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Note on the Equivalence of Special Norms on the Lebesgue Space

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Abstract: In this paper, we consider a norm based on the infinitesimal generator of the shift semigroup in a direction. The relevance of such a focus is guaranteed by an abstract representation of a uniformly elliptic operator by means of a composition of the corresponding infinitesimal generator. The main result of the paper is a theorem establishing equivalence of norms in functional spaces. Even without mentioning the relevance of this result for the constructed theory, we claim it deserves to be considered itself.

Keywords: equivalence of norms; compact embedding of spaces; infinitesimal generator; m-accretive operator; uniformly elliptic operator

MSC: 46E30; 46E40; 47B28; 20M05

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1. Introduction

In theoretical mathematics, such abstract notions as norm equivalence, embedding of spaces, and compact embedding of spaces have a significance comparable with the one of various natural phenomena in physics. The era of these abstract notions became after the time when Russian mathematician A.A. Dezin in 1953 had published the paper [1]. Ten years later, S.L. Sobolev appeared on the scientific scene, and his paper [2] became well-known all over the world and enormously increased the interest of the global scientific society to the topic. In 1971, R.A. Adams obtained a similar result comparable in its significance for unbounded domains [3]. Nowadays, many authors pay attention to the topic and many special results [4–6] as well as ones that are in the framework of the newly created theories of mathematics [7–11] have been recognized as relevant. Thus, inspired by the paper [12] written by I.A. Kiprianov, we want to make a modest contribution to the theory. However, not only has the popularity and beauty of classical mathematics have motivated us to write this, but also some of the facts that inspired us lie in the fractional calculus theory. Basically, an event in which a differential operator with a fractional derivative in final terms underwent a careful study [13,14] has played an important role in our research. It is remarkable that various approaches exist to study the operator and one of them is based on an opportunity to represent it in a sum. Here, we should note that this method works if one of the summands is selfadjoint or normal. Thus, in the case corresponding to a selfadjoint summand, we can partially solve the problem having applied the results of the perturbation theory, within the framework of which the following papers are well-known [15–20]. In other cases, we can use methods of [21], which are relevant if we deal with non-selfadjoint operators. In the paper [22], we explore a special operator class for which a number of spectral theory theorems can be applied. Furthermore, we construct an abstract model of a differential operator in terms of m-accretive operators and call it an m-accretive operator transform, and we find such conditions that, being imposed, guarantee that the transform belongs to the class. One of them is a compact embedding of a space generated by an m-accretive operator (infinitesimal generator) into the initial Hilbert space. Note that, in the case corresponding to the second order operator with the Kiprianov operator in final terms, we obtain the embedding mentioned above in the one-dimensional case only. In this paper, we try to reveal this problem, and the main result is a theorem establishing equivalence of norms in function spaces which provides a compact embedding of a space generated by the infinitesimal generator of the shift semigroup in a direction into the Lebsgue space. We claim that the result is novel, and we should also note the fact that gives us a rather abstract view on the issue, for we pursue exceptionally theoretical goals and the majority of them involve how to describe the uniformly elliptic and fractional integro-differential operators considered in [22] in terms of the infinitesimal generator of the shift semigroup in a direction. As for relevance that is more fundamental than applied, as it often occurs with such kind of results, we should turn to the series of papers by I.A. Kiprianov devoted to an alternative branch of the fractional calculus theory [12,23,24]. The author introduced a directional fractional derivative later represented in [21] as a fractional power of the shift semigroup in a direction. Modern results also exist; for instance, in the paper [25], the approach based on directional fractional integro-differentiation was implemented. The progress in this area of research brings us to the paper [22], the main result of which, by virtue of the results obtained in this paper, can be reformulated in terms of the infinitesimal generator of the shift semigroup in a direction. Eventually, we may say that an opportunity to apply spectral theorems [22] in the natural way becomes relevant not only due to the application part, but a better comprehension of the mathematical phenomenon.

The paper is organized as follows: In Section 1, a brief historical review as well as some facts that motivated the author to write the paper is presented. In Section 2, some denotations and notions that are used throughout the paper are presented. Section 3 is devoted to the central results of the paper that are rather fundamental; this also contains two subsections, and each of them is devoted to finding connections between the main results and a concrete mathematical concept, the relevant mathematical objects such as a uniformly elliptic operator, and a fractional integro-differential operator are considered.

2. Preliminaries

Let $C, C_i,\ i \in \mathbb{N}_0$ be real constants. We assume that a value of C is positive and can be different in various formulas, but values of C_i are certain. Furthermore, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\tilde{L}$ the *closure* of an operator L. Denote by $\mathrm{D}(L)$, $\mathrm{R}(L)$, $\mathrm{N}(L)$ the *domain*, the *range*, and the *kernel* or *null space* of an operator L, respectively. The deficiency (codimension) of $\mathrm{R}(L)$, dimension of $\mathrm{N}(L)$ are denoted by $\operatorname{def} T,\ \operatorname{nul} T$, respectively. In accordance with the terminology of the monograph [26], the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of an operator L. Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements, and we have a bounded embedding provided by the inequality $\|f\|_{\mathfrak{H}} \leq C_0 \|f\|_{\mathfrak{H}_+}, C_0 > 0,\ f \in \mathfrak{H}_+$. Moreover, any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$. An operator L is called *bounded from below* if the following relation holds $\mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq \gamma_L \|f\|^2_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \gamma_L \in \mathbb{R}$, where γ_L is called a lower bound of L. An operator L is called *accretive* if $\gamma_L = 0$. An operator L is called *strictly accretive* if $\gamma_L > 0$. An operator L is called *m-accretive* if the next relation holds $(L+\zeta)^{-1} \in \mathcal{B}(\mathfrak{H}),\ \|(L+\zeta)^{-1}\| \leq (\mathrm{Re}\zeta)^{-1},\ \mathrm{Re}\zeta > 0$. Assume that $T_t,\ (0 \leq t < \infty)$ is a semigroup of bounded linear operators on $\mathfrak{H}$, by definition put

$$Af = -\lim_{t \to +0} \left(\frac{T_t - I}{t} \right) f,$$

where $\mathrm{D}(A)$ is a set of elements for which the last limit exists in the sense of the norm $\mathfrak{H}$. In accordance with definition ([27,28], p. 1), the operator—A is called the *infinitesimal generator* of the semigroup T_t. Using notations of the paper [23], we assume that Ω is a convex domain, with a sufficient smooth boundary (C^3 class), of the n—dimensional Euclidean space $\mathbb{E}^n$,

