*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

then, for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the Riemann–Liouville fractional derivative of the aforementioned function, may be written as follows:

$$_aD_x^{\alpha} f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \tag{27}$$

Finally, applying the operator (22) with $a = 0$ to the function x^{μ}, with $\mu > -1$, from the **Proposition** 4, we obtain the following result:

$$_0D_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}. \tag{28}$$

3.2. Introduction to the Caputo Fractional Derivative

Michele Caputo published a book and introduced a new definition of fractional derivative, and he created this definition with the objective of modeling anomalous diffusion phenomena. The definition of Caputo had already been discovered independently by Gerasimov. This fractional derivative is of the utmost importance since it allows us to give a physical interpretation of the initial value problems, moreover being used to model fractional time. In some texts, it is known as the fractional derivative of Gerasimov–Caputo [15].

Let f be a function, such that f is n-times differentiable with $f^{(n)} \in L^1_{loc}(a,b)$, then the **(right) fractional derivative of Caputo** is defined as [14]

$$^C_aD_x^{\alpha} f(x) := {}_aI_x^{n-\alpha}\left(\frac{d^n}{dx^n} f(x)\right) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \tag{29}$$

where $n = \lceil \alpha \rceil$. It should be mentioned that the Caputo fractional derivative behaves as the inverse operator to the left of the Riemann–Liouville fractional integral, that is,

$$ {}_a^C D_x^\alpha \left({}_a I_x^\alpha f(x) \right) = f(x). $$

On the other hand, the relation between the fractional derivatives of Caputo and Riemann–Liouville is given by the following expression [14]:

$$ {}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right), $$

then, if $f^{(k)}(a) = 0 \ \forall k < n$, we obtain that

$$ {}_a^C D_x^\alpha f(x) = {}_a D_x^\alpha f(x), \tag{30} $$

considering the previous particular case, it is possible to unify the definitions of R–L fractional integral and Caputo fractional derivative as follows:

$$ {}_a^C D_x^\alpha f(x) := \begin{cases} {}_a I_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ {}_a I_x^{n-\alpha} \left(\dfrac{d^n}{dx^n} f(x) \right), & \text{if } \alpha \geq 0 \end{cases}, \tag{31} $$

where $n = \lceil \alpha \rceil$.

4. Fractional Newton–Raphson Method

We begin this section by mentioning that, although the interest in fractional calculus has mainly focused on the study and development of techniques to solve differential equation systems of order non-integer [8–14]. Over the years, iterative methods have also been developed that use the properties of fractional derivatives to solve algebraic equation systems [4,5,20–25]. These methods may be called **fractional iterative methods**; recently, these methods have been useful in the search for solutions to algebraic equation systems related to hybrid solar receivers [4,16]. It should be noted that, depending on the definition of fractional derivative used, fractional iterative methods have the particularity that they may be used of local form [20] or of global form [5].

Let $\mathbb{P}_n(\mathbb{R})$ be the space of polynomials of degree less than or equal to $n \in \mathbb{N}$ with real coefficients. The N–R method is characterized by the fact that, if it generates divergent sequences of complex numbers, they may lead to the creation of a fractal [26]. On the other hand, the order of the fractional derivatives seems to be closely related to the fractal dimension [8]; based on the above, a method was developed that makes use of the N–R method and the fractional derivatives. The N–R method is useful for finding the roots of a function $f \in \mathbb{P}_n(\mathbb{R})$. However, this method is limited because it cannot find roots $\xi \in \mathbb{C} \setminus \mathbb{R}$, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (14) has an initial condition $x_0 \in \mathbb{R}$. To solve this problem and develop a method that has the ability to find roots, both real and complex, of a polynomial if the initial condition x_0 is real, we propose a new method, which consists of the Newton–Raphson method with the implementation of the fractional derivatives. Before continuing, it is necessary to define the **fractional Jacobian matrix** of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$ f^{(\alpha)}(x) := \left([f]_{jk}^{(\alpha)}(x) \right), \tag{32} $$

where

$$ [f]_{jk}^{(\alpha)} = \partial_k^\alpha [f]_j(x) := \frac{\partial^\alpha}{\partial [x]_k^\alpha} [f]_j(x), \quad 1 \leq j,k \leq n. $$

with $[f]_j : \mathbb{R}^n \to \mathbb{R}$. The operator $\partial^\alpha / \partial [x]_k^\alpha$ denotes any fractional derivative, applied only to the variable $[x]_k$, which fulfills the following condition of continuity respect to the order of the derivative

$$\lim_{\alpha \to 1} \frac{\partial^\alpha}{\partial [x]_k^\alpha} [f]_j(x) = \frac{\partial}{\partial [x]_k} [f]_j(x), \quad 1 \leq j, k \leq n,$$

then, the matrix (32) fulfills that

$$\lim_{\alpha \to 1} f^{(\alpha)}(x) = f^{(1)}(x), \tag{33}$$

where $f^{(1)}(x)$ denotes the Jacobian matrix of the function f. Considering a function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$, then using as a basis the idea of the N–R method (14), and considering any fractional derivative that fulfills the condition (33), we can define the **Fractional Newton–Raphson Method** as follows [5,21]:

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - \left(f^{(\alpha)}(x_i) \right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots. \tag{34}$$

For the above expression to make sense, due to the part of the integral operator that fractional derivatives usually have, and that the F N–R method can be used in a wide variety of functions [5], we consider in the expression (34) that each fractional derivative is obtained for a real variable $[x]_k$, and, if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]_k$, that is,

$$f^{(\alpha)}(x_i) := f^{(\alpha)}(x) \Big|_{x \longrightarrow x_i}, \quad x \in \mathbb{R}^n, \quad x_i \in \mathbb{C}^n. \tag{35}$$

Convergence of the Fractional Newton–Raphson Method

It should be mentioned that, in general, in the F N–R method $\left\| \Phi^{(1)}(\alpha, \xi) \right\| \neq 0$ if $\| f(\xi) \| = 0$, and from **Corollary 1**, **Proposition 1**, **Proposition 2** and the condition (33), any sequence $\{x_i\}_{i=0}^\infty$ generated by the iterative method (34) has an order of convergence at least linear, that is, the F N–R method, considering the **Theorem 1**, may fulfill an equation analogous to Equation (15) with $p \geq 1$, which becomes more evident when considering $\alpha \in [1 - \epsilon, 1 + \epsilon] \setminus \{1\}$. The aforementioned, for the case in one dimension, may be observed in the following proposition:

