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Generalizations of Hermite–Hadamard Type Integral Inequalities for Convex Functions

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Abstract: In the paper, with the help of two known integral identities and by virtue of the classical Hölder integral inequality, the authors establish several new integral inequalities of the Hermite–Hadamard type for convex functions. These newly established inequalities generalize some known results.

Keywords: generalization; integral inequality; convex function; Hermite–Hadamard type

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1. Backgrounds and Motivations

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The equalities in (1) are valid if and only if $f(x)$ is a linear function on $[a,b]$, as can be seen in [1] (p. 59). In mathematical literature, the double inequality (1) is called the Hermite–Hadamard inequality, named after Charles Hermite (1822–1901) and Jacques Hadamard (1865–1963). The Hermite–Hadamard inequality (1) is a necessary and sufficient condition for a real function to be convex on a closed and bounded real interval. It was extensively studied and generalized over more than one century, since it was first published in [2,3]. Copies of these two papers are available on the Internet since they belong to the fundamental knowledge of the humankind. The monograph [1] is fundamental and can be freely downloaded from the Internet. Other four fundamental monographs are [4–7]. They present the directions of development of the research in this field until now. Since then, the double inequality (1) has attracted many mathematicians' attention. Especially, in the last three decades, numerous generalizations, variants and extensions of this double inequality have been presented. In particular, the Hermite–Hadamard-type inequalities associated with a variety of fractional integral operators have been provided in [8,9] and closely related references therein.

In the paper [10], the Hermite–Hadamard integral inequality (1) was generalized as the following theorems.

Theorem 1 ([10] (Lemma 3)). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of an interval I, with $a, b \in I$ and $a < b$. If f is a convex function on I, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{1}{4}\left[f\left(\frac{3b-a}{2}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{3a-b}{2}\right)\right] \tag{2}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\frac{f\left(\frac{a+b}{2}\right)}{2}\right|\le\left|\frac{f\left(\frac{3b-a}{2}\right)+f\left(\frac{3a-b}{2}\right)}{4}\right|. \tag{3}$$

After carefully verifying the above, we find that the convexity of f should be added to [10] (Theorem 3). The slightly amended version of [10] (Theorem 3) can be stated as follows.

Theorem 2 ([10] (Lemma 3))**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° with $a,b\in I$ and $a<b$, the second derivative $f'': \left[\frac{3a-b}{2},\frac{3b-a}{2}\right]\to\mathbb{R}$ be a continuous function on $\left[\frac{3a-b}{2},\frac{3b-a}{2}\right]$, and $q>1$. If f and $|f''|^q$ are convex on $\left[\frac{3a-b}{2},\frac{3b-a}{2}\right]$, then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-\frac{1}{4}\left[f\left(\frac{3b-a}{2}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a-b}{2}\right)\right]\right|\\ \le\frac{(b-a)^2}{3}\left[\frac{1}{2}\left(\left|f''\left(\frac{3b-a}{2}\right)\right|^q+\left|f''\left(\frac{3a-b}{2}\right)\right|^q\right)\right]^{1/q}.\end{aligned}$$

In this paper, with the help of two known integral identities (see Lemmas 1 and 2 in the next section) and by virtue of the classical Hölder integral inequality, we aim to generalize those inequalities in Theorems 1 and 2 to several new Hermite–Hadamard-type inequalities for convex functions.

2. Two Lemmas

For establishing new Hermite–Hadamard type inequalities for convex functions and generalizing those inequalities in Theorems 1 and 2, we need the following lemmas.

Lemma 1 ([11] (Lemma 2.1))**.** *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L_1([a,b])$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x=\frac{b-a}{2}\int_0^1(1-2t)f'(b+t(a-b))\,\mathrm{d}\,t. \tag{4}$$

Remark 1. *Since*

$$\int_0^{1/2}(1-2t)f'(b+t(a-b))\,\mathrm{d}\,t=\frac{1}{2}\int_0^1(1-u)f'\left(b+u\frac{a-b}{2}\right)\mathrm{d}\,u$$

and

$$\int_{1/2}^1(1-2t)f'(b+t(a-b))\,\mathrm{d}\,t=-\frac{1}{2}\int_0^1 uf'\left(\frac{a+b}{2}+u\frac{a-b}{2}\right)\mathrm{d}\,u,$$

the identity (4) is equivalent to

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\\ =\frac{b-a}{4}\left[\int_0^1(1-t)f'\left(b+t\frac{a-b}{2}\right)\mathrm{d}\,t-\int_0^1 tf'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\mathrm{d}\,t\right].\end{aligned}$$