P is a fixed point of the boundary $\partial\Omega$, $Q(r,\mathbf{e})$ is an arbitrary point of Ω; we denote by $\mathbf{e}$ a unit vector having a direction from P to Q, denote by $r=|P-Q|$ the Euclidean distance between the points P,Q, and use the shorthand notation $T:=P+\mathbf{e}t,\ t\in\mathbb{R}$. We consider the Lebesgue classes $L_p(\Omega),\ 1\leq p<\infty$ of complex valued functions. For the function $f\in L_p(\Omega)$, we have

$$\int\limits_{\Omega}|f(Q)|^p dQ=\int\limits_{\omega}d\chi\int\limits_{0}^{d(\mathbf{e})}|f(Q)|^p r^{n-1}dr<\infty, \tag{1}$$

where $d\chi$ is an element of a solid angle of the unit sphere surface (the unit sphere belongs to $\mathbb{E}^n$), and ω is a surface of this sphere, $d:=d(\mathbf{e})$ is the length of the segment of the ray going from the point P in the direction $\mathbf{e}$ within the domain Ω. We use a shorthand notation $P\cdot Q=P^iQ_i=\sum_{i=1}^n P_iQ_i$ for the inner product of the points $P=(P_1,P_2,...,P_n)$, $Q=(Q_1,Q_2,...,Q_n)$, which belong to $\mathbb{E}^n$. Denote by $D_i f$ a weak partial derivative of the function f with respect to a coordinate variable with index $1\leq i\leq n$. We assume that all functions have a zero extension outside of $\bar{\Omega}$. Furthermore, unless otherwise stated, we use notations of the papers [23,24,26].

Lemma 1. *Assuming that L is a closed densely defined operator, the following condition holds:*

$$\|(L+t)^{-1}\|_{\mathrm{R}\to\mathfrak{H}}\leq\frac{1}{t},\ t>0, \tag{2}$$

where a notation $\mathrm{R}:=\mathrm{R}(L+t)$ is used. Then, the operator L is m-accretive.

Proof. Using (2), consider

$$\|f\|^2_{\mathfrak{H}}\leq\frac{1}{t^2}\|(L+t)f\|^2_{\mathfrak{H}};\ \|f\|^2_{\mathfrak{H}}\leq\frac{1}{t^2}\Big\{\|Lf\|^2_{\mathfrak{H}}+2t\mathrm{Re}(Lf,f)_{\mathfrak{H}}+t^2\|f\|^2_{\mathfrak{H}}\Big\};$$

$$t^{-1}\|Lf\|^2_{\mathfrak{H}}+2\mathrm{Re}(Lf,f)_{\mathfrak{H}}\geq 0,\ f\in\mathrm{D}(L).$$

Let t be tended to infinity, then we obtain

$$\mathrm{Re}(Lf,f)_{\mathfrak{H}}\geq 0,\ f\in\mathrm{D}(L). \tag{3}$$

It means that the operator L is accretive. Due to (3), we have $\{\lambda\in\mathbb{C}:\ \mathrm{Re}\lambda<0\}\subset\Delta(L)$, where $\Delta(L)=\mathbb{C}\setminus\overline{\Theta(L)}$. Applying Theorem 3.2 ([26], p. 268), we obtain that $L-\lambda$ has a closed range and $\mathrm{nul}(L-\lambda)=0$, $\mathrm{def}(L-\lambda)=\mathrm{const},\ \forall\lambda\in\Delta(L)$. Let $\lambda_0\in\Delta(L),\ \mathrm{Re}\lambda_0<0$. Note that, as a consequence of inequality (3), we have

$$\mathrm{Re}(f,(L-\lambda)f)_{\mathfrak{H}}\geq-\mathrm{Re}\lambda\|f\|^2_{\mathfrak{H}},\ f\in\mathrm{D}(L). \tag{4}$$

Since the operator $L-\lambda_0$ has a closed range, then

$$\mathfrak{H}=\mathrm{R}(L-\lambda_0)\oplus\mathrm{R}(L-\lambda_0)^{\perp}.$$

We remark that the intersection of the sets $\mathrm{D}(L)$ and $\mathrm{R}(L-\lambda_0)^{\perp}$ is zero because, if we assume the contrary, then applying inequality (4), for arbitrary element $f\in\mathrm{D}(L)\cap\mathrm{R}(L-\lambda_0)^{\perp}$, we get

$$-\mathrm{Re}\lambda_0\|f\|^2_{\mathfrak{H}}\leq\mathrm{Re}(f,[L-\lambda_0]f)_{\mathfrak{H}}=0,$$

hence $f=0$. It implies that

$$(f,g)_{\mathfrak{H}}=0,\ \forall f\in\mathrm{R}(L-\lambda_0)^{\perp},\ \forall g\in\mathrm{D}(L).$$

Since $\mathrm{D}(L)$ is a dense set in $\mathfrak{H}$, then $\mathrm{R}(L-\lambda_0)^{\perp}=0$. It implies that $\mathrm{def}(L-\lambda_0)=0$, and, if we take into account Theorem 3.2 ([26], p. 268), then we come to the conclusion that $\mathrm{def}(L-\lambda)=0,\ \forall\lambda\in\Delta(L)$, hence the operator L is m-accretive. The proof is complete. □

Assume that $\Omega\subset\mathbb{E}^n$ is a convex domain, with a sufficient smooth boundary (C^3 class) of the n-dimensional Euclidian space. For the sake of the simplicity, we consider that Ω is bounded. Consider the shift semigroup in a direction acting on $L_2(\Omega)$ and defined as follows $T_t f(Q)=f(Q+\mathbf{e}t)$, where $Q\in\Omega$, $Q=P+\mathbf{e}r$. The following lemma establishes a property of the infinitesimal generator $-A$ of the semigroup T_t.

Lemma 2. *We claim that* $A=\tilde{A}_0$, $\mathrm{N}(A)=0$, *where* A_0 *is a restriction of* A *on the set* $C_0^{\infty}(\Omega)$.