Proposition 5. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then, any sequence $\{x_i\}_{i=0}^\infty$ generated by the iteration function of the F N–R method, such that $x_i \to \xi$, fulfills the following condition:*

$$|x_{i+1} - \xi| \leq \frac{\left| \Phi^{(p)}(\alpha, \xi) \right|}{p!} |x_i - \xi|^p, \tag{36}$$

where

$$p = \begin{cases} 1, & \text{if } \alpha \neq 1 \text{ and } f^{(\alpha)}(\xi) \neq 0 \\ 2, & \text{if } \alpha = 1 \text{ and } f^{(1)}(\xi) \neq 0 \end{cases}. \tag{37}$$

Proof. Considering the iteration function of the F N–R method

$$\Phi(\alpha, x) = x - \left(f^{(\alpha)}(x) \right)^{-1} f(x),$$

and calculating its first and second derivative

$$\Phi^{(1)}(\alpha, x) = 1 - \left(f^{(\alpha)}(x) \right)^{-1} f^{(1)}(x) + f(x) \left[\left(f^{(\alpha)}(x) \right)^{-2} D_x f^{(\alpha)}(x) \right],$$

$$
\begin{aligned}
\Phi^{(2)}(\alpha,x) =& f(x)\left[\left(f^{(\alpha)}(x)\right)^{-2} D_x^2 f^{(\alpha)}(x) - 2\left(f^{(\alpha)}(x)\right)^{-3}\left(D_x f^{(\alpha)}(x)\right)^2\right] \\
&+ 2\left(f^{(\alpha)}(x)\right)^{-2} f^{(1)}(x) D_x f^{(\alpha)}(x) - \left(f^{(\alpha)}(x)\right)^{-1} f^{(2)}(x),
\end{aligned}
$$

then, assuming that $f^{(\alpha)}(\xi) \neq 0 \; \forall \alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cup \{1\}$, and taking into account the condition (33) together with the fact that ξ is a zero of f, we obtain that

$$
\lim_{x \to \xi} \Phi(\alpha, x) = \xi,
$$

$$
\lim_{x \to \xi} \left|\Phi^{(1)}(\alpha, x)\right| = \begin{cases} \left|1 - \left(f^{(\alpha)}(\xi)\right)^{-1} f^{(1)}(\xi)\right| & , \text{ if } \alpha \neq 1 \\ 0 & , \text{ if } \alpha = 1 \end{cases},
$$

$$
\lim_{x \to \xi} \left|\Phi^{(2)}(\alpha, x)\right| = \begin{cases} \left|2\left(f^{(\alpha)}(\xi)\right)^{-2} f^{(1)}(\xi) D_x f^{(\alpha)}(\xi) - \left(f^{(\alpha)}(\xi)\right)^{-1} f^{(2)}(\xi)\right| & , \text{ if } \alpha \neq 1 \\ \left|\left(f^{(\alpha)}(\xi)\right)^{-1} f^{(2)}(\xi)\right| & , \text{ if } \alpha = 1 \end{cases}.
$$

As a consequence, from the **Theorem 1**, the F N–R method has an order of convergence at least linear, that is, fulfills Equation (36) with $p \geq 1$. □

From the above proposition, together with the **Proposition 1**, it may be obtained that almost any fractional iterative method that has a similar structure to the fractional Newton–Raphson method [5,22–25] has the ability to change from an order of convergence (at least) linear to an order of convergence (at least) quadratic, as long as the method fulfills the condition (33). An alternative to achieve the change in the order of convergence of some fractional iterative method, analogous to F N–R method, is to replace the constant value α in the order of the fractional derivatives by some function that guarantees that the condition (33) is fulfilled, that is,

$$
\alpha \in \mathbb{R} \setminus \mathbb{Z} \quad \longrightarrow \quad \alpha(x) : \mathbb{C}^n \to (\mathbb{R} \setminus \mathbb{Z}) \cup \{1\}. \tag{38}
$$

It is necessary to mention that an example of the aforementioned may be found in the **Fractional Newton Method**, which is defined as follows [5]:

$$
x_{i+1} := \Phi(\alpha, x_i) = x_i - \left(N_{\alpha_f}(x_i)\right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots, \tag{39}
$$

where $N_{\alpha_f}(x_i)$ is given by the following matrix:

$$
N_{\alpha_f}(x_i) := \left([N_{\alpha_f}]_{jk}(x_i)\right) = \left(\partial_k^{\alpha_f([x_i]_k, x_i)} [f]_j(x_i)\right). \tag{40}
$$

with $\delta > 0$, and $\alpha_f([x_i]_k, x_i)$ a function defined as follows:

$$
\alpha_f([x_i]_k, x_i) := \begin{cases} \alpha, & \text{if } |[x_i]_k| \neq 0 \text{ and } \|f(x_i)\| \geq \delta \\ 1, & \text{if } |[x_i]_k| = 0 \text{ or } \|f(x_i)\| < \delta \end{cases}, \tag{41}
$$

the difference between the methods (34) and (39), is that, just for the second method, there may exist a value $\delta > 0$, such that, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (39) converges to a zero ξ of f, there exists a value $k > 0$ such that $\forall i \geq k$, from **Proposition 1**, **Proposition 5** and condition (33), the sequence has an order of convergence (at least) quadratic in $B(\xi; \delta)$.

5. Aitken's Method

Due to not all fractional iterative methods fulfilling the condition (33), since not all methods have a similar structure to the F N–R method [4,5,16], an alternative such as that of Equation (38) to accelerate its order of convergence would not be suitable. However, an alternative that may be used in general in any fractional iterative method to accelerate its convergence is to combine the method with the **Aitken's method** [3,27].

Aitken's method, also known as the Δ^2- method of Aitken [3], is one of the first and simplest methods to accelerate the convergence of a given convergent sequence $\{x_i\}_{i=0}^{\infty}$, that is,

$$\lim_{i\to\infty}\|x_i-\xi\|\to 0,$$

this method allows for transforming the sequence $\{x_i\}_{i=0}^{\infty}$ to a sequence $\{x_i'\}_{i=0}^{\infty}$, which generally converges faster point ξ that the original sequence. Under certain circumstances, Aitken's method may accelerate the convergence of a method that has an order of convergence (at least) linear to an order of convergence almost quadratic, then it is generally used to accelerate the iterative methods used to find the zeros of a function f [1–3].