Lemma 2 ([12] (Lemma 2.1))**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$. If $f'\in L_1([a,b])$, then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x-f\left(\frac{a+b}{2}\right)\\ =(b-a)\left[\int_0^{1/2}tf'(b+t(a-b))\,\mathrm{d}\,t+\int_{1/2}^1(t-1)f'(b+t(a-b))\,\mathrm{d}\,t\right].\end{aligned} \tag{5}$$

Let $u, v \in \mathbb{R}$ with $u < v$ and $\lambda > \mu \geq 0$. For $t \in [0, 1]$, it is clear that

$$v + t(u - v) = \left(\frac{\lambda - \mu}{\lambda + \mu}t + \frac{\mu}{\lambda + \mu}\right)\frac{\lambda u - \mu v}{\lambda - \mu} + \left(\frac{\mu - \lambda}{\lambda + \mu}t + \frac{\lambda}{\lambda + \mu}\right)\frac{\lambda v - \mu u}{\lambda - \mu}. \quad (6)$$

3. New Integral Inequalities of Hermite–Hadamard Type

Now, with the help of integral identities (4) and (5), and by virtue of the classical Hölder integral inequality, we begin to establish several new integral inequalities of the Hermite–Hadamard type for convex functions on $\mathbb{R}$ and to generalize integral inequalities in the aforementioned Theorems 1 to 2.

In this section, we use the notations

$$I_{\lambda,\mu}(u, v) = \left[\frac{\lambda u - \mu v}{\lambda - \mu}, \frac{\lambda v - \mu u}{\lambda - \mu}\right] \quad \text{and} \quad I^{\circ}_{\lambda,\mu}(u, v) = \left(\frac{\lambda u - \mu v}{\lambda - \mu}, \frac{\lambda v - \mu u}{\lambda - \mu}\right). \quad (7)$$

Theorem 3. *Suppose that $\lambda > \mu \geq 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $f : I_{\lambda,\mu}(a, b) \to \mathbb{R}$ be a convex function. Then*

$$\begin{aligned} f\left(\frac{a + b}{2}\right) &\leq \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x \\ &\leq \frac{1}{2(\lambda + \mu)}\left[(\lambda - \mu)f\left(\frac{\lambda b - \mu a}{\lambda - \mu}\right) + 4\mu f\left(\frac{a + b}{2}\right) + (\lambda - \mu)f\left(\frac{\lambda a - \mu b}{\lambda - \mu}\right)\right] \end{aligned} \quad (8)$$

and

$$\left|\frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x - \frac{2\mu}{\lambda + \mu}f\left(\frac{a + b}{2}\right)\right| \leq \frac{\lambda - \mu}{2(\lambda + \mu)}\left|f\left(\frac{\lambda b - \mu a}{\lambda - \mu}\right) + f\left(\frac{\lambda a - \mu b}{\lambda - \mu}\right)\right|, \quad (9)$$

where the equalities in (8) and (9) are valid if $f(x)$ is a linear function on $[a, b]$.

Proof. Using the change of the variable $x = \frac{\lambda}{\lambda+\mu}t + \frac{\mu}{\lambda+\mu}(a + b)$ for $t \in \left[\frac{\lambda a - \mu b}{\lambda}, \frac{\lambda b - \mu a}{\lambda}\right]$ and the convexity of f on $I_{\lambda,\mu}(a, b)$, we have

