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Pseudo-Lucas Functions of Fractional Degree and Applications

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Abstract: In a recent article, the first and second kinds of multivariate Chebyshev polynomials of fractional degree, and the relevant integral repesentations, have been studied. In this article, we introduce the first and second kinds of pseudo-Lucas functions of fractional degree, and we show possible applications of these new functions. For the first kind, we compute the fractional Newton sum rules of any orthogonal polynomial set starting from the entries of the Jacobi matrix. For the second kind, the representation formulas for the fractional powers of a $r \times r$ matrix, already introduced by using the pseudo-Chebyshev functions, are extended to the Lucas case.

Keywords: orthogonal polynomials; recurrence relations; integral representations; generalized lucas functions of fractional degree; matrix roots; fractional newton sum rules

MSC: Primary 33C99; Secondary 15A15; 11B83; 30E20; 65Q30

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1. Introduction

Applications of special functions and polynomials can be found in the solution to every problem in mathematical physics, engineering, statistics, biology, and in general, in applied mathematics. Among the numerous contributions, we want to mention here only a few articles in support of this pleonastic statement [1–4]. An important class of these functions is given by the hypergeometric functions, which unifies, through the introduction of suitable parameters, almost all special functions (see, e.g., [5–10]).

The most widely used in this area are the Chebyshev polynomials and the functions that constitute their extension to the case of the fractional index, which was recently introduced [11,12].

The classical Chebyshev polynomials have been generalized to the multivariate case [13–17]. A simple connection between the multivariate Lucas [18,19] and the Chebyshev polynomials has been shown in [20], and integral representations and generating functions [21] were derived.

Multivariate versions of such polynomials have also proved important. The case of pseudo-Chebyshev functions of fractional degree, generalizing the results in [11,12], have been considered in [22,23].

Multivariate Lucas polynomials of the second kind have proved useful for computing integer-exponent powers of a matrix [24]. In this regard it will not be useless to remember, once again, that every holomorphic function of a matrix, and in particular the exponential, is nothing but the polynomial interpolating the function on the matrix's spectrum and then it is useless to consider expansions in series of the given matrix. However, this definition is also used today, despite it having been known of since the time of Sylvester [25]. Another useful observation in this area is the fact that when writing the powers of a matrix using the second kind of Lucas polynomials, only the invariants of the matrix are used—that is, the elements of the matrix and not the eigenvalues (and sometimes the eigenvectors, as in the case of the exponential). This results in a considerable economy of calculation.

Therefore, the advantage of considering Lucas polynomials consists of reducing the computational cost of such operations, and can be applied also in related problems, such as the solutions of linear dynamical systems [4] and the computation of sums of the powers of the zeros of a set of orthogonal polynomials [26].

Chebyshev multivariate functions of the second kind with a fractional index have recently been used for computing the roots of nonsingular complex matrices [22], and those of the first kind for computing the moments, with fractional exponent, of the density of zeros of a set of orthogonal polynomials [23]

It remains to define the multivariate Lucas polynomials of the first and second kind of fractional index (called for simplicity, pseudo-Lucas functions). This topic is considered in this paper to complement the results achieved previously.

In particular, since the computation of roots of a non-singular complex matrix by using the second kind of pseudo-Lucas functions is rather equivalent to that recently proposed exploiting the second kind of pseudo-Chebyshev functions [22,23,27], we limit ourselves to recall the the case of the square root a 3×3 matrix, but the obtained formulas can be extended to more general roots, with the due cautions indicated in [23], in relation to the multiplicities.

The first kind of pseudo-Lucas functions are applied, in what follows, to derive the fractional Newton sum rules for a general set of orthogonal polynomials, defined by a three-term recurrence relation (through its Jacobi matrix), extending the equations proven in [26,28–30].

2. Basic Definitions

We consider the $r \times r$, matrix $\mathcal{A} = (a_{ij})$, , whose invariants are

$$u_1 := tr\,\mathcal{A}, \quad u_2 := \sum_{i<j}^{1,r} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \quad \cdots, \quad u_r := det\,\mathcal{A}. \tag{1}$$

Using, for shortness, $\mathbf{u} := (u_1, u_2, \ldots, u_r)$, we denote by $P(\lambda) = P(\mathbf{u}; \lambda)$, its *characteristic polynomial*, so that the characteristic equation is:

$$P(\mathbf{u};\lambda) := det(\lambda\mathcal{I} - \mathcal{A}) = \lambda^r - u_1\lambda^{r-1} + u_2\lambda^{r-2} + \ldots + (-1)^r u_r = 0. \tag{2}$$

2.1. Recalling the Multivariate Lucas Polynomials

The $F_{k,n}(\mathbf{u})$ functions are the basis of the r-dimensional vectorial space of solutions of the homogeneous linear bilateral recurrence relation with constant coefficients u_k $(k = 1, 2, \ldots, r)$, (with $u_r \neq 0$):

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r}, \qquad (n \in \mathbf{Z}), \tag{3}$$

whose characteristic equation coincides with (2).