To illustrate Aitken's method for the case in one dimension, suppose that the sequence $\{x_i\}_{i=0}^{\infty}$ converges to the point ξ as a geometric sequence with factor k such that $|k|<1$, that is,

$$x_{i+1}-\xi=k(x_i-\xi),\quad i=0,1,2,\cdots, \tag{42}$$

where the value of ξ may be determined using the following system of equations:

$$x_{i+1}-\xi=k(x_i-\xi), \tag{43}$$
$$x_{i+2}-\xi=k(x_{i+1}-\xi), \tag{44}$$

subtracting Equation (43) from Equation (44), we obtain the value of k

$$k=\frac{x_{i+2}-x_{i+1}}{x_{i+1}-x_i},$$

placing ξ on the left side of Equation (43)

$$\xi=\frac{kx_i-x_{i+1}}{k-1}=\frac{(k-1+1)x_i-x_{i+1}}{k-1}=x_i-\frac{x_{i+1}-x_i}{k-1},$$

and substituting the value of k in the previous expression

$$\xi=x_i-\frac{(x_{i+1}-x_i)(x_{i+1}-x_i)}{(x_{i+2}-x_{i+1})-(x_{i+1}-x_i)}=x_i-\frac{(x_{i+1}-x_i)^2}{x_{i+2}-2x_{i+1}+x_i},$$

defining the difference operator

$$\Delta x_i:=x_{i+1}-x_i,$$

then

$$\Delta^2 x_i=\Delta x_{i+1}-\Delta x_i=x_{i+2}-2x_{i+1}+x_i.$$

Therefore, we obtain that the value of ξ is given by the following expression:

$$\xi=x_i-\frac{(\Delta x_i)^2}{\Delta^2 x_i}. \tag{45}$$

Aitken's method is considered taking into account Equation (45). The Δ^2- method of Aitken consists of generating a new sequence $\{x'_i\}_{i=0}^{\infty}$, where

$$x'_i = x_i - \frac{(x_{i+1} - x_i)^2}{x_{i+2} - 2x_{i+1} + x_i}, \tag{46}$$

such that

$$\lim_{i\to\infty} |x'_i - \xi| \to 0.$$

On the other hand, to note that the sequence $\{x'_i\}_{i=0}^{\infty}$ converges more quickly to value ξ than the sequence $\{x_i\}_{i=0}^{\infty}$, consider the following proposition:

Proposition 6. *Let $\{x_i\}_{i=0}^{\infty}$ be a sequence, such that $x_i \to \xi$. Then, the sequence $\{x'_i\}_{i=0}^{\infty}$ generated by Aitken's method, given by* (46)*, has a speed of convergence greater than the original sequence.*

Proof. The proof may be found in Appendix A.6. □

From the above proposition, it follows that any fractional iterative method, given by the following expression

$$x_{i+1} := \Phi(\alpha, x_i), \quad i = 0, 1, 2, \cdots, \tag{47}$$

may accelerate its speed of convergence using Aitken's method, giving rise to **Fractional Steffensen's Method**, which is defined as follows:

$$x_{i+1} := \Psi(\alpha, x_i), \quad i = 0, 1, 2, \cdots, \tag{48}$$

where we use the function $\Psi(\alpha, x)$ to denote the implementation of Aitken's method to any fractional iterative method for the case of one variable [3] and for the case of several variables [27].

Results of the Fractional Newton–Raphson Method with Aitken's Method

Examples of the implementation of the F N–R method and Aitken's method for the multidimensional case may be found in the references [5,27], respectively. However, to maintain an illustrative character, the following examples are solved for the case in one dimension using the R–L fractional derivative and the Caputo fractional derivative through Equations (28) and (30). Instructions for implementing the F N–R method, along with information to provide values $\alpha \in [0.7, 1.3] \setminus \{1\}$, are found in the reference [5]. For rounding reasons, for the examples, the following function is defined:

$$\mathrm{Rnd}_m(x) := \left\{ \begin{array}{cl} \mathrm{Re}(x), & \text{if } |\mathrm{Im}(x)| \le 10^{-m} \\ x, & \text{if } |\mathrm{Im}(x)| > 10^{-m} \end{array} \right. . \tag{49}$$

Combining the function (49) with the methods (34) and (48), the following iterative methods are defined:

$$x_{i+1} := \mathrm{Rnd}_5(\Phi(\alpha, x_i)), \; i = 0, 1, 2 \cdots, \tag{50}$$

$$x_{i+1} := \mathrm{Rnd}_5(\Psi(\alpha, x_i)), \; i = 0, 1, 2 \cdots, \tag{51}$$

it should be mentioned that the methods (50) and (51) may be implemented through recursive programming in a way analogous to that presented in the reference [28].

Example 1. *Let f be a function, with*

$$f(x) = -85.86x^{14} + 19.3x^{13} - 92.34x^{12} + 3.13x^{11} + 64.75x^{10} - 54.17x^9 - 17.7x^8 \\ - 13.05x^7 - 56.82x^6 - 56.93x^5 - 94.95x^4 - 95.09x^3 - 84.16x^2.$$

Then, the initial condition $x_0 = 9.86$ is chosen to use the iterative methods given by (50) and (51). Consequently, we obtain the results of Tables 1 and 2.

- *F N–R method without Aitken's method*

Table 1. Results obtained using the iterative method (50).

	α	x_n	$\lVert x_n - x_{n-1}\rVert_2$	$\lVert f(x_n)\rVert_2$	n
1	1.00161	0.00007156	6.94600E-05	4.31004E-07	46

- *F N–R method with Aitken's method*

Table 2. Results obtained using the iterative method (51).