$$\begin{aligned} \frac{1}{b - a}\int_a^b f(x)\,\mathrm{d}\,x &= \frac{\lambda}{(\lambda + \mu)(b - a)}\int_{(\lambda a - \mu b)/\lambda}^{(\lambda b - \mu a)/\lambda} f\left(\frac{\lambda}{\lambda + \mu}t + \frac{\mu}{\lambda + \mu}(a + b)\right)\mathrm{d}\,t \\ &= \frac{\lambda}{(\lambda + \mu)(b - a)}\int_{(\lambda a - \mu b)/\lambda}^{(\lambda b - \mu a)/\lambda} f\left(\frac{\lambda - \mu}{\lambda + \mu}\frac{\lambda}{\lambda - \mu}t + \frac{2\mu}{\lambda + \mu}\left(\frac{a + b}{2}\right)\right)\mathrm{d}\,t \\ &\leq \frac{\lambda}{(\lambda + \mu)(b - a)}\int_{(\lambda a - \mu b)/\lambda}^{(\lambda b - \mu a)/\lambda}\left[\frac{\lambda - \mu}{\lambda + \mu}f\left(\frac{\lambda}{\lambda - \mu}t\right) + \frac{2\mu}{\lambda + \mu}f\left(\frac{a + b}{2}\right)\right]\mathrm{d}\,t \\ &= \frac{(\lambda - \mu)^2}{(\lambda + \mu)^2(b - a)}\int_{(\lambda a - \mu b)/(\lambda - \mu)}^{(\lambda b - \mu a)/(\lambda - \mu)} f(t)\,\mathrm{d}\,t + \frac{2\mu}{\lambda + \mu}f\left(\frac{a + b}{2}\right) \end{aligned} \quad (10)$$

and

$$\frac{(\lambda - \mu)^2}{(\lambda + \mu)^2(b - a)}\int_{(\lambda a - \mu b)/(\lambda - \mu)}^{(\lambda b - \mu a)/(\lambda - \mu)} f(t)\,\mathrm{d}\,t \leq \frac{\lambda - \mu}{2(\lambda + \mu)}\left[f\left(\frac{\lambda b - \mu a}{\lambda - \mu}\right) + f\left(\frac{\lambda a - \mu b}{\lambda - \mu}\right)\right]. \quad (11)$$

Substituting the inequality (11) into the inequality (10), we have

$$
\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{2(\lambda+\mu)}\left[(\lambda-\mu)f\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)+4\mu f\left(\frac{a+b}{2}\right)+(\lambda-\mu)f\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right]-f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{2(\lambda+\mu)}\left[(\lambda-\mu)f\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)-2(\lambda-\mu)f\left(\frac{a+b}{2}\right)+(\lambda-\mu)f\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right].
\end{aligned}
$$

Therefore, the inequalities (8) and (9) hold.

It is straightforward to verify that, if $f(x) = cx + d$ on $[a, b]$ for c, d being constants, the equalities in (8) and (9) are valid. Theorem 3 is thus proven. □

Remark 2. *If setting $\lambda = 1$ and $\mu = 0$ in Theorem 3, then we recover the double inequality (1).*

If letting $\lambda = 3$ and $\mu = 1$ in Theorem 3, we derive the above inequalities (2) and (3) obtained in [10] (Lemma 3).

Theorem 4. *Suppose that $\lambda > \mu \ge 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $f : I_{\lambda,\mu}(a, b) \to \mathbb{R}$ be a differentiable mapping on $I^\circ_{\lambda,\mu}(a, b)$. If $|f'|^q$ for $q \ge 1$ is a convex function on $I_{\lambda,\mu}(a, b)$, then*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{b-a}{8}\left\{\left[\frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q+\frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]^{1/q}\right. \\
&\quad\left.+\left[\frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q+\frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]^{1/q}\right\}.
\end{aligned}
\tag{12}
$$

Proof. By Lemma 2 and the Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \\
&\le (b-a)\left[\int_0^{1/2} t|f'(b+t(a-b))|\,\mathrm{d}\,t+\int_{1/2}^1(1-t)|f'(b+t(a-b))|\,\mathrm{d}\,t\right] \\
&\le (b-a)\left\{\left(\int_0^{1/2} t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^{1/2} t|f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad\left.+\left(\int_{1/2}^1(1-t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_{1/2}^1(1-t)|f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}
\tag{13}
$$

Since $(\mu-\lambda)t+\lambda \ge 0$ and

$$
\left(\frac{\lambda-\mu}{\lambda+\mu}t+\frac{\mu}{\lambda+\mu}\right)+\left(\frac{\mu-\lambda}{\lambda+\mu}t+\frac{\lambda}{\lambda+\mu}\right)=1
$$

for $t \in [0, 1]$, letting $u = a$ and $v = b$ in the identity (6) and using the convexity of $|f'|^q$ arrive at

$$
|f'(b+t(a-b))|^q \le \frac{(\lambda-\mu)t+\mu}{\lambda+\mu}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q+\frac{(\mu-\lambda)t+\lambda}{\lambda+\mu}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q. \tag{14}
$$

