axioms

MDPI

Article

An Extension of Beta Function by Using Wiman's Function

Rahul Goyal [1], Shaher Momani [2,3], Praveen Agarwal [1,3,4,5] and Michael Th. Rassias [6,*]

[1] Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India; rahul.goyal01@anandice.ac.in (R.G.); praveen.agarwal@anandice.ac.in or goyal.praveen2011@gmail.com (P.A.)
[2] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan; S.Momani@ju.edu.jo
[3] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
[4] Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211019, India
[5] International Center for Basic and Applied Sciences, Jaipur 302029, India
[6] Institute for Advanced Study, Program in Interdisciplinary Studies, 1 Einstein Dr, Princeton, NJ 08540, USA
* Correspondence: michail.rassias@math.uzh.ch

Abstract: The main purpose of this paper is to study extension of the extended beta function by Shadab et al. by using 2-parameter Mittag-Leffler function given by Wiman. In particular, we study some functional relations, integral representation, Mellin transform and derivative formulas for this extended beta function.

Keywords: classical Euler beta function; gamma function; Gauss hypergeometric function; confluent hypergeometric function; Mittag-Leffler function

MSC: 33B15; 33C05; 33C15; 33E12

check for updates

Citation: Goyal, R.; Momani, S.; Agarwal, P.; Rassias, M.T. An Extension of Beta Function by Using Wiman's Function. *Axioms* **2021**, *10*, 187. https://doi.org/10.3390/axioms10030187

Academic Editor: Chris Goodrich

Received: 03 August 2021
Accepted: 12 August 2021
Published: 16 August 2021

1. Introduction and Preliminaries

Special functions play important roles in science and engineering due its variety of applications; that is why many researchers have been working on them for many years [1–8]. The classical beta function is one of the most important member of the class of special functions, and it is also known as the Euler Integral of first kind. It has a wide range of applications in science, especially in engineering mathematics. In the last few years, the extension of the classical Euler beta function and gamma function has been an interesting topic for researchers due to their pivot role in advanced research.

The classical Euler beta function defined as follows [9].

$$B(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt, \qquad \Re(x_1), \Re(x_2) > 0. \tag{1}$$

The gamma function defined as follows [9].

$$\Gamma(x_1) = \int_0^\infty t^{x_1-1}e^{-t}dt, \qquad \Re(x_1) > 0. \tag{2}$$

The study of the extension of the special function began in early 1990. Chaudhry and Zubair extended the gamma function [10] in 1994 for the first time, as defined in the following.

$$\Gamma(x_1; r) = \int_0^\infty t^{x_1-1}e^{-t-rt^{-1}}dt, \qquad \Re(x_1) > 0, r > 0. \tag{3}$$

In the sequence, Chaudhry et al. extended the classical Euler beta function [11] in 1997 defined as follows:

$$B(x_1, x_2; r) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}e^{\frac{-r}{t(1-t)}}dt, \tag{4}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0$ and $\Re(r) > 0$.

Remark 1. *If we set $r = 0$ in (3) and (4), we obtain gamma function and beta function given by (1) and (2), respectively:*

$$\Gamma(x_1; 0) = \Gamma(x_1) \tag{5}$$

and

$$B(x_1, x_2; 0) = B(x_1, x_2). \tag{6}$$

In the above extension, Chaudhry and Zubair have used an exponential function as kernel to extend the classical Euler beta function and gamma function in terms of integrals. After few years, many researchers have used confluent hypergeometric function ${}_1F_1$ defined as [9] in order to generalize the classical Euler beta function and gamma function.