Proof. Let us show that T_t is a strongly continuous semigroup (C_0 semigroup). It can be easily established due to the continuity in average property. Using the Minkowskii inequality, we have

$$\left\{\int\limits_{\Omega}|f(Q+\mathbf{e}t)-f(Q)|^2dQ\right\}^{\frac{1}{2}}\leq\left\{\int\limits_{\Omega}|f(Q+\mathbf{e}t)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}$$

$$+\left\{\int\limits_{\Omega}|f(Q)-f_m(Q)|^2dQ\right\}^{\frac{1}{2}}+\left\{\int\limits_{\Omega}|f_m(Q)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}$$
$$=I_1+I_2+I_3<\varepsilon,$$

where $f\in L_2(\Omega)$, $\{f_n\}_1^{\infty}\subset C_0^{\infty}(\Omega)$; m is chosen so that $I_1,I_2<\varepsilon/3$, and t is chosen so that $I_3<\varepsilon/3$. Thus, there exists such a positive number t_0 that

$$\|T_t f-f\|_{L_2}<\varepsilon,\ t<t_0,$$

for arbitrary small $\varepsilon>0$. Hence, in accordance with the definition, T_t is a C_0 semigroup. Using the assumption that all functions have the zero extension outside $\bar{\Omega}$, we have $\|T_t\|\leq 1$. Hence, we conclude that T_t is a C_0 semigroup of contractions (see [27]). Hence, by virtue of Corollary 3.6 ([27], p. 11), we have

$$\|(\lambda+A)^{-1}\|\leq\frac{1}{\mathrm{Re}\lambda},\ \mathrm{Re}\lambda>0. \tag{5}$$

Inequality (5) implies that A is m-accretive. It is a well-known fact that an infinitesimal generator $-A$ is a closed operator, hence A_0 is closeable. It is not hard to prove that $\tilde{A}_0$ is an m-accretive operator. For this purpose, let us rewrite relation (5) in the form

$$\|(\lambda+\tilde{A}_0)^{-1}\|_{\mathrm{R}\to\mathfrak{H}}\leq\frac{1}{\mathrm{Re}\lambda},\ \mathrm{Re}\lambda>0.$$

Applying Lemma 1, we obtain that $\tilde{A}_0$ is an m-accretive operator. Note that there does not exist an accretive extension of an m-accretive operator (see [26]). On the other hand, it is clear that $\tilde{A}_0\subset A$. Thus, we conclude that $\tilde{A}_0=A$. Consider an operator

$$Bf(Q)=\int_0^r f(P+\mathbf{e}[r-t])dt,\ f\in L_2(\Omega).$$

It is not hard to prove that $B\in\mathcal{B}(L_2)$; applying the generalized Minkowskii inequality, we get

$$\|Bf\|_{L_2}\leq\int\limits_0^{\mathrm{diam}\,\Omega}dt\left(\int\limits_{\Omega}|f(P+\mathbf{e}[r-t])|dQ\right)^{1/2}\leq C\|f\|_{L_2}.$$

Note that the fact $A_0^{-1} \subset B$, follows from the properties of the one-dimensional integral defined on smooth functions. Using Theorem 2 ([29], p. 555), the proved above fact $\tilde{A}_0 = A$, we deduce that $A^{-1} = \widetilde{A_0^{-1}}$, hence $A^{-1} \subset B$. The proof is complete. □

3. Main Results

Consider a linear space $\mathbb{L}_2^n(\Omega) := \{f = (f_1, f_2, \dots, f_n),\, f_i \in L_2(\Omega)\}$, endowed with the inner product

$$(f, g)_{\mathbb{L}_2^n} = \int_\Omega (f, g)_{\mathbb{E}^n} dQ,\ f, g \in \mathbb{L}_2^n(\Omega).$$

It is clear that this pair forms a Hilbert space and let us use the same notation $\mathbb{L}_2^n(\Omega)$ for it. Consider a sesquilinear form

$$t(f,g) := \sum_{i=1}^{n} \int_\Omega (f, \mathbf{e_i})_{\mathbb{E}^n} \overline{(g, \mathbf{e_i})_{\mathbb{E}^n}} dQ,\ f, g \in \mathbb{L}_2^n(\Omega),$$

where $\mathbf{e_i}$ corresponds to $P_i \in \partial\Omega,\ i = 1, 2, ..., n$ (i.e., $Q = P_i + \mathbf{e_i} r$).

Lemma 3. *The points $P_i \in \partial\Omega$, $i = 1, 2, ..., n$ can be chosen so that the form t generates an inner product.*

Proof. It is clear that we should only establish an implication $t(f, f) = 0 \Rightarrow f = 0$. Since $\Omega \in \mathbb{E}^n$, then, without a loss of generality, we can assume that there exists $P_i \in \partial\Omega$, $i = 1, 2, ..., n$, such that

$$\Delta = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} & P_{n2} & ... & P_{nn} \end{vmatrix} \neq 0, \tag{6}$$

where $P_i = (P_{i1}, P_{i2}, ..., P_{in})$. It becomes clear if we remind readers that, in the contrary case, for arbitrary set of points $P_i \in \partial\Omega$, $i = 1, 2, ..., n$, we have

$$P_n = \sum_{k=1}^{n-1} c_k P_k,\ c_k = \text{const}.$$

From what follows this, we can consider Ω at least as a subset of $\mathbb{E}^{n-1}$. Continuing this line of reasoning, we can find such a dimension p that a corresponding $\Delta \neq 0$ and further assume that $\Omega \in \mathbb{E}^p$. Consider a relation

$$\sum_{i=1}^{n} \int_\Omega |(\psi, \mathbf{e_i})_{\mathbb{E}^n}|^2 dQ = 0,\ \psi \in \mathbb{L}_2^n(\Omega).$$

It follows that $(\psi(Q), \mathbf{e_i})_{\mathbb{E}^n} = 0$ a.e. $i = 1, 2, ..., n$. Note that every P_i corresponds to the set $\vartheta_i := \{Q \subset \vartheta_i :\ (\psi(Q), \mathbf{e_i})_{\mathbb{E}^n} \neq 0\}$. Considering $\Omega' = \Omega \setminus \bigcup_{i=1}^{n} \vartheta_i$, it is clear that $\operatorname{mess}\left(\bigcup_{i=1}^{n} \vartheta_i\right) = 0$. Note that, due to this construction created, we can reformulate the above relation obtained in the coordinate form

$$\begin{cases} (P_{11} - Q_1)\psi_1(Q) + (P_{12} - Q_2)\psi_2(Q) + ... + (P_{1n} - Q_n)\psi_n(Q) = 0 \\ (P_{21} - Q_1)\psi_1(Q) + (P_{22} - Q_2)\psi_2(Q) + ... + (P_{2n} - Q_n)\psi_n(Q) = 0 \\ \qquad ... \qquad\qquad ... \qquad\qquad ... \\ (P_{n1} - Q_1)\psi_1(Q) + (P_{n2} - Q_2)\psi_2(Q) + ... + (P_{nn} - Q_n)\psi_n(Q) = 0 \end{cases},$$

where $\psi = (\psi_1, \psi_2, ..., \psi_n)$, $Q = (Q_1, Q_2, ..., Q_n)$, $Q \in \Omega'$. Therefore, if we prove that

$$\Lambda(Q) = \begin{vmatrix} P_{11} - Q_1 & P_{12} - Q_2 & ... & P_{1n} - Q_n \\ P_{21} - Q_1 & P_{22} - Q_2 & ... & P_{2n} - Q_n \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix} \neq 0 \ a.e.,$$