All the properties of these functions and their applications can be found in [19,20].

The bilateral sequence $\{F_{1,n}(\mathbf{u})\}_{n \in \mathbf{Z}}$ was called, by É. Lucas [18], the *fundamental solution* of the recursion (3) (in French "fonction fondamentale").
It satisfies the initial conditions:

$$F_{1,-1} = 0, \; F_{1,0} = 0, \; \ldots, \; F_{1,r-3} = 0, \; F_{1,r-2} = 1. \tag{4}$$

In what follows, we write:

$$\Phi_n(\mathbf{u}) := F_{1,n}(\mathbf{u}) = F_{1,n}(u_1, u_2, \ldots, u_r), \; (n \in \mathbf{Z}), \tag{5}$$

for denoting the multivariate second kind of Lucas polynomials, (short form—MSK Lucas polynomials).

For the integer powers of the matrix $\mathcal{A}$, in [20,24] the representation formula was shown:

$$\mathcal{A}^n = F_{1,n-1}(\mathbf{u})\mathcal{A}^{r-1} + F_{2,n-1}(\mathbf{u})\mathcal{A}^{r-2} + \cdots + F_{r,n-1}(\mathbf{u})\mathcal{I}\,. \tag{6}$$

Since all the $F_{k,n}$ can be expressed in terms of the $F_{1,n}$ polynomials, the polynomials $\Phi_n(\mathbf{u})$, defined by putting

$$\Phi_n(\mathbf{u}) := F_{1,n}(\mathbf{u}) = F_{1,n}(u_1, u_2, \ldots, u_r)\,,\ (n \in \mathbf{Z})\,, \tag{7}$$

appear as the natural tool for representing the integer powers of $\mathcal{A}$, [20].

Another basic solution of the recursion (3) is the sum of integer powers of the roots of the characteristic Equation (2). This solution is denoted by $\{\Psi(\mathbf{u})\}_{n\in\mathbf{Z}}$, and is called the primordial solution (in French "fonction primordiale" by É. Lucas [18]):

$$\Psi_n(\mathbf{u}) := \Psi_n(u_1, u_2, \ldots, u_r)\,,\ (n \in \mathbf{Z})\,. \tag{8}$$

We use for them the acronym MFK Lucas polynomials.

These polynomials are computed by using the initial conditions derived from the Newton formulas, [20]. More precisely, denoting by $\lambda_1, \lambda_2, \ldots, \lambda_r$ the roots of the characteristic Equation (2), we put:

$$\begin{cases} \Psi_{r-1} = u_1 = \displaystyle\sum_{h=1}^{r} \lambda_h \\ \Psi_r = u_1\Psi_{r-1} - 2u_2 = u_1^2 - 2u_2 = \displaystyle\sum_{h=1}^{r} \lambda_h^2 \\ \ldots \\ \Psi_{2r-2} = u_1\Psi_{2r-3} - u_2\Psi_{2r-4} + \cdots + (-1)^{r-2}u_{r-1}\Psi_{r-1} + (-1)^{r-1}ru_r = \displaystyle\sum_{h=1}^{r} \lambda_h^r\,, \end{cases} \tag{9}$$

For $n > 2r - 2$ we get:

$$\Psi_n = u_1\Psi_{n-1} - u_2\Psi_{n-2} + \cdots + (-1)^{r-1}u_r\Psi_{n-r} = \sum_{h=1}^{r} \lambda_h^{n-r+2} \tag{10}$$

We have, in particular:

$$\Psi_{r-2} = r = \sum_{h=1}^{r} \lambda_h^0\,.$$

The choice of the initial value of the indices is done in such a way that in the particular case $r = 2$, and by having $u_1 = 2x$ and $u_2 = 1$, the classical univariate Chebyshev polynomials, with the same indexes, are recovered.