The confluent hypergeometric function defined as follows [9]:

$${}_1F_1(r_1; r_2; z) = \sum_{n=0}^{\infty} \frac{(r_1)_n}{(r_2)_n} \frac{z^n}{n!}, \tag{7}$$

where $(r)_n$ represents the Pochhammer symbol defined as follows [9].

$$(r)_n := \frac{\Gamma(r+n)}{\Gamma(r)} = \begin{cases} 1 & (n = 0; r \in \mathbb{C} \setminus \{0\}) \\ r(r+1)\cdots(r+n-1) & (n \in \mathbb{N}; r \in \mathbb{C}). \end{cases}$$

In 2011, Özergin et al. [12] have used the confluent hypergeometric function to generalize the classical Euler beta function and gamma function as follows:

$$\Gamma_r^{(r_1,r_2)}(x_1) = \int_0^{\infty} t^{x_1-1} {}_1F_1\left(r_1; r_2; -t - \frac{r}{t}\right) dt, \tag{8}$$

where $\Re(x_1) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ and $\Re(r) > 0$. The following is also the case:

$$B_r^{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} {}_1F_1\left(r_1; r_2; \frac{-r}{t(1-t)}\right) dt, \tag{9}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0, \min\{\Re(r_1), \Re(r_2)\} > 0$ and $\Re(r) > 0$.

Remark 2. *(1) If we set $r_1 = r_2$ in (8) and (9), we obtain extended gamma function and extended beta function given by (3) and (4), respectively:*

$$\Gamma_r^{(r_1,r_1)}(x_1) = \Gamma(x_1; r) \tag{10}$$

and

$$B_r^{(r_1,r_1)}(x_1, x_2) = B(x_1, x_2; r). \tag{11}$$

(2) If we set $r_1 = r_2$ and $r = 0$ in (8) and (9), we obtain gamma function and classical Euler beta function given by (1) and (2), respectively:

$$\Gamma_0^{(r_1,r_1)}(x_1) = \Gamma(x_1) \tag{12}$$

and

$$B_0^{(r_1,r_1)}(x_1, x_2) = B(x_1, x_2). \tag{13}$$

In the above extension, Özergin et al. have used confluent hypergeomteric function ${}_1F_1$ as a kernel to extend the classical Euler beta function and gamma function in terms of integrals. After a few years, some researchers have used the Mittag-Leffler function defined as [13] to modify the classical Euler beta function and gamma function.

One parameter Mittag-Leffler function is defined as follows [13].

$$E_{r_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(kr_1 + 1)}, \quad \Re(r_1) \geq 0, z \in C. \tag{14}$$

In 2017, Pucheta [14] had introduced a new modified extension of the gamma and beta functions by using one parameter Mittag-Leffler function $E_{r_1}(z)$, as follows:

$$\Gamma^{r_1}(x_1) = \int_0^{\infty} t^{x_1-1} E_{r_1}(-t)dt, \quad \Re(x_1) > 0, \Re(r_1) > 0. \tag{15}$$

$$B_r^{r_1}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1}(rt(1-t))dt, \tag{16}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r) > 0$ and $\Re(r_1) > 0$.

Remark 3. *(1) If we set $r_1 = 1$ in (15), then we obtain gamma function given by the (2).*

$$\Gamma^1(x_1) = \Gamma(x_1). \tag{17}$$

(2) If we take $r_1 = 1$ and $r = 0$ in (16), then we obtain the beta function given by (1).

$$B_0^1(x_1, x_2) = B(x_1, x_2). \tag{18}$$

In the above extension, Pucheta had used one parameter Mittag-Leffler function $E_{r_1}(z)$ as a kernel to modify the classical Euler beta function and gamma function in terms of integrals. One year later in January 2018, Shadab et al. [15] have adopted as similar method used by Pucheta [14] to extend the classical Euler beta function by using one parameter Mittag-Leffler function $E_{r_1}(z)$.

Another Extension of classical Euler beta function defined as follows [15]:

$$B_{r_1}^r(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1}\left(-rt^{-1}(1-t)^{-1}\right)dt, \tag{19}$$

where $\Re(x_1), \Re(x_2) > 0, \Re(r) > 0$ and $\Re(r_1) > 0$.