	α	x_n	$\lVert x_n - x_{n-1}\rVert_2$	$\lVert f(x_n)\rVert_2$	n
1	0.87703	−0.00449671 + 1.2464767i	5.57148E-05	7.01925E-05	8
2	0.87821	−0.06554663 + 0.93609376i	9.93578E-05	1.07146E-05	7
3	0.88376	−0.06554664 − 0.93609378i	4.06133E-05	1.74330E-06	8
4	0.91922	−0.90057347 + 0.22444635i	9.90788E-05	2.07375E-05	5
5	0.92610	−0.56043513 + 0.57983003i	8.24879E-06	1.68512E-06	7
6	0.92643	0.59293293 + 0.81897545i	9.31195E-06	9.18616E-06	6
7	0.92659	1.05051127 + 0.38407315i	6.46220E-07	3.79293E-05	7
8	0.92668	1.05051127 − 0.38407314i	1.92354E-07	8.61954E-06	8
9	1.00161	−0.00000009	9.47400E-05	6.81696E-13	2
10	1.08086	0.59293292 − 0.81897547i	8.09817E-05	2.50406E-05	6
11	1.08184	−0.56043512 − 0.57983002i	2.39767E-05	2.50817E-06	7
12	1.11378	−0.00449673 − 1.24647667i	7.61577E-08	6.43359E-05	9
13	1.17623	−0.90057347 − 0.2244463i	9.28255E-05	2.29317E-05	7

Example 2. *Let f be a function, with*

$$f(x) = 88.43x^{16} - 61.92x^{15} + 24.94x^{14} + 95.51x^{13} - 94.75x^{12} + 40.88x^{11} \\ + 65.89x^{10} + 85.7x^9 + 28.55x^8 + 31.37x^7 + 31.13x^6 \\ + 12.48x^5 - 95.28x^4 - 59.44x^3 - 7.31x^2.$$

Then, the initial condition $x_0 = -9.86$ is chosen to use the iterative methods given by (50) and (51). Consequently, we obtain the results of Tables 3 and 4.

- *F N–R method without Aitken's method*

Table 3. Results obtained using the iterative method (50).

	α	x_n	$\lVert x_n - x_{n-1}\rVert_2$	$\lVert f(x_n)\rVert_2$	n
1	1.00393	−1.11795723	1.30000E-07	2.23178E-06	40
2	1.04143	−0.43822992	7.54300E-05	6.21616E-05	52
3	1.05194	−0.16991479	6.50004E-05	1.26695E-05	53
4	1.15095	−0.35589097 + 0.80514169i	1.51327E-07	5.71986E-05	67

- *F N–R method with Aitken's method*

Table 4. Results obtained using the iterative method (51).

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.87132	−0.35589093 - 0.80514174i	7.86160E-05	2.32788E-05	7
2	0.87264	0.48722967− 0.92230783i	1.13741E-06	4.48612E-06	10
3	0.87366	0.28979196 −1.12272312i	4.79113E-06	1.62050E-05	11
4	0.89238	-0.35589092 + 0.80514178i	6.16682E-06	6.68576E-07	8
5	0.89568	0.48722967 + 0.92230782i	3.89880E-05	7.47107E-06	10
6	0.89766	1.0660797 + 0.56313314i	3.00491E-05	5.93475E-05	9
7	1.00393	0.00000008	3.36500E-05	4.67840E-14	2
8	1.01491	0.87919885	4.09653E-05	1.68448E-05	5
9	1.02115	−0.16993135	9.72851E-05	1.39749E-08	3
10	1.04143	1.0660797 −0.56313315i	6.75680E-05	6.95220E-05	5
11	1.05194	−0.43824114	1.88156E-05	2.58740E-06	3
12	1.12328	−0.71363729 − 0.41959459i	3.26455E-06	4.52706E-06	5
13	1.12610	−0.71363727 + 0.41959459i	8.75710E-06	1.99343E-07	8
14	1.13498	−1.11795723	1.23100E-05	2.23178E-06	4
15	1.15095	0.28979195 + 1.12272311i	2.35722E-06	3.11810E-05	10

Example 3. *Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of functions, with*

$$f_k(x) = \sum_{m=1}^{k} \frac{(-1)^{m+1}x^{2m+1}}{(2m+1)\Gamma(2m+2)} \underset{k\to\infty}{\longrightarrow} x - \frac{\pi}{2} + \int_x^{\infty} \frac{\sin(t)}{t}dt.$$

Then, considering the value $k = 50$, the initial condition $x_0 = -17.28$ is chosen to use the iterative methods given by (50) and (51). Consequently, we obtain the results of Tables 5 and 6.

- *F N–R method without Aitken's method*

Table 5. Results obtained using the iterative method (50).

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f_{50}(x_n)\|_2$	n
1	0.70163	−14.94772136 + 6.14653734i	1.30298E-05	9.32400E-05	25
2	0.85274	−8.33609528 − 5.06388182i	1.69580E-05	3.33020E-05	12
3	1.00181	−0.00013924 + 0.00001328i	6.81766E-05	1.52026E-13	25
4	1.15221	8.33610117 + 5.06387543i	2.38905E-05	4.04742E-05	16

- *F N–R method with Aitken's method*

Table 6. Results obtained using the iterative method (51).

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f_{50}(x_n)\|_2$	n
1	0.70006	21.39353648 − 6.837026i	4.12311E-08	1.31841E-08	9
2	0.70021	14.94772196 + 6.14653076i	1.12581E-05	6.28944E-08	8
3	0.70130	21.39353648 + 6.83702599i	3.60555E-08	2.00623E-07	9
4	0.70163	−27.77675536 − 7.34778011i	3.05941E-07	4.08123E-06	8
5	0.72911	−21.39353648 + 6.837026i	1.74642E-07	1.31841E-08	6
6	0.72933	−14.94772196 + 6.14653076i	1.98086E-05	6.28944E-08	5
7	0.72969	−21.39353648 − 6.837026i	2.65981E-05	1.31841E-08	8
8	0.73214	−8.33609941 + 5.06388042i	9.58390E-05	3.99288E-08	5
9	0.80714	8.33609941 + 5.06388043i	8.67685E-05	3.93431E-08	8
10	0.81260	−34.12862021 + 7.7539021i	4.21598E-05	9.52275E-05	8
11	0.85274	−8.33609941 − 5.06388042i	9.72662E-05	3.99288E-08	5
12	0.89041	−14.94772197− 6.14653076i	8.26951E-05	1.22005E-07	7
13	1.00181	−0.0000215 + 0.00002121i	7.36947E-05	1.53039E-15	4
14	1.10820	−27.77675547 + 7.34778007i	5.05482E-05	2.74956E-06	7
15	1.15221	27.77675547 + 7.34778014i	4.12311E-08	2.42256E-06	8
16	1.15395	14.94772196 − 6.14653077i	3.48421E-06	9.26415E-08	7
17	1.15404	8.33609941− 5.06388043i	2.37921E-05	3.93431E-08	8

Example 4. *Let f be a function, with*

$$f(x) = \sin\left(x^2\right),$$

and assuming that

$$f^{(\alpha)}(x) \approx \sum_{k=0}^{40} \frac{(-1)^k \Gamma(4k+3)}{\Gamma(2k+2)\Gamma(4k-\alpha+3)} x^{4k+2-\alpha}.$$

Then, the initial condition $x_0 = 3.29$ *is chosen to use the iterative methods given by* (50) *and* (51)*. Consequently, we obtain the results of Tables 7 and 8.*

- *F N–R method without Aitken's method*

Table 7. Results obtained using the iterative method (50).