then we obtain $\psi = 0$ a.e. Assume, on the contrary, i.e., that there exists such a set $\mathrm{Y} \subset \Omega$, mess $\mathrm{Y} \neq 0$, that $\Lambda(Q) = 0$, $Q \in \mathrm{Y}$. We have

$$\begin{vmatrix} P_{11} - Q_1 & P_{12} - Q_2 & ... & P_{1n} - Q_n \\ P_{21} - Q_1 & P_{22} - Q_2 & ... & P_{2n} - Q_n \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix} = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} - Q_1 & P_{22} - Q_2 & ... & P_{2n} - Q_n \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix}$$

$$- \begin{vmatrix} Q_1 & Q_2 & ... & Q_n \\ P_{21} - Q_1 & P_{22} - Q_2 & ... & P_{2n} - Q_n \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix} = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix}$$

$$- \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ Q_1 & Q_2 & ... & Q_n \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix} - \begin{vmatrix} Q_1 & Q_2 & ... & Q_n \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} - Q_1 & P_{n2} - Q_2 & ... & P_{nn} - Q_n \end{vmatrix}$$

$$= \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} & P_{n2} & ... & P_{nn} \end{vmatrix} - \sum_{j=1}^{n} \Delta_j = 0,$$

where

$$\Delta_j = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{j-1\,1} & P_{j-1\,2} & ... & P_{j-1\,n} \\ Q_1 & Q_2 & ... & Q_n \\ P_{j+1\,1} & P_{j+1\,2} & ... & P_{j+1\,n} \\ ... & ... & ... & ... \\ P_{n1} & P_{n2} & ... & P_{nn} \end{vmatrix}.$$

Therefore, we have

$$\sum_{j=1}^{n} \Delta_j / \Delta = 1,$$

since $\Delta \neq 0$. Hence, we can treat the above matrix constructions in the way that gives us the following representation:

$$\sum_{j=1}^{n} \alpha_j P_j = Q, \ \sum_{j=1}^{n} \alpha_j = 1, \ \alpha_j = \Delta_j / \Delta.$$

Now, let us prove that Y belongs to a hyperplane in $\mathbb{E}^n$, and we have

$$\begin{vmatrix} P_{11} - Q_1 & P_{12} - Q_2 & ... & P_{1n} - Q_n \\ P_{21} - P_{11} & P_{22} - P_{12} & ... & P_{2n} - P_{1n} \\ ... & ... & ... & ... \\ P_{n1} - P_{n-1\,1} & P_{n2} - P_{n-1\,2} & ... & P_{nn} - P_{n-1\,n} \end{vmatrix} = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} & P_{n2} & ... & P_{nn} \end{vmatrix}$$

$$-\begin{vmatrix} Q_1 & Q_2 & \dots & Q_n \\ P_{21}-P_{11} & P_{22}-P_{12} & \dots & P_{2n}-P_{1n} \\ \dots & \dots & \dots & \dots \\ P_{n1}-P_{n-1\,1} & P_{n2}-P_{n-1\,2} & \dots & P_{nn}-P_{n-1\,n} \end{vmatrix}$$

$$=\begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} - \begin{vmatrix} \sum\limits_{j=1}^{n}\alpha_j P_{j1} & \sum\limits_{j=1}^{n}\alpha_j P_{j2} & \dots & \sum\limits_{j=1}^{n}\alpha_j P_{jn} \\ P_{21}-P_{11} & P_{22}-P_{12} & \dots & P_{2n}-P_{1n} \\ \dots & \dots & \dots & \dots \\ P_{n1}-P_{n-1\,1} & P_{n2}-P_{n-1\,2} & \dots & P_{nn}-P_{n-1\,n} \end{vmatrix}$$

$$=\begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} - \sum_{j=1}^{n}\alpha_j \begin{vmatrix} P_{j1} & P_{j2} & \dots & P_{jn} \\ P_{21}-P_{11} & P_{22}-P_{12} & \dots & P_{2n}-P_{1n} \\ \dots & \dots & \dots & \dots \\ P_{n1}-P_{n-1\,1} & P_{n2}-P_{n-1\,2} & \dots & P_{nn}-P_{n-1\,n} \end{vmatrix}$$

$$=\begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} - \begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} \sum_{j=1}^{n}\alpha_j$$

$$=\begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} - \begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{vmatrix} = 0.$$

Hence, Y belongs to a hyperplane generated by the points P_i, $i = 1, 2, ..., n$. Therefore, messY $= 0$, and we obtain $\psi = 0$ a.e. The proof is complete. □

Consider a pre Hilbert space $\mathbf{L}_2^n(\Omega) := \{f : f \in \mathbb{L}_2^n(\Omega)\}$ endowed with the inner product

$$(f, g)_{\mathbf{L}_2^n} := \sum_{i=1}^{n} \int\limits_{\Omega} (f, \mathbf{e_i})_{\mathbb{E}^n} \overline{(g, \mathbf{e_i})_{\mathbb{E}^n}} dQ,\ f, g \in \mathbb{L}_2^n(\Omega),$$

where $\mathbf{e_i}$ corresponds to $P_i \in \partial\Omega,\ i = 1, 2, ..., n$ condition (6) holds. The following theorem establishes a norm equivalence:

Theorem 1. *The norms $\|\cdot\|_{\mathbb{L}_2^n}$ and $\|\cdot\|_{\mathbf{L}_2^n}$ are equivalent.*

Proof. Consider the space $\mathbb{L}_2^n(\Omega)$ and a functional $\varphi(f) := \|f\|_{\mathbf{L}_2^n}$, $f \in \mathbb{L}_2^n(\Omega)$. Let us prove that $\varphi(f) \geq C$, $f \in$ U, where U $:= \{f \in \mathbb{L}_2^n(\Omega), \|f\|_{\mathbb{L}_2^n} = 1\}$. Assume, on the contrary, then there exists such a sequence $\{\psi_k\}_1^\infty \subset$ U, that $\varphi(\psi_k) \to 0$, $k \to \infty$. Since the sequence $\{\psi_k\}_1^\infty$ is bounded, then we can extract a weekly convergent subsequence $\{\psi_{k_j}\}_1^\infty$ and claim that the week limit ψ of the sequence $\{\psi_{k_j}\}_1^\infty$ belongs to U. Consider a functional

$$\mathcal{L}_g(f) := \sum_{i=1}^{n} \int\limits_{\Omega} (f, \mathbf{e_i})_{\mathbb{E}^n} \overline{(g, \mathbf{e_i})_{\mathbb{E}^n}} dQ,\ f, g \in \mathbb{L}_2^n(\Omega).$$