Remark 4. *If we take $r_1 = 1$ in (19), then we obtain the equation extended beta function given by (4).*

$$B_1^r(x_1, x_2) = B(x_1, x_2; r). \tag{20}$$

In the above extension, Shadab et al. have used one parameter Mittag-Leffler function $E_{r_1}(z)$ as a kernel to modify the classical Euler beta function in terms of integrals. After some time in November 2018, Atash et al. [16] received motivation from Shadab and Pucheta and introduced a new extension of gamma function and classical Euler beta function by using a 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$ (Wiman's function).

The 2-parameter Mittag-Leffler function(known as Wiman's function) is defined as follows [13].

$$E_{r_1,r_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(kr_1 + r_2)}, \quad \Re(r_1) \geq 0, \Re(r_2) \geq 0, z \in C. \tag{21}$$

The new extended gamma function is defined as follows [16]:

$$\Gamma_r^{(r_1,r_2)}(x_1) = \int_0^{\infty} t^{x_1-1} E_{r_1,r_2}\left(-t - rt^{-1}\right)dt, \tag{22}$$

where $\Re(x_1) > 0, \min\{\Re(r_1), \Re(r_2)\} > 0$ and $r \geq 0$.

The new extended beta function is defined as follows [16]:

$$B_{r,q}^{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t}\right) E_{r_1,r_2}\left(-q(1-t)^{-1}\right)dt, \tag{23}$$

where $\Re(x_1), \Re(x_2) > 0, \Re(r_1), \Re(r_2) > 0, r$ and $q \geq 0$.

Remark 5. *(1) If we take $r = 0$ in (22), then we obtain another extended gamma function defined as follows:*

$$\Gamma_0^{(r_1,r_2)}(x_1) = \Gamma^{(r_1,r_2)}(x_1) = \int_0^\infty t^{x_1-1} E_{r_1,r_2}(-t)dt, \tag{24}$$

where $\Re(x_1) > 0, \Re(r_1) > 0$ and $\Re(r_2) > 0$.

*(2) If we take $r_1 = r_2 = 1$ in (22), then we obtain the extended gamma function given by (**??**).*

$$\Gamma_r^{(1,1)}(x_1) = \Gamma(x_1; r). \tag{25}$$

(3) If we set $r_2 = 1$ and $r = 0$ in (22), then we obtain the modified gamma function given by (15).

$$\Gamma_0^{(r_1,1)}(x_1) = \Gamma^{r_1}(x_1). \tag{26}$$

(4) If we substitute $r_1 = 1 = r_2$ and $r = 0$ in (22), then we obtain the gamma function given by (2).

$$\Gamma_0^{(1,1)}(x_1) = \Gamma(x_1). \tag{27}$$

(5) If we take $r_1 = r_2 = 1$ and $r = q$ in (23), then we obtain the extended beta function given by (4).

$$B_{r,r}^{(1,1)}(x_1,x_2) = B(x_1,x_2;r). \tag{28}$$

(6) If we set $r_1 = r_2 = 1$ and $r = q = 0$ in (23), then we obtain the beta function given by (1).

$$B_{0,0}^{(1,1)}(x_1,x_2) = B(x_1,x_2). \tag{29}$$

In the above extension, Atash et al. have used 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$(Wiman's function) as a kernel to modify the classical Euler beta function in terms of integrals. As given above, many researchers have worked on various extensions and modifications of gamma and classical Euler beta function in terms of integrals for which the kernel contains the exponential function, confluent hypergeometric function, Mittag-Leffler function and 2-parameter Mittag-Leffler function. Furthermore, they studied their various properties and applications.

2. Results

In this paper, motivated by Pucheta [14], Shadab et al. [15] and Atash et al. [16], we extend the extended beta function defined as (19) by using the 2-parameter Mittag-leffler function $E_{r_1,r_2}(z)$ (Wiman's function) given by (21).

Definition 1. *An extension of extended beta function $B_{(r_1,r_2)}^{(r)}(x_1,x_2)$ with $r \geq 0$ is defined by the following:*

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right)dt, \tag{30}$$

where $\min\{\Re(x_1), \Re(x_2)\} > 0$, $\Re(r_1) > 0, \Re(r_2) > 0$ and $E_{r_1,r_2}(z)$ is the 2-parameter Mittag-Leffler function given by (21).