Straightforward computation yields

$$
\begin{aligned}
&\int_0^{1/2} t|f'(b+t(a-b))|^q \,\mathrm{d}\,t \\
&\le \int_0^{1/2} t\left[\frac{(\lambda-\mu)t+\mu}{\lambda+\mu}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{(\mu-\lambda)t+\lambda}{\lambda+\mu}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]\mathrm{d}\,t \\
&= \frac{\lambda+2\mu}{24(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{2\lambda+\mu}{24(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
&\int_{1/2}^{1} (1-t)|f'(b+t(a-b))|^q \,\mathrm{d}\,t \\
&\le \int_{1/2}^{1} (1-t)\left[\frac{(\lambda-\mu)t+\mu}{\lambda+\mu}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{(\mu-\lambda)t+\lambda}{\lambda+\mu}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]\mathrm{d}\,t \\
&= \frac{2\lambda+\mu}{24(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{\lambda+2\mu}{24(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q.
\end{aligned}
\tag{16}
$$

It is easy to see that

$$
\int_0^{1/2} t\,\mathrm{d}\,t = \int_{1/2}^{1}(1-t)\,\mathrm{d}\,t = \frac{1}{8}. \tag{17}
$$

Applying inequalities (15), (16), and (17) into the inequality (13) gives

$$
\begin{aligned}
\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| &\le \frac{b-a}{8}\left\{\left[\int_0^{1/2} t|f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q}\right. \\
&\quad \left.+\left[\int_{1/2}^{1}(1-t)|f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le \frac{b-a}{8}\left\{\left[\frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]^{1/q}\right. \\
&\quad \left.+\left[\frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\right]^{1/q}\right\}.
\end{aligned}
$$

The proof of Theorem 4 is complete. □

Corollary 1. *Under conditions of Theorem 4,*

1. *if $q=1$, then*

$$
\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left[\frac{\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right| + \left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|}{2}\right];
$$

2. *if $\lambda=1$ and $\mu=0$, then*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \\
&\qquad \le \frac{b-a}{8}\left\{\left[\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right]^{1/q} + \left[\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right]^{1/q}\right\}.
\end{aligned}
$$

Theorem 5. *Suppose that $\lambda > \mu \geq 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $f : I_{\lambda,\mu}(a, b) \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}_{\lambda,\mu}(a, b)$, where $I_{\lambda,\mu}(a, b)$ and $I^{\circ}_{\lambda,\mu}(a, b)$ are defined as in (7). If $|f'|^q$ for $q > 1$ is a convex function on $I_{\lambda,\mu}(a, b)$, then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\Bigg\{\Bigg[\frac{\lambda+3\mu}{4(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q \\ +\frac{3\lambda+\mu}{4(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\Bigg]^{1/q} + \Bigg[\frac{3\lambda+\mu}{4(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q \\ +\frac{\lambda+3\mu}{4(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q\Bigg]^{1/q}\Bigg\}.\end{aligned} \tag{18}$$

Proof. Similar to the proof of the inequality (12) in Theorem 4, making use of Lemma 2 and the Hölder integral inequality reveals

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - f\left(\frac{a+b}{2}\right)\right| \\ &\leq (b-a)\Bigg\{\left[\int_0^{1/2} t^{q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^{1/2}|f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q} \\ &+\left[\int_{1/2}^1 (1-t)^{q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_{1/2}^1 |f'(b+t(a-b))|^q\,\mathrm{d}\,t\right]^{1/q}\Bigg\},\end{aligned} \tag{19}$$

where

$$\int_0^{1/2} t^{q/(q-1)}\,\mathrm{d}\,t = \int_{1/2}^1 (1-t)^{q/(q-1)}\,\mathrm{d}\,t = \frac{q-1}{2q-1}\left(\frac{1}{2}\right)^{(2q-1)/(q-1)}. \tag{20}$$

From the inequality (14) and by the convexity of $|f'|^q$, we obtain

$$\int_0^{1/2}|f'(b+t(a-b))|^q\,\mathrm{d}\,t \leq \frac{\lambda+3\mu}{8(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{3\lambda+\mu}{8(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q \tag{21}$$

and

$$\int_{1/2}^1|f'(b+t(a-b))|^q\,\mathrm{d}\,t \leq \frac{3\lambda+\mu}{8(\lambda+\mu)}\left|f'\left(\frac{\lambda a-\mu b}{\lambda-\mu}\right)\right|^q + \frac{\lambda+3\mu}{8(\lambda+\mu)}\left|f'\left(\frac{\lambda b-\mu a}{\lambda-\mu}\right)\right|^q. \tag{22}$$