	α	x_n	$\lVert x_n - x_{n-1} \rVert_2$	$\lVert f(x_n) \rVert_2$	n
1	0.84175	1.77244862	2.77800E-05	1.85430E-05	9
2	0.88194	3.54491603	3.76700E-05	5.90454E-05	8
3	0.88428	3.06996642	6.84700E-05	8.41408E-05	10
4	0.98182	2.50663097	9.01300E-05	1.35126E-05	3
5	1.09634	−2.50662031	2.44798E-05	3.99287E-05	4
6	1.10015	−4.68946323	2.10983E-05	8.31896E-05	9
7	1.10056	−5.01324797	2.00345E-05	8.60200E-05	9
8	1.10117	−5.60498334	4.21000E-06	8.82942E-05	21
9	1.14221	5.60499912	3.67000E-06	8.85993E-05	7
10	1.14547	5.31735876	7.47000E-06	2.96998E-05	7
11	1.15097	5.01325276	1.00200E-05	3.79931E-05	7
12	1.15908	4.68946617	1.54700E-05	5.56156E-05	7
13	1.17640	4.34160246	1.23700E-05	4.40009E-05	4

- *F N–R method with Aitken's method*

Table 8. Results obtained using the iterative method (51).

	α	x_n	$\lVert x_n - x_{n-1} \rVert_2$	$\lVert f(x_n) \rVert_2$	n
1	0.80229	3.54490771	8.13400E-05	5.80583E-08	2
2	0.84175	3.9633273	3.07800E-05	1.89763E-08	2
3	0.88143	−5.31736155	2.80000E-06	2.88897E-08	3
4	0.88194	−4.34160716	1.38420E-05	3.18978E-06	3
5	0.88247	−3.54490738	2.57466E-05	2.28158E-06	3
6	0.88428	−1.77245385	2.22000E-06	3.20997E-09	3
7	0.88821	−0.00001155	6.94304E-05	1.33402E-10	4
8	0.90399	1.77245385	8.16500E-05	3.20997E-09	2
9	0.92421	3.06998012	8.79300E-05	2.35742E-08	3
10	0.98182	2.50662827	9.91500E-05	2.32164E-08	2
11	1.09634	−2.50662975	8.33108E-05	7.39641E-06	2
12	1.10015	−4.68948099	8.10371E-05	8.33804E-05	3
13	1.10056	−5.01326496	7.11525E-05	8.43304E-05	3
14	1.10117	−5.60499585	6.68979E-05	5.19426E-05	4
15	1.14221	5.60499119	5.32600E-05	2.95920E-07	2
16	1.14547	5.31736154	1.01200E-05	1.35237E-07	2
17	1.15097	5.01325654	5.61300E-05	9.28656E-08	2
18	1.15908	4.68947211	1.28200E-05	9.53393E-08	2
19	1.17640	4.34160754	5.99300E-05	1.09846E-07	2

In all the examples shown, there is a decrease in the number of iterations necessary to converge to the solutions when implementing Aitken's method, which translates into the sequences generated showing an acceleration in their speed of convergence, which was to be expected given the **Proposition** 6. On the other hand, although it is not explicitly mentioned, the implementation of Aitken's method in any iterative method causes changes in the slopes of the lines that cross the x-axis to generate the sequences that converge to the solutions. A consequence of the aforementioned is that, if an iterative method is combined with Aitken's method, and the resulting method converges to a solution ξ given an initial condition x_0, when the original method is implemented with the same initial condition, it does not necessarily converge to the same solution ξ. However, in a fractional iterative method where the initial condition generally remains fixed, the same principle applies but with the order α of the derivatives, a fact that can be seen in the different examples presented.

The fractional iterative methods, such as the fractional Newton–Raphson method, can find multiple zeros of a function using a single initial condition. This partially solves the intrinsic problem of classical iterative methods, which is that, in general, it is necessary to provide N initial conditions to find N zeros of a function. Due to the fractional operators implemented, these methods can be considered non-local parametric iterative methods, so they have two important characteristics: (i) The initial condition does not necessarily need to be close to the searched values due to the non-local nature of fractional operators [17]. (ii) When working in a space of N dimensions, it is necessary to change the initial condition, unlike the classical iterative methods where, in the worst case, it is necessary to vary the N entries of the initial condition until obtaining a suitable value, and it is enough to vary the parameter α of the fractional operators before opting to change the initial condition, until a suitable value is found that allows for generating a sequence that converges to a searched value [5].

The above features make fractional iterative methods an ideal numerical tool for working with nonlinear algebraic equation systems that vary with respect to time-dependent parameters, such as the system obtained by studying the temperatures and efficiencies of a hybrid solar receiver [4,16]. When working in N dimensions with a nonlinear system that evolves due to time-dependent parameters, as a consequence of nonlinearity, the solutions can change their position in space considerably between each time step, so the use of a classical iterative method may require the task of determining a suitable initial condition for each new time step, which may be a complicated task when it is not clear in which region of space a solution is found, and it is necessary to vary all the entries of the initial condition until finding values that are suitable. However, when using a fractional iterative method, it is enough to vary the parameter α of the fractional operators to generate the search for solutions in different regions of space regardless of the number of dimensions [4].