When $r > 2$, as it was remarked in [20], the MSK and MFK Lucas polynomials are related, in a very simple way, to the multivariate Chebyshev polynomials, introduced by R. Lidl and C. Wells [16], R. Lidl [15], and M. Bruschi and P.E. Ricci [21], and studied by K.B. Dunn and R. Lidl [14], and R.J. Beerends [13]).

Integral Representations

An integral form of the SKM and FKM Lucas polynomials has been proven in [22,23].

Theorem 1. *The SKM Lucas polynomials are represented by the integral formula:*

$$\Phi_n(\mathbf{u}) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\lambda^{n+1}}{P(\lambda)}\, d\lambda\,. \tag{11}$$

Theorem 2. *The FKM Lucas polynomials are represented by the integral formula:*

$$\Psi_n(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{n-r+2}\,P'(\lambda)}{P(\lambda)}\,d\lambda\,. \tag{12}$$

The integral representations (11) and (12) have been used in [22,23] for defining the multivariate pseudo-Chebyshev functions of fractional order.

3. The Multivariate Pseudo-Lucas Functions

Using the integral representations (11) and (12), it is possible to define the second (first) kind of multivariate Lucas functions for rational values of their indexes, which will be denoted with the acronyms SKMP-L functions and FKMP-L functions. In fact we can say by definition:

Definition 1. *The SKMP-L functions are defined by the integral representation:*

$$\Phi_{\frac{p}{q}}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}+1}}{P(\lambda)}\,d\lambda\,. \tag{13}$$

The FKMP-L functions are defined by the integral representation:

$$\Psi_{\frac{p}{q}}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{\frac{p}{q}-r+2}\,P'(\lambda)}{P(\lambda)}\,d\lambda\,. \tag{14}$$

Note that the computation of the integrals in Equations (11)–(14) can be obtained without knowing the roots $\lambda_1, \lambda_2, \dots, \lambda_r$, using the Gershgorin circle theorem and choosing as γ a circle, centered at the origin, surrounding all the eigenvalues.

3.1. Applications of the SKMP-L Functions

The SKMP-L functions can be applied in computing matrix roots, (and in particular, square roots), essentially with the same method used in [22,23], where the second kind of pseudo-Chebyshev (shortly SKMP-C) functions have been exploited. (See also [27], where the multiplicity problem of the roots was analyzed).

Here we show, for simplicity, only the case of the square root of a 3×3 non-singular complex matrix $\mathcal{A}$, applying the SKMP-L functions to the particular case considered in Section 2.1.

Using definition (13), assuming $p/q = 1/2$, Equation (6) becomes:

$$\mathcal{A}^{1/2} = \Phi_{-1/2}(\mathbf{u})\mathcal{A}^2 + [-u_2\Phi_{-3/2}(\mathbf{u}) + u_3\Phi_{-5/2}(\mathbf{u})]\mathcal{A} + u_3\Phi_{-3/2}(\mathbf{u})\mathcal{I}\,, \tag{15}$$

where

$$\Phi_{-1/2}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{1/2}}{P(\lambda)}\,d\lambda\,, \quad \Phi_{-3/2}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{-1/2}}{P(\lambda)}\,d\lambda\,,$$

$$\Phi_{-5/2}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_\gamma \frac{\lambda^{-3/2}}{P(\lambda)}\,d\lambda\,.$$

The above equation can be obviously generalized to an $r \times r$ matrix and to a general fractional exponent, according to the results proven in [22,23] using the SKMP-C functions.

3.2. Applications of the FKMP-L Functions

In [22,23] the SKMP-L functions have been applied in computing matrix roots (in particular, square roots) (see also [27], where the multiplicity problem of the roots was analyzed).

In this Section we consider possible applications of the FKMP-L functions. As a matter of fact, in an old article (see [31] and the references therein), the FKM Lucas polynomials, according to Equation (12), have been used in order to represent the moments of the density of zeros of an orthogonal polynomial set, defined by a three term recurrence relation. The definition of the nth moment of the density of zeros of a polynomial of degree r

$$\mu_n^{(r)} := \frac{1}{r} \sum_{k=1}^{r} \lambda_k^n \,,$$

can be extended to the fractional degree by the following one, which could be useful in applied mathematics and statistics:

Definition 2. *The fractional moments of the density of zeros of the polynomial $P(\lambda)$, whose zeros are given by $\lambda_1, \lambda_2, \ldots, \lambda_r$, can be defined as*

$$\mu_{\frac{p}{q}}^{(r)} := \frac{1}{r} \sum_{k=1}^{r} \lambda_k^{\frac{p}{q}} \,, \tag{16}$$

and are represented as

$$\mu_{\frac{p}{q}}^{(r)} = \frac{1}{r} \Psi_{\frac{p}{q}+r-2}(\mathbf{u}) = \frac{1}{2\,\pi\,\mathrm{i}} \oint_{\gamma} \frac{\lambda^{\frac{p}{q}} \, P'(\lambda)}{r\,P(\lambda)} \, d\lambda = \frac{1}{r} \left[\lambda_1^{\frac{p}{q}} + \lambda_2^{\frac{p}{q}} + \cdots + \lambda_r^{\frac{p}{q}} \right]. \tag{17}$$

4. A Recursion for the Coefficients of Orthogonal Polynomial Sets

We consider a polynomial set defined through a three-term recurrence relation:

$$\begin{cases} P_{-1} = 0, \quad P_0 = 1\,, \\ P_n(x) = (x - \alpha_n)\, P_{n-1}(x) - \beta_{n-1}^2 \, P_{n-2}(x)\,, \quad (n \geq 1)\,, \end{cases} \tag{18}$$

with real parameters α_n and β_n, and $\beta_{n-1} \neq 0$. According to a theorem by Favard (see e.g., [32]), the corresponding polynomials belong to a set orthogonal with respect to a suitable measure.

The parameters are usually arranged in the tridiagonal symmetric Jacobi matrix J_n:

$$J_n = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & \ldots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & \beta_{n-1} & \alpha_n \end{pmatrix}, \tag{19}$$

characterizing the considered set of orthogonal polynomials. In several articles the Newton sum rules of such polynomials have been determined, without computing the relevant zeros, in different frameworks. This was done starting from the Jacobi matrix in [31]; from the ordinary differential equation in [28]; from the three term recurrence relation in [30]. Additionally, the case of associated and co-recursive polynomials has been considered in [29].

In what follows, in order to limit our approach, we will only consider the case of polynomials generated by the recurrence (18).

In analogy to the preceding formulas, we put: $\mathbf{u}_{(n)} := (u_{(1,n)}, u_{(2,n)} \ldots, u_{(n,n)})$, and

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \, u_{(k,n)} \, x^{n-k} \,, \qquad (n = 1, 2, \ldots)\,, \tag{20}$$

and denote by

$$x_{1,n},\, x_{2,n},\, \dots,\, x_{n,n}\,, \tag{21}$$

the zeros of $P_n(x)$, and by

$$y_h(n) = \sum_{k=1}^{n} x_{k,n}^h\,, \qquad (h = 1, 2, \dots)\,, \tag{22}$$

the Newton sum rules corresponding to the zeros (21).

In [30] the following results were proven:

Theorem 3. *For the coefficients $u_{(k,n)}$ appearing in Equation (20), the following recursion holds:*

$$u_{(k,n)} = u_{(k,n-1)} + \alpha_n\, u_{(k-1,n-1)} - \beta_{n-1}^2\, u_{(k-2,n-2)}\,, \tag{23}$$

$(k = 1, 2, \dots, n;\quad n = 1, 2, \dots)$, *where, by definition,*

$$u_{(n,n-1)} = u_{(-1,n-1)} = u_{(-1,n-2)} = u_{(-2,n-2)} = 0\,.$$

According to the Newton–Girard formulas, the starting values of the recursion in Equation (23) are given by:

$$\begin{aligned}
&u_{(1,n)} = y_1(n)\,,\\
&u_{(2,n)} = \Big[u_{(1,n)}\, y_1(n) - y_2(n)\Big]/2 = \big[y_1^2(n) - y_2(n)\big]/2\,,\\
&u_{(3,n)} = \Big[-u_{(1,n)}\, y_2(n) + u_{(2,n)}\, y_1(n) + y_3(n)\Big]/3 = \big[y_1^2(n) - y_2(n)\big]/6\,,\\
&\dots\\
&u_{(n,n)} = \Big[(-1)^n u_{(1,n)}\, y_{n-1}(n) + (-1)^{n-1} u_{(2,n)}\, y_{n-2}(n) + \dots\\
&\qquad\qquad\qquad + u_{(n-1,n)}\, y_1(n) + (-1)^{n-1} y_n(n)\Big]/n\,.
\end{aligned} \tag{24}$$