Due to the following obvious chain of the inequalities

$$|\mathcal{L}_g(f)| \leq \sum_{i=1}^{n} \left\{ \int\limits_{\Omega} |(f, \mathbf{e_i})_{\mathbb{E}^n}|^2 dQ \right\}^{\frac{1}{2}} \left\{ \int\limits_{\Omega} |(g, \mathbf{e_i})_{\mathbb{E}^n}|^2 dQ \right\}^{\frac{1}{2}}$$
$$\leq n \|f\|_{\mathbb{L}_2^n} \|g\|_{\mathbb{L}_2^n},\ f, g \in \mathbb{L}_2^n(\Omega), \tag{7}$$

we see that $\mathcal{L}_g$ is a linear bounded functional on $\mathbb{L}_2^n(\Omega)$. Therefore, by virtue of the weak convergence of the sequence $\{\psi_{k_j}\}$, we have $\mathcal{L}_g(\psi_{k_j}) \to \mathcal{L}_g(\psi)$, $k_j \to \infty$. On the other hand, recall that, since it was supposed that $\varphi(\psi_k) \to 0$, $k \to \infty$, then we have $\varphi(\psi_{k_j}) \to 0$, $k \to \infty$. Hence, applying (3), we conclude that $\mathcal{L}_g(\psi_{k_j}) \to 0$, $k_j \to \infty$. Combining the given above results, we obtain

$$\mathcal{L}_g(\psi) = \sum_{i=1}^{n} \int_{\Omega} (\psi, \mathbf{e_i})_{\mathbb{E}^n} \overline{(g, \mathbf{e_i})}_{\mathbb{E}^n} dQ = 0, \forall g \in \mathbb{L}_2^n(\Omega). \tag{8}$$

Taking into account (8) and using the ordinary properties of Hilbert space, we obtain

$$\sum_{i=1}^{n} \int_{\Omega} |(\psi, \mathbf{e_i})_{\mathbb{E}^n}|^2 dQ = 0.$$

Hence, in accordance with Lemma 3, we get $\psi = 0$ a.e. Notice that, by virtue of this fact, we come to the contradiction with the fact $\|\psi\|_{\mathbb{L}_2^n} = 1$. Hence, the following estimate is true: $\varphi(f) \geq C$, $f \in \mathrm{U}$. Having applied the Cauchy Schwartz inequality to the Euclidian inner product, we can also easily obtain $\varphi(f) \leq \sqrt{n}\|f\|_{\mathbb{L}_2^n}$, $f \in \mathbb{L}_2^n(\Omega)$. Combining the above inequalities, we can rewrite these two estimates as follows: $C_0 \leq \varphi(f) \leq C_1$, $f \in \mathrm{U}$. To make the issue clear, we can rewrite the previous inequality in the form

$$C_0\|f\|_{\mathbb{L}_2^n} \leq \varphi(f) \leq C_1\|f\|_{\mathbb{L}_2^n}, \; f \in \mathbb{L}_2^n(\Omega), \; C_0, C_1 > 0. \tag{9}$$

The proof is complete. □

Consider a pre Hilbert space

$$\widetilde{\mathfrak{H}}_A^n := \{f, g \in C_0^\infty(\Omega), \, (f, g)_{\widetilde{\mathfrak{H}}_A^n} = \sum_{i=1}^{n} (A_i f, A_i g)_{L_2}\},$$

where $-A_i$ is an infinitesimal generator corresponding to the point P_i. Here, we should point out that the form $(\cdot, \cdot)_{\widetilde{\mathfrak{H}}_A^n}$ generates an inner product due to the fact $\mathrm{N}(A_i) = 0$, $i = 1, 2, ..., n$ proved in Lemma 2. Let us denote a corresponding Hilbert space by $\mathfrak{H}_A^n$.

Corollary 1. *The norms $\|\cdot\|_{\mathfrak{H}_A^n}$ and $\|\cdot\|_{H_0^1}$ are equivalent, and we have a bounded compact embedding*

$$\mathfrak{H}_A^n \subset\subset L_2(\Omega).$$

Proof. Let us prove that

$$Af = -(\nabla f, \mathbf{e})_{\mathbb{E}^n}, \; f \in C_0^\infty(\Omega).$$

Using the Lagrange mean value theorem, we have

$$\int_{\Omega} \left| \left(\frac{T_t - I}{t} \right) f(Q) - (\nabla f, \mathbf{e})_{\mathbb{E}^n}(Q) \right|^2 dQ = \int_{\Omega} |(\nabla f, \mathbf{e})_{\mathbb{E}^n}(Q_\xi) - (\nabla f, \mathbf{e})_{\mathbb{E}^n}(Q)|^2 dQ,$$

where $Q_\xi = Q + \mathbf{e}\xi$, $0 < \xi < t$. Since the function $(\nabla f, \mathbf{e})_{\mathbb{E}^n}$ is continuous on $\bar{\Omega}$, then it is uniformly continuous on $\bar{\Omega}$. Thus, for arbitrary $\varepsilon > 0$, a positive number $\delta > 0$ can be chosen so that

$$\int_{\Omega} |(\nabla f, \mathbf{e})_{\mathbb{E}^n}(Q_\xi) - (\nabla f, \mathbf{e})_{\mathbb{E}^n}(Q), |^2 dQ < \varepsilon, \; t < \delta,$$

from what follows the desired result. Taking it into account, we obtain

$$\|Af\|_{L_2} = \left\{ \int\limits_{\Omega} |(\nabla f, \mathbf{e})_{\mathbb{E}^n}|^2 dQ \right\}^{1/2} \leq \left\{ \int\limits_{\Omega} \|\mathbf{e}\|^2_{\mathbb{E}^n} \sum_{i=1}^{n} |D_i f|^2 dQ \right\}^{1/2} = \|f\|_{H_0^1}, \, f \in C_0^\infty(\Omega). \tag{10}$$

Using this estimate, we easily obtain $\|f\|_{\mathfrak{H}_A^n} \leq C\|f\|_{H_0^1}$, $f \in C_0^\infty(\Omega)$. On the other hand, as a particular case of Formula (9), we obtain $C_0\|f\|_{H_0^1} \leq \|f\|_{\mathfrak{H}_A^n}$, $f \in C_0^\infty(\Omega)$. Thus, we can combine the previous inequalities and rewrite them as follows: $C_0\|f\|_{H_0^1} \leq \|f\|_{\mathfrak{H}_A^n} \leq C\|f\|_{H_0^1}$, $f \in C_0^\infty(\Omega)$. Passing to the limit at the left-hand and right-hand side of the last inequality, we get

$$C_0\|f\|_{H_0^1} \leq \|f\|_{\mathfrak{H}_A^n} \leq C\|f\|_{H_0^1}, \, f \in H_0^1(\Omega).$$