6. Conclusions

In this paper, it is shown that F N–R with Aitken's converges faster than the simple F N–R. In summary, the following results are presented: In **Corollary 1**, an alternative way is obtained to demonstrate when an iterative method has an order of convergence at least linear. Considering **Proposition 1** together with **Proposition 2**, it is proved that Newton's method fulfills a necessary but not sufficient condition to have an order of convergence at least quadratic. In **Proposition 3**, the radical differences that may there exist between the results of the conventional calculus and the fractional calculus when obtaining the derivative of a function are exposed, which is a consequence of dependency of the integer parameter $n(\alpha)$, which generally has the fractional derivative. In **Proposition 4**, it is proved that, under certain conditions, the results when calculating the derivative of a function in the fractional calculus are analogous to those obtained in the conventional calculus. In **Proposition 5**, it is proved that the F N–R method has an order of convergence at least linear, but it follows that it has the ability to gradually change to an order of convergence at least quadratic as the value α approaches the value of one. It also follows that the change in the order of convergence in the F N–R method may be achieved by implementing a function on the order of the fractional derivatives. In **Proposition 6**, it is proved that any succession may accelerate its speed of convergence through the implementation of Aitken's method, with which it follows that it is an ideal alternative to accelerate the speed of convergence of any fractional iterative method that does not have a structure similar to the F N–R method.

Taking into account the results in this paper, although there are surely different alternatives to accelerate the speed of convergence of the fractional iterative methods, take, for example, the strategy of changing the constant order α of the fractional derivative by a function and giving rise to the method (39), Aitken's method is a simple and efficient method to accelerate the speed of convergence of any fractional iterative method, in particular for the F N–R method, due to it presenting an order of convergence at least linear for the case in which the order of the derivative is different from one. Then, in conjunction with the Aitken method, it is concluded that the F N–R method becomes an efficient iterative method to calculate the largest possible number of zeros of a function.

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Appendix A

Appendix A.1 Proof of the Theorem 1

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function, and let $\{\hat{e}_k\}_{k=1}^n$ be the canonical basis of $\mathbb{R}^n$. Considering the following index notation (Einstein notation)

$$\Phi(x) = \sum_{k=1}^{n} [\Phi]_k(x)\hat{e}_k := [\Phi]_k(x)\hat{e}_k = \hat{e}_k[\Phi]_k(x),$$

and using the Taylor series expansion of a vector-valued function in multi-index notation, we obtain two cases:

(i) Case $p \geq 2$:

$$\begin{aligned}\Phi(x_i) =& \Phi(\xi) + \sum_{|\gamma|=1}^{p} \frac{1}{\gamma!}\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i-\xi)^\gamma + \hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right)\\ =& \Phi(\xi) + \sum_{m=1}^{p}\left(\sum_{|\gamma|=m} \frac{1}{\gamma!}\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i-\xi)^\gamma\right) + \hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right),\end{aligned}$$

then

$$\begin{aligned}\|\Phi(x_i)-\Phi(\xi)\| \leq& \sum_{m=1}^{p}\left(\sum_{|\gamma|=m}\frac{1}{\gamma!}\left\|\hat{e}_k\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i-\xi)^\gamma\right\|\right) + \left\|\hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right)\right\|\\ \leq& \sum_{m=1}^{p}\left(\sum_{|\gamma|=m}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\|\right)\|x_i-\xi\|^m + o\left(\|x_i-\xi\|^p\right),\end{aligned}$$

assuming that ξ is a fixed point of Φ and that $\dfrac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma} = 0\ \forall k \geq 1$ and $\forall |\gamma| < p$ is fulfilled, the previous expression implies that

$$\frac{\|\Phi(x_i)-\Phi(\xi)\|}{\|x_i-\xi\|^p} = \frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|^p} \leq \sum_{|\gamma|=p}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\| + \frac{o\left(\|x_i-\xi\|^p\right)}{\|x_i-\xi\|^p},$$

therefore

$$\lim_{i\to\infty}\frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|^p} \leq \sum_{|\gamma|=p}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\|,$$

as a consequence, if the sequence $\{x_i\}_{i=0}^\infty$ generated by (3) converges to ξ, there exists a value $k > 0$ such that

$$\|x_{i+1}-\xi\| \leq \left(\sum_{|\gamma|=p} \frac{1}{\gamma!} \left\| \frac{\partial^{\gamma}[\Phi]_k(\xi)}{\partial x^{\gamma}} \hat{e}_k \right\| \right) \|x_i-\xi\|^p, \quad \forall i \geq k,$$

then Φ is (locally) convergent of (at least) order p.

(ii) Case $p = 1$:

$$\begin{aligned}\Phi(x_i) =& \Phi(\xi) + \sum_{|\gamma|=1} \frac{1}{\gamma!} \hat{e}_k \frac{\partial^{\gamma}[\Phi]_k(\xi)}{\partial x^{\gamma}} (x_i-\xi)^{\gamma} + \hat{e}_k[o]_k \left(\max_{|\gamma|=1} \{(x_i-\xi)^{\gamma}\} \right) \\ =& \Phi(\xi) + \Phi^{(1)}(x_i)(x_i-\xi) + \hat{e}_k[o]_k \left(\max_{|\gamma|=1} \{(x_i-\xi)^{\gamma}\} \right),\end{aligned}$$

then

$$\|\Phi(x_i) - \Phi(\xi)\| \leq \left\| \Phi^{(1)}(\xi) \right\| \|x_i-\xi\| + o(\|x_i-\xi\|),$$

assuming that ξ is a fixed point of Φ, the previous expression implies that

$$\frac{\|\Phi(x_i)-\Phi(\xi)\|}{\|x_i-\xi\|} = \frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|} \leq \left\| \Phi^{(1)}(\xi) \right\| + \frac{o(\|x_i-\xi\|)}{\|x_i-\xi\|},$$

therefore

$$\lim_{i\to\infty} \frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|} \leq \left\| \Phi^{(1)}(\xi) \right\|.$$

As a consequence, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (3) converges to ξ, there exists a value $k > 0$ such that

$$\|x_{i+1}-\xi\| \leq \left\| \Phi^{(1)}(\xi) \right\| \|x_i-\xi\|, \quad \forall i \geq k,$$

considering $m \geq 1$, from the previous inequality, we obtain that

$$\|x_{i+m}-\xi\| \leq \left\| \Phi^{(1)}(\xi) \right\| \|x_{i+m-1}-\xi\| \leq \left\| \Phi^{(1)}(\xi) \right\|^2 \|x_{i+m-2}-\xi\| \leq \cdots \leq \left\| \Phi^{(1)}(\xi) \right\|^m \|x_i-\xi\|,$$

and assuming that $\left\| \Phi^{(1)}(\xi) \right\| < 1$ is fulfilled

$$\lim_{m\to\infty} \|x_{i+m}-\xi\| \leq \lim_{m\to\infty} \left\| \Phi^{(1)}(\xi) \right\|^m \|x_i-\xi\| \to 0,$$

then Φ is (locally) convergent of order (at least) linear.