Theorem 4. *The coefficients of $P_n(x)$ are recursively linked to the entries of the Jacobi matrix (19) by Equation (23), and the relevant Newton sum rules are represented as*

$$y_h(n) = \sum_{k=1}^{n} x_{k,n}^h = \Psi_{h+n-2}\Big(\mathbf{u}_{(n)}\Big)\,, \quad (h = 1, 2, \dots)\,. \tag{25}$$

5. The Case of Fractional Moments

Comparing Equations (9)–(12) with (14), the following result immediately follows:

Theorem 5. *For any fixed rational numbers p/q, $q \neq 0$, and for any polynomial degree n, the rational moments of the density of zeros of the polynomial $P_n(x)$ defined in (20) can be computed as*

$$\begin{aligned}
\mu_{\frac{p}{q}}^{(n)} &:= \frac{1}{n} \sum_{k=1}^{n} x_{k,n}^{\frac{p}{q}}\\
= \frac{y_{\frac{p}{q}}(n)}{n} &= \frac{1}{n}\Psi_{\frac{p}{q}+n-2}(\mathbf{u}_{(n)}) = \frac{1}{2\,n\,\pi\,\mathrm{i}} \oint_\gamma \frac{\xi^{\frac{p}{q}}\, P_n'(\xi)}{P_n(\xi)}\, d\xi\,.
\end{aligned} \tag{26}$$

Using Equation (26), the computation of rational moments is performed without knowing the zeros of $P_n(x)$, since it is sufficient to assume as γ a circle, centered at the origin, surrounding all the zeros of $P_n(x)$. The value of the first member follows from the computation of the contour integral.

Remark 1. *Recalling Cauchy's bounds for the roots of polynomials [33,34], it immediately follows that the radius of the circle γ could be chosen as*

$$2+\max\left\{\frac{|u_{(n-1,n)}|}{|u_{(n,n)}|},\frac{|u_{(n-2,n)}|}{|u_{(n,n)}|},\cdots,\frac{1}{|u_{(n,n)}|}\right\}. \tag{27}$$

Remark 2. *Note that the fractional powers have several determinations, depending on the denominator q, but in the examined case, dealing with orthogonal polynomials, all the zeros are real, so in the preceding equations the roots are supposed to be the arithmetical values, and in the contour integral the powers of ξ are intended to be the principal values.*

6. Numerical Computations

In this section a few checks of the preceding Equation (26) have been performed by the second author, considering particular Jacobi polynomials as the second kind of Chebyshev polynomial $U_4(z)$, the Legendre polynomial $P_5(z)$, the third kind of Chebyshev polynomial $V_4(z)$, and the fourth kind of Chebyshev polynomial $W_5(z)$ (see [35]).

The numerical computations confirm the results of Section 6, unless an error is smaller than the machine error.

6.1. Second Kind of Chebyshev Polynomial $U_4(z)$

We have: $U_4(z) = 16z^4 - 12z^2 + 1$. By choosing $p/q = 2/3$, and denoting by ξ_k, $(k = 1,2,3,4)$ the zeros of $U_4(z)$, we find:

$$\mu_{\frac{2}{3}}^{(4)} = \frac{1}{8\pi \mathrm{i}}\oint_\gamma \frac{\xi^{\frac{2}{3}}(64\xi^3-24\xi)}{16\xi^4-12\xi^2+1}\,d\xi = \frac{1}{4}\sum_{k=1}^{4}\mathrm{Res}\left[\frac{\xi^{\frac{2}{3}}(4\xi^3-\frac{2}{3}\xi)}{\xi^4-\frac{3}{4}\xi+\frac{1}{16}},\xi_k\right]$$
$$= 0.662656547986741 = \frac{1}{4}\sum_{k=1}^{4}\xi_k^{\frac{2}{3}}\,.$$