Substituting inequalities (20), (21) and (22) into the inequality (19) yields the inequality (18). The proof of Theorem 5 is complete. □

Theorem 6. *Suppose that $\lambda > \mu \geq 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $f : I_{\lambda,\mu}(a, b) \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}_{\lambda,\mu}(a, b)$, where $I_{\lambda,\mu}(a, b)$ and $I^{\circ}_{\lambda,\mu}(a, b)$ are defined as in (7). If $|f'|^q$ for $q \geq 1$ is a convex function on $I_{\lambda,\mu}(a, b)$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{8}\Bigg\{\Bigg[\frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda a-(2\mu-\lambda)b}{2(\lambda-\mu)}\right)\right|^q \\ +\frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)b-\mu a}{2(\lambda-\mu)}\right)\right|^q\Bigg]^{1/q} + \Bigg[\frac{2\lambda+\mu}{3(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q \\ +\frac{\lambda+2\mu}{3(\lambda+\mu)}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q\Bigg]^{1/q}\Bigg\}.\end{aligned} \tag{23}$$

Proof. By Lemma 1 and the Hölder integral inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{b-a}{4}\left[\int_0^1 (1-t)\left|f'\left(b+t\frac{a-b}{2}\right)\right|\mathrm{d}\,t+\int_0^1 t\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|\mathrm{d}\,t\right] \\
&\le \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 (1-t)\left|f'\left(b+t\frac{a-b}{2}\right)\right|^q \mathrm{d}\,t\right]^{1/q}\right. \\
&\quad \left. +\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|^q \mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}
\tag{24}
$$

For $t\in[0,1]$, putting $u=a$ and $v=\frac{a+b}{2}$ in the identity (6) and using the convexity of $|f'|^q$ result in

$$
\begin{aligned}
&\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|^q \\
&\quad \le \frac{(\lambda-\mu)t+\mu}{\lambda+\mu}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q+\frac{(\mu-\lambda)t+\lambda}{\lambda+\mu}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q.
\end{aligned}
$$

Accordingly, we have

$$
\begin{aligned}
\int_0^1 t\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|^q \mathrm{d}\,t &\le \int_0^1 t\left[\frac{(\lambda-\mu)t+\mu}{\lambda+\mu}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q\right. \\
&\quad \left. +\frac{(\mu-\lambda)t+\lambda}{\lambda+\mu}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q\right]\mathrm{d}\,t \\
&= \frac{2\lambda+\mu}{6(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q+\frac{\lambda+2\mu}{6(\lambda+\mu)}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q.
\end{aligned}
\tag{25}
$$

Similarly, taking $u=\frac{a+b}{2}$ and $v=b$ in the identity (6) gives

$$
\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(b+t\frac{a-b}{2}\right)\right|^q \mathrm{d}\,t \\
&\quad \le \frac{\lambda+2\mu}{6(\lambda+\mu)}\left|f'\left(\frac{\lambda a-(2\mu-\lambda)b}{2(\lambda-\mu)}\right)\right|^q+\frac{2\lambda+\mu}{6(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)b-\mu a}{2(\lambda-\mu)}\right)\right|^q.
\end{aligned}
\tag{26}
$$

Substituting inequalities (25) and (26) into inequality (24) yields (23). The proof of Theorem 6 is complete. □

Corollary 2. *Under conditions of Theorem 6, if* $q=1$, $\lambda=1$, *and* $\mu=0$, *then*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|}{3}\right].
$$

Theorem 7. *Suppose that $\lambda > \mu \geq 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $f : I_{\lambda,\mu}(a,b) \to \mathbb{R}$ be a differentiable mapping on $I^\circ_{\lambda,\mu}(a,b)$, where $I_{\lambda,\mu}(a,b)$ and $I^\circ_{\lambda,\mu}(a,b)$ are defined as in (7). If $|f'|^q$ for $q > 1$ is a convex function on $I_{\lambda,\mu}(a,b)$ and $0 \leq \ell \leq q$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{b-a}{4}\left[\frac{q-1}{2q-(\ell+1)}\right]^{1-1/q} \\ \times\left\{\left[\frac{\lambda+(\ell+1)\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{\lambda a-(2\mu-\lambda)b}{2(\lambda-\mu)}\right)\right|^q\right.\right. \\ \left.+\frac{(\ell+1)\lambda+\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)b-\mu a}{2(\lambda-\mu)}\right)\right|^q\right]^{1/q} \\ +\left[\frac{(\ell+1)\lambda+\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q\right. \\ \left.\left.+\frac{\lambda+(\ell+1)\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q\right]^{1/q}\right\}.\end{aligned} \tag{27}$$