Appendix A.2 Proof of the Proposition 1

From the **Theorem 1**, we have that an iteration function has an order of convergence (at least) quadratic if it fulfills the following condition:

$$\lim_{x\to\xi} \frac{\partial[\Phi]_k(x)}{\partial[x]_j} = 0, \quad \forall j,k \leq n,$$

which may be written equivalently as follows:

$$\lim_{x\to\xi} \left\| \Phi^{(1)}(x) \right\| = 0. \tag{A1}$$

Then, we may assume that we have a function $f(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with a zero $\xi \in \Omega$, such that all of its first partial derivatives are defined in ξ, and taking the iteration function Φ given by (11), the k-th component of the iteration function may be written as

$$[\Phi]_k(x) = [x]_k - \sum_{j=1}^{n} [A]_{kj}(x)[f]_j(x),$$

then

$$\begin{aligned}\partial_l[\Phi]_k(x) =&\delta_{lk} - \sum_{j=1}^{n} \left([A]_{kj}(x)\partial_l[f]_j(x) + \left(\partial_l[A]_{kj}(x)\right)[f]_j(x)\right)\\ =&\delta_{kl} - \sum_{j=1}^{n} \left([A]_{kj}(x)[f]_{jl}^{(1)}(x) + \left(\partial_l[A]_{kj}(x)\right)[f]_j(x)\right),\end{aligned}$$

where δ_{kl} is the Kronecker delta, which is defined as

$$\delta_{kl} = \delta_{lk} = \left\{ \begin{array}{ll} 1, & \text{si } l = k \\ 0, & \text{si } l \neq k \end{array} \right. .$$

Assuming that condition (A1),

$$\partial_l[\Phi]_k(\xi) = \delta_{kl} - \sum_{j=1}^{n} [A]_{kj}(\xi)[f]_{jl}^{(1)}(\xi) = 0 \quad \Rightarrow \quad \sum_{j=1}^{n} [A]_{kj}(\xi)[f]_{jl}^{(1)}(\xi) = \delta_{kl},\ \forall l,k \leq n$$

is fulfilled; then, the above expression may be written in matrix form as follows:

$$A(\xi)f^{(1)}(\xi) = I_n \quad \Rightarrow \quad A(\xi) = \left(f^{(1)}(\xi)\right)^{-1},$$

where I_n denotes the identity matrix of $n \times n$. Then, any matrix $A(x)$ that fulfills the following condition

$$\lim_{x \to \xi} A(x) = \left(f^{(1)}(\xi)\right)^{-1}$$

guarantees that $\delta > 0$ exists, such that iteration function Φ given by (11), fulfills a necessary (but not sufficient) condition to be (locally) convergent of order (at least) quadratic in $B(\xi;\delta)$.

Appendix A.3 Proof of the Proposition 2

Considering that the form of the function f is not explicitly determined, it is possible to consider two possibilities:

(i) Assuming the function may be written as $f(x) = (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, then

$$f^{(1)}(x) = (x-\xi)^{m-1}\left[(x-\xi)g^{(1)}(x) + mg(x)\right].$$

As a consequence, the iteration function of N–R method takes the following form:

$$\Phi(x) = x - (x-\xi)h(x)g(x),$$

with

$$h(x) = \left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-1},$$

then

$$\Phi^{(1)}(x) = 1 - h(x)\left[(x-\xi)g^{(1)}(x) + g(x)\right] - (x-\xi)h^{(1)}(x)g(x),$$

where

$$h^{(1)}(x) = -\left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-2}\left[(1+m)g^{(1)}(x) + (x-\xi)g^{(2)}(x)\right],$$

therefore

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = |1-h(\xi)g(\xi)| = \left|1-\frac{1}{m}\right| < 1, \tag{A2}$$

and, from the **Theorem 1**, the Newton–Raphson method has an order of convergence at least linear, that is, it fulfills Equation (15) with $p=1$.

(ii) Assuming that $f(x) \neq (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, the first derivative of the iteration function of Newton–Raphson method takes the following form:

$$\Phi^{(1)}(x) = f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(2)}(x)\right],$$

and, if it is fulfilled that $f^{(1)}(\xi) \neq 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = 0, \tag{A3}$$

and, from the **Theorem 1**, the Newton–Raphson method has an order of convergence at least quadratic, that is, it fulfills Equation (15) with $p=2$. On other hand, the second derivative of the iteration function of Newton–Raphson method takes the following form:

$$\Phi^{(2)}(x) = \left(f^{(1)}(x)\right)^{-1} f^{(2)}(x) + f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(3)}(x) - 2\left(f^{(1)}(x)\right)^{-3}\left(f^{(2)}(x)\right)^{2}\right],$$

and if it is fulfilled that $f^{(1)}(\xi) \neq 0$ and $f^{(2)}(\xi) = 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(2)}(x)\right| = 0, \tag{A4}$$

and, from the **Theorem 1**, the Newton–Raphson method has an order of convergence at least cubic, that is, it fulfills Equation (15) with $p=3$. Finally, the third derivative of the iteration function of the Newton–Raphson method takes the following form:

$$\begin{aligned}\Phi^{(3)}(x) =& f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(4)}(x)\right] + 2\left(f^{(1)}(x)\right)^{-1} f^{(3)}(x) - 3\left(f^{(1)}(x)\right)^{-2}\left(f^{(2)}(x)\right)^{2}\\ &+ 6f(x)\left[\left(f^{(1)}(x)\right)^{-4}\left(f^{(2)}(x)\right)^{3} - \left(f^{(1)}(x)\right)^{-3} f^{(2)}(x) f^{(3)}(x)\right],\end{aligned}$$

and, if it is fulfilled that $f^{(1)}(\xi) \neq 0$, $f^{(2)}(\xi) = 0$ and $f^{(3)}(\xi) = 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(2)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(3)}(x)\right| = 0, \tag{A5}$$

and, from the **Theorem 1**, the Newton–Raphson method has an order of convergence at least tetrahedral, that is, it fulfills Equation (15) with $p=4$.