Combining the fact $H_0^1(\Omega) \subset\subset L_2(\Omega)$, (Rellich–Kondrashov theorem) with the lower estimate in the previous inequality, we complete the proof. □

Remark 1. *Note that the following relation follows directly from the definition*

$$\mathbb{L}_2^n(\Omega) = \mathbf{L}_2^n(\Omega) = L_2(\Omega), \, n = 1.$$

By virtue of (10), *it is also clear that*

$$\|Af\|_{L_2(\Omega)} = \left\{ \int\limits_{\Omega} |(\nabla f, \mathbf{e})_{\mathbb{E}^n}|^2 dQ \right\}^{1/2} = \|f\|_{H_0^1(\Omega)}, \, f \in C_0^\infty(\Omega), \, n = 1,$$

from what follows that $\mathfrak{H}_A^n(\Omega) = H_0^1(\Omega)$, $n = 1$.

Furthermore, we aim to represent some known operators in terms of the infinitesimal generator of the shift semigroup in a direction and apply the obtained results to the established representations. In this way, we come to natural conditions in terms of the infinitesimal generator of the shift semigroup in a direction that allows us to apply theorems (**A**)–(**C**) [22].

3.1. Uniformly Elliptic Operator in the Divergent Form

Consider a uniformly ecliptic operator

$$-\mathcal{T} := -D_j(a^{ij}D_i\cdot), \, a^{ij}(Q) \in C^2(\bar{\Omega}), \, a^{ij}\xi_i\xi_j \geq \gamma_a|\xi|^2, \, \gamma_a > 0, \, i,j = 1,2,...,n,$$

$$\mathrm{D}(\mathcal{T}) = H^2(\Omega) \cap H_0^1(\Omega).$$

The following theorem gives us a key to apply results of the paper [22] in accordance with which a number of spectral theorems can be applied to the operator $-\mathcal{T}$. Moreover, the conditions established bellow are formulated in terms of the operator A, which reveals a mathematical nature of the operator $-\mathcal{T}$.

Theorem 2. *We claim that*

$$-\mathcal{T} = \frac{1}{n}\sum_{i=1}^{n} A_i^* G_i A_i, \tag{11}$$

and the following relations hold:

$$-\mathrm{Re}(\mathcal{T}f, f)_{L_2} \geq C\|f\|_{\mathfrak{H}_A^n}; \, |(\mathcal{T}f, g)_{L_2}| \leq C\|f\|_{\mathfrak{H}_A^n}\|g\|_{\mathfrak{H}_A^n}, \, f, g \in C_0^\infty(\Omega),$$

where G_i are some operators corresponding to the operators A_i.

Proof. It is easy to prove that

$$\|A_i f\|_{L_2} \leq C\|f\|_{H_0^1},\ f \in H_0^1(\Omega). \tag{12}$$

For this purpose, we should use a representation $A_i f(Q) = -(\nabla f, \mathbf{e_i})_{\mathbb{E}^n}, f \in C_0^\infty(\Omega)$. Applying the Cauchy–Schwarz inequality, we get

$$\|A_i f\|_{L_2} \leq \left\{ \int\limits_\Omega |(\nabla f, \mathbf{e_i})_{\mathbb{E}^n}|^2 dQ \right\}^{1/2} \leq \left\{ \int\limits_\Omega \|\nabla f\|^2_{\mathbb{E}^n} \|\mathbf{e_i}\|^2_{\mathbb{E}^n} dQ \right\}^{1/2} = \|f\|_{H_0^1},\ f \in C_0^\infty(\Omega).$$

Passing to the limit at the left-hand and right-hand side, we obtain (12). Thus, we get $H_0^1(\Omega) \subset \mathrm{D}(A_i)$. Let us find a representation for the operator G_i. Consider the operators

$$B_i f(Q) = \int_0^r f(P_i + \mathbf{e}[r-t])dt,\ f \in L_2(\Omega),\ i = 1, 2, ...n.$$

It is obvious that

$$\int\limits_\Omega A_i(B_i \mathcal{T} f \cdot g) dQ = \int\limits_\Omega A_i B_i \mathcal{T} f \cdot g\, dQ + \int\limits_\Omega B_i \mathcal{T} f \cdot A_i g\, dQ,\ f \in C^2(\bar{\Omega}), g \in C_0^\infty(\Omega). \tag{13}$$

Using the divergence theorem, we get

$$\int\limits_\Omega A_i(B_i \mathcal{T} f \cdot g)\, dQ = \int\limits_S (\mathbf{e_i}, \mathbf{n})_{\mathbb{E}^n} (B_i \mathcal{T} f \cdot g)(\sigma) d\sigma, \tag{14}$$

where S is the surface of Ω. Taking into account that $g(S) = 0$ and combining (13) and (14), we get

$$-\int\limits_\Omega A_i B_i \mathcal{T} f \cdot \bar{g}\, dQ = \int\limits_\Omega B_i \mathcal{T} f \cdot \overline{A_i g}\, dQ,\ f \in C^2(\bar{\Omega}), g \in C_0^\infty(\Omega). \tag{15}$$

Supposing that $f \in H^2(\Omega)$, then there exists a sequence $\{f_n\}_1^\infty \subset C^2(\bar{\Omega})$ such that $f_n \xrightarrow{H^2} f$ (see ([29], p. 346)). Using this fact, it is not hard to prove that $\mathcal{T} f_n \xrightarrow{L_2} \mathcal{T} f$. Therefore, $A_i B_i \mathcal{T} f_n \xrightarrow{L_2} \mathcal{T} f$, since $A_i B_i \mathcal{T} f_n = \mathcal{T} f_n$. It is also clear that $B_i \mathcal{T} f_n \xrightarrow{L_2} B_i \mathcal{T} f$, since B_i is continuous (see proof of Lemma 2). Using these facts, we can extend relation (15) to the following:

$$-\int\limits_\Omega \mathcal{T} f \cdot \bar{g}\, dQ = \int\limits_\Omega B_i \mathcal{T} f\, \overline{A_i g}\, dQ,\ f \in \mathrm{D}(\mathcal{T}),\ g \in C_0^\infty(\Omega). \tag{16}$$