Remark 6. *(1) If we set $r_2 = 1$ in (30), then we obtain another extension of the beta function given by (19).*

$$B_{(r_1,1)}^{(r)}(x_1,x_2) = B_{r_1}^{r}(x_1,x_2). \tag{31}$$

(2) If we take $r_1 = 1 = r_2$ *in* (30)*, then we obtain the extended beta function given by* (4)*.*

$$B^{(r)}_{(1,1)}(x_1, x_2) = B(x_1, x_2; r). \tag{32}$$

(3) If we set $r_1 = 1 = r_2$ *and* $r = 0$ *in* (30)*, then we obtain the beta function given by* (1)*.*

$$B^{(0)}_{(1,1)}(x_1, x_2) = B(x_1, x_2). \tag{33}$$

In this paper we also study some of the results of an extended beta function defined as (30).

Theorem 1. *Let* $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ *and* $\Re(r_2) > 0$*, then extended beta function defined as* (30) *satisfies the Symmetric relation.*

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = B^{(r)}_{(r_1,r_2)}(x_2, x_1). \tag{34}$$

Proof of Theorem 1. From (30) we have
$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$

By substituting $t = (1-x)$ in the above and interchanging cross-ponding variables, we obtain the following.
$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 (1-x)^{x_1-1} x^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{x(1-x)}\right) dx.$

Then, by definition of an extension of the extended beta function, we obtain our desired result. $B^{(r)}_{(r_1,r_2)}(x_1, x_2) = B^{(r)}_{(r_1,r_2)}(x_2, x_1)$. □

Theorem 2. *Let* $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ *and* $\Re(r_2) > 0$*, then the extended beta function defined as* (30) *satisfies the Functional relation.*

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2). \tag{35}$$

Proof of Theorem 2. We apply the definition of (30) to right hand side of the (35) and perform some manipulations with the terms. Then, we obtain our desired result.
$B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2) =$
$\int_0^1 t^{x_1-1}(1-t)^{x_2} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt + \int_0^1 t^{x_1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$
$B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2) = \int_0^1 [t^{-1} + (1-t)^{-1}] t^{x_1}(1-t)^{x_2} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$
$B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$
$B^{(r)}_{(r_1,r_2)}(x_1, x_2+1) + B^{(r)}_{(r_1,r_2)}(x_1+1, x_2) = B^{(r)}_{(r_1,r_2)}(x_1, x_2).$
□

Theorem 3. *Let* $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ *and* $\Re(r_2) > 0$*, then extended beta function defined as* (30) *satisfies the Summation relation .*

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \sum_{n=0}^{\infty} B^{(r)}_{(r_1,r_2)}(x_1+n, x_2+1). \tag{36}$$

Proof of Theorem 3. From (30) we have the following.
$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$

After re-arranging the terms, we have the following.
$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2}(1-t)^{-1} E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right) dt.$

Then, we have the the Binomial expansion defined as follows.

$$(1-t)^{-1} = \sum_{n=0}^{\infty} t^n, |t| < 1. \tag{37}$$

Using (37) in the above expression, we obtain the following.
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2}\sum_{n=0}^{\infty} t^n E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$
Upon interchanging order of integration and summation in the above, we obtain the following:
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \sum_{n=0}^{\infty}\int_0^1 t^{x_1+n-1}(1-t)^{x_2}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$
Then, we obtain our desired result from (30).
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \sum_{n=0}^{\infty} B^{(r)}_{(r_1,r_2)}(x_1+n, x_2+1).$ □

Theorem 4. *Let $r \geq 0, \min\{\Re(x_1), \Re(1-x_2)\} > 0, \Re(r_1) > 0$ and $\Re(r_2) > 0$, then the extended beta function defined as (30) satisfies another Summation relation.*

$$B^{(r)}_{(r_1,r_2)}(x_1, 1-x_2) = \sum_{n=0}^{\infty}\frac{(x_2)_n}{n!}B^{(r)}_{(r_1,r_2)}(x_1+n, 1). \tag{38}$$