Proof. Similar to the proof of the inequality (23) in Theorem 6 from Lemma 1 and the Hölder integral inequality, we derive

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ &\leq \frac{b-a}{4}\left[\int_0^1 (1-t)\left|f'\left(b+t\frac{a-b}{2}\right)\right|\mathrm{d}\,t + \int_0^1 t\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|\mathrm{d}\,t\right] \\ &\leq \frac{b-a}{4}\left\{\left[\int_0^1 (1-t)^{(q-\ell)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 (1-t)^\ell\left|f'\left(b+t\frac{a-b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right. \\ &\quad\left.+\left[\int_0^1 t^{(q-\ell)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^\ell\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}.\end{aligned} \tag{28}$$

It is obvious that

$$\int_0^1 (1-t)^{(q-\ell)/(q-1)}\,\mathrm{d}\,t = \int_0^1 t^{(q-\ell)/(q-1)}\,\mathrm{d}\,t = \frac{q-1}{2q-(\ell+1)}. \tag{29}$$

By the identity (6) and the convexity of $|f'|^q$, we obtain

$$\begin{aligned}\int_0^1 (1-t)^\ell\left|f'\left(b+t\frac{a-b}{2}\right)\right|^q\mathrm{d}\,t \leq \frac{\lambda+(\ell+1)\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{(\lambda a-(2\mu-\lambda)b}{2(\lambda-\mu)}\right)\right|^q \\ +\frac{(\ell+1)\lambda+\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)b-\mu a}{2(\lambda-\mu)}\right)\right|^q\end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_0^1 t^\ell\left|f'\left(\frac{a+b}{2}+t\frac{a-b}{2}\right)\right|^q\mathrm{d}\,t \leq \frac{(\ell+1)\lambda+\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{(2\lambda-\mu)a-\mu b}{2(\lambda-\mu)}\right)\right|^q \\ +\frac{\lambda+(\ell+1)\mu}{(\ell+1)(\ell+2)(\lambda+\mu)}\left|f'\left(\frac{\lambda b-(2\mu-\lambda)a}{2(\lambda-\mu)}\right)\right|^q.\end{aligned} \tag{31}$$

Substituting inequalities (29), (30) and (31) into the inequality (28) concludes the inequality (27). The proof of Theorem 7 is complete. □

4. Remarks

In this section, we provide several remarks on our main results and related ones

Remark 3. *The facts that inequalities (8) and (9) in Theorem 3 are sharp were observed and pointed out by an anonymous referee.*

Remark 4. *In fact, the new inequalities in this paper are obtained by using the computation techniques inspired by the papers [11,12] and generalizing ideas from the paper [10]. Similar types of inequalities, or particular cases, are obtained in the literature by other techniques. One may see the Hermite–Hadamard type inequalities from [13]. These texts are excerpted and adapted from valuable comments of an anonymous referee of this paper.*

Remark 5. *The new inequalities for convex and differentiable functions in this paper have particular cases in [10] and some properties make them distinctive from other existing inequalities of the Hermite–Hadamard type under similar hypotheses (for example those from [11]). These texts are excerpted and adapted from valuable comments of an anonymous referee of this paper.*

Remark 6. *The new inequalities (8) and (9) for convex functions are sharp. But the inequalities involving differentiable functions having derivatives with convexity properties lose the property of sharpness within the class of linear functions. For example, the inequality (12) in Theorem 4, the inequality (18) in Theorem 5, the inequality (23) in Theorem 6, and the inequality (27) in Theorem 7 are not sharp for linear functions, as the classical Hermite-Hadamard inequality (1) and the inequalities from [11] (for similar types of functions), but they are sharp for constant functions. This solves the problem of sharpness easily. These texts are excerpted and adapted from valuable comments of an anonymous referee of this paper.*

5. Conclusions

In this paper, with the help of two known integral identities and by virtue of the Hölder integral inequality, in Theorems 3–7, and their corollaries, we established several new integral inequalities of the Hermite–Hadamard type for convex functions. These newly established inequalities generalize corresponding ones in the paper [10].

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