Note that it was previously proved that $A_i^{-1} \subset B_i$ (see the proof of Lemma 2), $H_0^1(\Omega) \subset \mathrm{D}(A_i)$. Hence, $G_i A_i f = B_i \mathcal{T} f,\ f \in \mathrm{D}(\mathcal{T})$, where $G_i := B_i \mathcal{T} B_i$. Using this fact, we can rewrite relation (16) in a form

$$-\int\limits_\Omega \mathcal{T} f \cdot \bar{g}\, dQ = \int\limits_\Omega G_i A_i f\, \overline{A_i g}\, dQ,\ f \in \mathrm{D}(\mathcal{T}),\ g \in C_0^\infty(\Omega). \tag{17}$$

Note that, in accordance with Lemma 2, we have

$$\forall g \in \mathrm{D}(A_i),\ \exists \{g_n\}_1^\infty \subset C_0^\infty(\Omega),\ g_n \underset{A_i}{\longrightarrow} g.$$

Therefore, we can extend relation (17) to the following:

$$-\int\limits_\Omega \mathcal{T} f \cdot \bar{g}\, dQ = \int\limits_\Omega G_i A_i f\, \overline{A_i g}\, dQ,\ f \in \mathrm{D}(\mathcal{T}),\ g \in \mathrm{D}(A_i). \tag{18}$$

Relation (18) indicates that $G_iA_if \in \mathrm{D}(A_i^*)$, and it is clear that $-\mathcal{T} \subset A_i^*G_iA_i$. On the other hand, in accordance with Chapter VI, Theorem 1.2 [30], we have that $-\mathcal{T}$ is a closed operator. Using the divergence theorem, we get

$$-\int\limits_{\Omega} D_j(a^{ij}D_if)\bar{g}dQ = \int\limits_{\Omega} a^{ij}D_if\,\overline{D_jg}dQ,\ f \in C^2(\Omega),\ g \in C_0^\infty(\Omega).$$

Passing to the limit at the left-hand and right-hand side of the last inequality, we can extend it to the following:

$$-\int\limits_{\Omega} D_j(a^{ij}D_if)\,\bar{g}dQ = \int\limits_{\Omega} a^{ij}D_if\,\overline{D_jg}dQ,\ f \in H^2(\Omega),\ g \in H_0^1(\Omega).$$

Therefore, using the uniformly elliptic property of the operator $-\mathcal{T}$, we get

$$-\operatorname{Re}(\mathcal{T}f,f)_{L_2} \geq \gamma_a \int\limits_{\Omega} \sum_{i=1}^{n} |D_if|^2\,dQ = \gamma_a\|f\|^2_{H_0^1},\ f \in \mathrm{D}(\mathcal{T}). \tag{19}$$

Using the Poincaré-Friedrichs inequality, we get $-\operatorname{Re}(\mathcal{T}f,f)_{L_2} \geq C\|f\|^2_{L_2},\ f \in \mathrm{D}(\mathcal{T})$, Applying the Cauchy–Schwarz inequality to the left-hand side, we can easily deduce that the conditions of Lemma 1 are satisfied. Thus, the operator $-\mathcal{T}$ is m-accretive. In particular, it means that there does not exist an accretive extension of the operator $-\mathcal{T}$. Let us prove that $A_i^*G_iA_i$ is accretive, for this purpose, combining (17) and (19), we get $(G_iA_if, A_if)_{L_2} \geq 0, f \in C_0^\infty(\Omega)$. Due to the relation $\tilde{A}_0 = A$, proved in Lemma 2, the previous inequality can be easily extended to $(G_iA_if, A_if)_{L_2} \geq 0, f \in \mathrm{D}(G_iA_i)$. In its own turn, it implies that $\left(A_i^*G_iA_if, f\right)_{L_2} \geq 0, f \in \mathrm{D}(A_i^*G_iA_i)$. Thus, we have obtained the desired result. Therefore, taking into account the facts given above, we deduce that $-\mathcal{T} = A_i^*G_iA_i,\ i = 1, 2, \dots n$ and obtain (11). Note that, in accordance with the accepted form of writing, we have

$$a^{ij}D_if\,\overline{D_jg} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}D_if \cdot \overline{D_jg},$$

where $a_{ij} := a^{ij}$. In these terms, applying the Cauchy–Schwarz inequality to the sums twice, we have

$$\left|a^{ij}D_if\,\overline{D_jg}\right| := \left|\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}D_if \cdot \overline{D_jg}\right| \leq \sum_{j=1}^{n}|D_jg|\sum_{i=1}^{n}\left|a_{ij}D_if\right|$$

$$\leq \sqrt{\sum_{i=1}^{n}|D_if|^2}\sum_{j=1}^{n}|D_jg|\sqrt{\sum_{i=1}^{n}|a_{ij}|^2} \leq \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}|^2}\sqrt{\sum_{i=1}^{n}|D_if|^2}\sqrt{\sum_{j=1}^{n}|D_jg|^2}$$

$$= \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}|^2} \cdot \|\nabla f\|_{\mathbb{E}^n}\,\|\nabla g\|_{\mathbb{E}^n},\ f,g \in C_0^\infty(\Omega).$$

Using the integration by parts formula, the previously obtained relation, applying the Cauchy–Schwarz inequality to the integrals, then, using Corollary 1, we obtain

$$\left|\int\limits_{\Omega}\mathcal{T}f \cdot \bar{g}\,dQ\right| = \left|\int\limits_{\Omega} a^{ij}D_if\,\overline{D_jg}\,dQ\right| \leq \int\limits_{\Omega}\left|a^{ij}D_if\,\overline{D_jg}\,dQ\right| \leq a_1\int\limits_{\Omega}\|\nabla f\|_{\mathbb{E}^n}\,\|\nabla g\|_{\mathbb{E}^n}\,dQ$$

$$\leq a_1\|f\|_{H_0^1}\|g\|_{H_0^1} \leq C\|f\|_{\mathfrak{H}_A^n}\|g\|_{\mathfrak{H}_A^n},\ f,g \in C_0^\infty(\Omega),$$

where

$$a_1 = \sup_{Q\in\bar{\Omega}} \sqrt{\sum_{i,j=1}^{n} |a^{ij}(Q)|^2}.$$

On the other hand, applying (12) and (19), we get

$$-\mathrm{Re}(\mathcal{T}f, f) \geq C\|f\|^2_{\mathfrak{H}^n_A},\ f \in C_0^\infty(\Omega).$$

The proof is complete. □

Thus, by virtue of Corollary 1 and Theorem 2, we are able to claim that theorems (**A**)–(**C**) [22] can be applied to the operator $-\mathcal{T}$.