Appendix A.4 Proof of the Proposition 3

The Riemann–Liouville fractional derivative of the function $f(x)$, through the Equation (22), presents two cases:

(i) If $\alpha < 0$, then :

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1}(t-c)^\mu dt, $$

taking the change of variable $t = c + (x-c)u$ in the previous expression

$$ {}_aD_x^\alpha f(x) = \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \int_{\frac{a-c}{x-c}}^1 (1-u)^{-\alpha-1}u^\mu du, $$

the above result may be rewritten in terms of the Beta function and the incomplete Beta function as follows:

$$ \begin{aligned} {}_aD_x^\alpha f(x) =& \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \left(B(\mu+1,-\alpha) - B_{\frac{a-c}{x-c}}(\mu+1,-\alpha) \right) \\ =& B(\mu+1,-\alpha) \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \left(1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,-\alpha)}{B(\mu+1,-\alpha)} \right), \end{aligned} $$

and, considering (25), we obtain that

$$ {}_aD_x^\alpha (x-c)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (x-c)^{\mu-\alpha} G_{-\alpha}\left(\frac{a-c}{x-c}, \mu+1 \right). \tag{A6} $$

(ii) If $\alpha \geq 0$, then:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1}(t-c)^\mu dt, $$

taking the change of variable $t = c + (x-c)u$ in the previous expression

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \int_{\frac{a-c}{x-c}}^1 (1-u)^{n-\alpha-1}u^\mu du \right], $$

the above result may be rewritten in terms of the Beta function and the incomplete Beta function as follows:

$$ \begin{aligned} {}_aD_x^\alpha f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \left(B(\mu+1,n-\alpha) - B_{\frac{a-c}{x-c}}(\mu+1,n-\alpha) \right) \right] \\ =& \frac{B(\mu+1,n-\alpha)}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \left(1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,n-\alpha)}{B(\mu+1,n-\alpha)} \right) \right], \end{aligned} $$

and, considering (25), we obtain that

$$ \begin{aligned} {}_aD_x^\alpha f(x) =& \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha+1)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} G_{n-\alpha}\left(\frac{a-c}{x-c}, \mu+1 \right) \right] \\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha+1)} \sum_{k=0}^n \binom{n}{k} \left(\frac{d^k}{dx^k}(x-c)^{\mu+n-\alpha} \right) G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1 \right), \end{aligned} $$

taking into account that, in the classical calculus,

$$ \frac{d^k}{dx^k}(x-c)^\mu = \frac{\mu!}{(\mu-k)!}(x-c)^{\mu-k} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)}(x-c)^{\mu-k}, $$

therefore

$$ {}_aD_x^\alpha (x-c)^\mu = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)} (x-c)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1\right). \tag{A7} $$

Appendix A.5 Proof of the Proposition 4

To prove the validity of the proposition for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, it is necessary to note that, from the **Proposition 3**, the following limits may be obtained:

$$ {}_aD_x^\alpha (x-a)^\mu = \lim_{c\to a} {}_aD_x^\alpha (x-c)^\mu, $$

$$ \lim_{c\to a} G_\alpha\left(\frac{a-c}{x-c}, m+1\right) = G_\alpha(0, \mu+1) = 1, $$

then, consider two cases:

(i) If $\alpha < 0$, from Equation (A6), we obtain that

$$ \begin{aligned} {}_aD_x^\alpha (x-a)^\mu =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \lim_{c\to a}\left((x-c)^{\mu-\alpha} G_{-\alpha}\left(\frac{a-c}{x-c}, \mu+1\right)\right) \\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (x-a)^{\mu-\alpha} G_{-\alpha}(0, \mu+1) \\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (x-a)^{\mu-\alpha}. \end{aligned} $$

(ii) If $\alpha \geq 0$, from Equation (A7), we obtain that

$$ \begin{aligned} {}_aD_x^\alpha (x-a)^\mu =& \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)} \lim_{c\to a}\left((x-c)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1\right)\right) \\ =& \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)} (x-a)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}(0, \mu+1) \\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (x-a)^{\mu-\alpha}. \end{aligned} $$

Appendix A.6 Proof of the Proposition 6

Suppose for Equation (42) that the k value fulfills the following conditions:

$$ k = k_0 + \delta_i, \quad \lim_{i\to\infty} \delta_i = 0, \quad |k| < 1, $$

then, from Equation (42),

$$ \begin{aligned} x_{i+1} - x_i =& (x_{i+1} - \xi) - (x_i - \xi) \\ =& (k-1)(x_i - \xi), \end{aligned} \tag{A8} $$

analogously

$$ \begin{aligned} x_{i+2} - x_{i+1} =& (k-1)(x_{i+1} - \xi) \\ =& (k-1)(x_{i+1} - x_i) + (k+1)(x_i - \xi) \\ =& \left[(k-1)^2 + (k+1) \right] (x_i - \xi), \end{aligned} $$

whereby

$$\begin{aligned} x_{i+2} - 2x_{i+1} + x_i =& (k-1)^2 (x_i - \xi) \\ =& \left[(k_0 - 1)^2 + \mu_i \right] (x_i - \xi), \end{aligned} \tag{A9}$$

where

$$\lim_{i \to \infty} \mu_i = 0,$$

finally substituting the Equations (A8) and (A9) in Equation (46), we obtain that

$$x_i' - \xi = (x_i - \xi) - \frac{[(k_0 - 1 + \delta_i)(x_i - \xi)]^2}{[(k_0-1)^2 + \mu_i](x_i - \xi)},$$

then

$$\frac{x_i' - \xi}{x_i - \xi} = 1 - \frac{(k_0 - 1 + \delta_i)^2}{(k_0 - 1)^2 + \mu_i},$$

therefore

$$\lim_{i \to \infty} \frac{|x_i' - \xi|}{|x_i - \xi|} = 0, \tag{A10}$$

which shows that, in general, the speed of convergence of the sequence $\{x_i'\}_{i=0}^{\infty}$ is greater than that of the original sequence.

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