6.2. Legendre Polynomial $P_5(z)$

We have: $P_5(z) = \frac{1}{8}\left(63z^5 - 70z^3 + 15z\right)$. By choosing $p/q = 2/5$, and denoting by ξ_k, $(k = 1,2,3,4,5)$ the zeros of $P_5(z)$, we find:

$$\mu_{\frac{2}{5}}^{(5)} = \frac{1}{10\pi \mathrm{i}}\oint_\gamma \frac{\xi^{\frac{2}{5}}\left(5\xi^4-\frac{10}{3}\xi^2+\frac{5}{21}\xi\right)}{\xi^5-\frac{10}{9}\xi^3+\frac{5}{21}\xi}\,d\xi = \frac{1}{5}\sum_{k=1}^{5}\mathrm{Res}\left[\frac{\xi^{\frac{2}{5}}\left(5\xi^4-\frac{10}{3}\xi^2+\frac{5}{21}\xi\right)}{\xi^5-\frac{10}{9}\xi^3+\frac{5}{21}\xi},\xi_k\right]$$
$$= 0.696809509722981 = \frac{1}{5}\sum_{k=1}^{5}\xi_k^{\frac{2}{5}}\,.$$

6.3. Third Kind of Chebyshev Polynomial $V_4(z)$

We have: $V_4(z) = P_4^{(1/2,-1/2)}(z) = \frac{35}{128}\left(16^4 + 8z^3 - 12z^2 - 4z + 1\right)$. By choosing $p/q = 2/3$, and denoting by ξ_k, $(k = 1,2,3,4)$ the zeros of $V_4(z)$, we find:

$$\mu_{\frac{2}{3}}^{(4)} = \frac{1}{8\pi \mathrm{i}}\oint_\gamma \frac{\xi^{\frac{2}{3}}(4\xi^3+\frac{3}{2}\xi^2-\frac{3}{2}\xi-\frac{1}{4})}{\xi^4+\frac{1}{2}\xi^3-\frac{3}{4}\xi^2-\frac{1}{4}\xi+\frac{1}{16}}\,d\xi$$
$$= \frac{1}{4}\sum_{k=1}^{4}\mathrm{Res}\left[\frac{\xi^{\frac{2}{3}}\left(4\xi^3+\frac{3}{2}\xi^2-\frac{3}{2}\xi-\frac{1}{4}\right)}{\xi^4+\frac{1}{2}\xi^3-\frac{3}{4}\xi^2-\frac{1}{4}\xi+\frac{1}{16}},\xi_k\right] = 0.6844515633130234 = \frac{1}{4}\sum_{k=1}^{4}\xi_k^{\frac{2}{3}}\,.$$

6.4. Fourth Kind Chebyshev Polynomial $W_5(z)$

We have: $W_5(z) = P_5^{(-1/2,1/2)}(z) = \frac{63}{256}\left(32z^5 - 16z^4 - 32z^3 + 12z^2 + 6z - 1\right)$. By choosing $p/q = 6/5$, and denoting by ξ_k, $(k = 1,2,3,4,5)$ the zeros of $W_5(z)$, we find:

$$\mu_{\frac{6}{5}}^{(5)} = \frac{1}{10\pi i} \oint_\gamma \frac{\xi^{\frac{6}{5}}\left(5\xi^4 - 2\xi^3 - 3\xi^2 + \frac{3}{4}\xi + \frac{3}{16}\right)}{\xi^5 - \frac{1}{2}\xi^4 - \xi^3 + \frac{3}{8}\xi^2 + \frac{3}{16}\xi - \frac{1}{32}}\, d\xi$$

$$= \frac{1}{5}\sum_{k=1}^{5} \mathrm{Res}\left[\frac{\xi^{\frac{6}{5}}\left(5\xi^4 - 2\xi^3 - 3\xi^2 + \frac{3}{4}\xi + \frac{3}{16}\right)}{\xi^5 - \frac{1}{2}\xi^4 - \xi^3 + \frac{3}{8}\xi^2 + \frac{3}{16}\xi - \frac{1}{32}}, \xi_k\right] = 0.5621593212493105$$

$$= \frac{1}{5}\sum_{k=1}^{5} \xi_k^{\frac{6}{5}}\,.$$

7. Conclusions

By using the integral representation formulas for the multivariate second (first) kind of Lucas polynomials, it was possible to define the second (first) kind of pseudo-Lucas functions of fractional order. An application of the second kind of pseudo-Lucas functions have been touched on in the problem of computing matrix roots, according to the same technique exploited by using the second kind of pseudo-Chebyshev functions. Furthermore, recalling preceding papers on the sum rules for zeros of polynomials, we have used the first kind of pseudo-Lucas functions of fractional degree in order to compute the fractional moments of the density of zeros for an orthogonal polynomial set, starting from its Jacobi matrix and avoiding the knowledge of zeros.

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