3.2. Fractional Integro-Differential Operator

In this paragraph we assume that $\alpha \in (0,1)$. In accordance with the definition given in the paper [31], we consider a directional fractional integral. By definition, put

$$(\mathfrak{I}^\alpha_{0+} f)(Q) := \frac{1}{\Gamma(\alpha)} \int\limits_0^r \frac{f(P + t\mathbf{e})}{(r-t)^{1-\alpha}} \left(\frac{t}{r}\right)^{n-1} dt,\ f \in L_p(\Omega),\ 1 \leq p \leq \infty.$$

In addition, we consider an auxiliary operator, the so-called truncated directional fractional derivative (see [31]). By definition, put

$$(\mathfrak{D}^\alpha_{d-,\varepsilon} f)(Q) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{r+\varepsilon}^d \frac{f(Q) - f(P + \mathbf{e}t)}{(t-r)^{\alpha+1}} dt + \frac{f(Q)}{\Gamma(1-\alpha)} (d-r)^{-\alpha},\ 0 \leq r \leq d - \varepsilon,$$

$$(\mathfrak{D}^\alpha_{d-,\varepsilon} f)(Q) = \frac{f(Q)}{\alpha} \left(\frac{1}{\varepsilon^\alpha} - \frac{1}{(d-r)^\alpha} \right),\ d - \varepsilon < r \leq d.$$

Now, we can define a directional fractional derivative as follows:

$$\mathfrak{D}^\alpha_{d-} f = \lim_{\substack{\varepsilon\to 0\\ (L_p)}} \mathfrak{D}^\alpha_{d-,\varepsilon} f,\ 1 \leq p \leq \infty.$$

The properties of these operators are described in detail in the paper [31]. We suppose $\mathfrak{I}^0_{0+} = I$. Nevertheless, this fact can be easily established by virtue of the reasonings corresponding to the one-dimensional case and given in [32]. We also consider an integral operator with a weighted factor (see [32], p. 175) defined by the following formal construction:

$$(\mathfrak{I}^\alpha_{0+} \mu f)(Q) := \frac{1}{\Gamma(\alpha)} \int\limits_0^r \frac{(\mu f)(P + t\mathbf{e})}{(r-t)^{1-\alpha}} \left(\frac{t}{r}\right)^{n-1} dt,$$

where μ is a real-valued function.

Consider a linear combination of an uniformly elliptic operator given in Theorem 2 and a composition of a fractional integro-differential operator, where the fractional differential operator is understood as the adjoint operator regarding the Kipriyanov operator (see [14,23,24])

$$L := -\mathcal{T} + \mathfrak{I}^\sigma_{0+} \rho\, \mathfrak{D}^\alpha_{d-},\ \rho \in L_\infty(\Omega),\ \sigma \in [0,1),$$

$$\mathrm{D}(L) = H^2(\Omega) \cap H^1_0(\Omega),$$

Theorem 3. *We claim that*

$$L = \frac{1}{n} \sum_{i=1}^{n} A_i^* G_i A_i + F A_1^\alpha, \tag{20}$$

where F is a bounded operator, $P_1 := P$, and G_i are the same as in Theorem 2. Moreover if γ_a is sufficiently large in comparison with $\|\rho\|_{L_\infty}$, then the following relations hold:

$$\mathrm{Re}(Lf,f)_{L_2} \geq C\|f\|_{\mathfrak{H}_A^n};\ |(Lf,g)_{L_2}| \leq C\|f\|_{\mathfrak{H}_A^n}\|g\|_{\mathfrak{H}_A^n},\ f,g \in C_0^\infty(\Omega).$$

Proof. The proof follows obviously from Theorem 2, Theorem 3 [22], and Corollary 1. □

Combining the fact $\mathfrak{H}_A^n \subset\subset L_2(\Omega)$ established in Corollary 1 and Theorem 3, we claim that theorems $(\mathbf{A})$–$(\mathbf{C})$ [22] can be applied to the operator L.

4. Discussion

In this paper, we have established the equivalence of the norm generated by the infinitesimal generator of the shift semigroup in a direction and the norm of the Nicodemus space. Firstly, we should turn again to the series of papers by I.A. Kiprianov devoted to an alternative branch of the fractional calculus theory, and we say an alternative branch since the author developed his own approach to classical questions of the operator theory by virtue of which such a novel notion as a directional fractional derivative was appeared. Throughout the papers [12,23,24], in technical reasonings, the author solely used the spherical coordinates in the n-dimensional Euclidean space. This rather insignificant feature, at first sight, determined many newly appeared constructions, for instance the spaces of fractionally differentiable functions introduced in [23]. The embedding theorems for the defined spaces were formulated afterwards in [12], and the attempt to consider fractional powers of operators was made in [24]. Note that, by virtue of the obtained results of the paper, we can offer an efficient approach to the issue, the central point of which is how to connect the Kiprianov's results with the classical ones, and we claim that Lemma 3 allows us to establish this connection. Thus, we have an opportunity to reveal more fully a true mathematical nature of differential operator as a notion. As a concrete achievement, we have a compact embedding of the space generated by the infinitesimal generator into the Lebsgue space. The considered particular cases correspond to a uniformly elliptic operator and its linear combination with the fractional integro-differential operator. We stress that, in the first case, the operator is not selfadjoint under the minimal assumptions regarding its coefficients, and, in the second case, the one is represented by a sum of two non-selfadjoint summands. Note that there are not many results devoted to the topic and this is why an opportunity to apply spectral theorems in the natural way becomes relevant. Here, we should explain that our aim is to describe the operators considered in [22] in terms of the infinitesimal generator of the shift semigroup in a direction. We should note that it has been done regarding the uniformly elliptic operator in the one-dimensional case. In this respect, the following question appears if we deal with the multidimensional case—Is there any simpler way to construct a suitable Hilbert space $\mathfrak{H}_+$ that is defined in terms of the infinitesimal generator and such that $\mathfrak{H}_+ \subset\subset L_2$? To answer the question, we should note that a conjecture $\|f\|_{H_0^1} \leq C\|Af\|_{L_2},\ f \in \mathrm{D}(A)$, which guarantees the desired result, seems to be wrong in the multidimensional case. Certainly, a counterexample is required. However, it is clear (see Remark 1) that the previous relation holds in the one-dimensional case. As a theoretical achievement of the offered approach, we can study differential and fractional integro-differential operators in terms of the semigroup theory considering more general notions. In this regard (see [22]), we have the following results: an asymptotic equivalence between the real component of a resolvent and the resolvent of the real component was established for the fractional integro-differential operator; a classification, in accordance with resolvent belonging to the Schatten–von Neumann class, was obtained; a sufficient condition of completeness of the root vectors system was formulated; and an asymptotic formula for the eigenvalues was obtained. Along with all these, we claim that, by virtue of popularity and the well-known applicability of the Lebesgue spaces theory, the result related to the norm equivalence deserves to be considered itself.

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