Proof of Theorem 4. From (30), we have the following.
$B^{(r)}_{(r_1,r_2)}(x_1,1-x_2) = \int_0^1 t^{x_1-1}(1-t)^{-x_2}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$
From Binomial expansion defined as follows :

$$(1-t)^{-x_2} = \sum_{n=0}^{\infty}\frac{(x_2)_n}{n!}t^n, |t| < 1. \tag{39}$$

Using (39) in above expression, we have the following.
$B^{(r)}_{(r_1,r_2)}(x_1,1-x_2) = \int_0^1 t^{x_1-1}\sum_{n=0}^{\infty}\frac{(x_2)_n}{n!}t^n E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$
Upon interchanging the order of integration and summation in the above, we obtain the following
$B^{(r)}_{(r_1,r_2)}(x_1,1-x_2) = \sum_{n=0}^{\infty}\frac{(x_2)_n}{n!}\int_0^1 t^{x_1+n-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$
Then, from (30), we obtain our desired result.
$B^{(r)}_{(r_1,r_2)}(x_1,1-x_2) = \sum_{n=0}^{\infty}\frac{(x_2)_n}{n!}B^{(r)}_{(r_1,r_2)}(x_1+n,1).$ □

Theorem 5. *Let $r \geq 0, \min\{\Re(x_1), \Re(x_2)\} > 0, \Re(r_1) > 0$ and $\Re(r_2) > 0$, then the extended beta function defined as (30) has a relation with beta function defined as (1).*

$$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma(r_2+nr_1)}B(x_1-n, x_2-n). \tag{40}$$

Proof of Theorem 5. From (21), substiute 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$ to (30), then we obtain the following.
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}\sum_{n=0}^{\infty}\frac{(-r)^n}{t^n(1-t)^n\Gamma(r_2+nr_1)}dt.$
Upon interchanging the order of integration and summation from the above, we obtain the following.
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma(r_2+nr_1)}\int_0^1 t^{x_1-n-1}(1-t)^{x_2-n-1}dt.$
Then, from (1), we obtain our desired result.
$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma(r_2+nr_1)}B(x_1-n,x_2-n).$ □

Theorem 6. *Let $r \geq 0, \Re(x_1+s) > 0, \Re(x_2+s) > 0, \Re(r_1) > 0, \Re(r_2) > 0$ and $\Re(s) > 0$, then the Mellin Transform of an extended beta function defined as (30) is given by the following.*

$$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr = B(x_1+s,x_2+s)\Gamma_0^{(r_1,r_2)}(s). \tag{41}$$

Proof of Theorem 6. Upon multiplying an extended beta function defined as (30) by $r^{(s-1)}$ and integrating it with respect to r from limit $r=0$ to $r=\infty$, we obtain the following.
$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr = \int_0^\infty r^{(s-1)}\left(\int_0^1 t^{x_1-1}(1-t)^{x_2-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt\right)dr.$
$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}\left(\int_0^\infty r^{(s-1)}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dr\right)dt.$

Substitute the above $v = \frac{r}{t(1-t)}$, then we obtain the following.
$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr = \int_0^1 t^{x_1+s-1}(1-t)^{x_2+s-1}\left(\int_0^\infty v^{s-1}E_{r_1,r_2}(-v)dv\right)dt.$

Then, from (1) and (24), we obtain our desired result.
$\int_0^\infty r^{(s-1)} B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr = B(x_1+s,x_2+s)\Gamma_0^{(r_1,r_2)}(s).$ □

Theorem 7. *Let $r \geq 0, \min\{\Re(x_1),\Re(x_2)\} > 0, \Re(r_1) > 0$ and $\Re(r_2) > 0$, then the extended beta function defined as (30) has another integral representation.*

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = 2\int_0^{\frac{\pi}{2}} (\cos(x))^{2x_1-1}(\sin(x))^{2x_2-1}E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2(\sin(x))^2}\right)dx. \tag{42}$$

Proof of Theorem 7. From (30), we obtain the following.
$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}E_{r_1,r_2}\left(\frac{-r}{t(1-t)}\right)dt.$

Upon substituting $t = (\cos(x))^2$ in the above and rearranging the terms, we obtain our result.
$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = 2\int_0^{\frac{\pi}{2}} (\cos(x))^{2x_1-1}(\sin(x))^{2x_2-1}E_{r_1,r_2}\left(\frac{-r}{(\cos(x))^2(\sin(x))^2}\right)dx.$ □

Theorem 8. *Let $r \geq 0, \min\{\Re(x_1),\Re(x_2)\} > 0, \Re(r_1) > 0, \Re(r_2) > 0$ and $k \in N$, then the extended beta function defined as (30) has the following relation.*

$$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \sum_{n=0}^{k}\frac{k!}{n!(k-n)!}B_{(r_1,r_2)}^{(r)}(x_1+n,x_2+k-n). \tag{43}$$

Proof of Theorem 8. From (35), we have the following.
$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = B_{(r_1,r_2)}^{(r)}(x_1,x_2+1) + B_{(r_1,r_2)}^{(r)}(x_1+1,x_2).$

Then, apply the same result of (35) on each term of the right hand side of the above equation. We obtain the following. $B_{(r_1,r_2)}^{(r)}(x_1,x_2) = B_{(r_1,r_2)}^{(r)}(x_1+2,x_2) + 2B_{(r_1,r_2)}^{(r)}(x_1,x_2) + B_{(r_1,r_2)}^{(r)}(x_1,x_2+2).$

Upon continuing the same process on the right hand side of the above equation and applying mathematical induction on k, we obtain the following.
$B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \sum_{n=0}^{k}\frac{k!}{n!(k-n)!}B_{(r_1,r_2)}^{(r)}(x_1+n,x_2+k-n).$ □

Derivative formulas of (30). Certain derivative formulas for the extended beta function (30) are established in the following theorem.

Theorem 9. *Each of the following derivative formulas holds true for $m \in N_0$ and $n \in N_0$.*

$$\frac{\partial^m}{\partial {x_1}^m}B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 t^{x_1-1}\ln^m(t)(1-t)^{x_2-1}E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right)dt. \tag{44}$$

$$\frac{\partial^n}{\partial {x_2}^n}B_{(r_1,r_2)}^{(r)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}\ln^n(1-t)E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right)dt. \tag{45}$$

$$\frac{\partial^{m+n}}{\partial {x_1}^m \partial {x_2}^n} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1} \ln^m (t)(1-t)^{x_2-1} \ln^n (1-t) E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt. \quad (46)$$

Proof of Theorem 9. In order to prove first part of the theorem, differentiating the extended beta function (30) with respect to the variable x_1 under the integral and applying the Leibniz rule for differentiation under integral sign are necessary procedures.

Then, we obtain the following.

$\frac{\partial^m}{\partial {x_1}^m} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \frac{\partial^m}{\partial {x_1}^m} \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt.$

$\frac{\partial^m}{\partial {x_1}^m} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 \left(\frac{\partial^m}{\partial {x_1}^m} t^{x_1}\right) t^{-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt.$

Substitute the value $\frac{\partial^m}{\partial {x_1}^m} t^{x_1} = t^{x_1} \ln^m (t)$ in the above equation, we obtain our desired result.

$\frac{\partial^m}{\partial {x_1}^m} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1} \ln^m (t)(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt.$

In a similar manner, we can obtain the remaining two parts. □

Theorem 10. *The derivative formula hold true.*

$\frac{\partial}{\partial r} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \left(\frac{1}{r}\right) \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \sum_{n=0}^{\infty} \frac{(-r)^n n}{t^n (1-t)^n \Gamma(r_1 n + r_2)} dt.$

Proof of Theorem 10. By differentiating the extended beta function (30) with respect to the variable r under the integral and applying the Leibniz rule for differentiation under integral sign, we have the following.

$\frac{\partial}{\partial r} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \frac{\partial}{\partial r} \int_0^1 t^{x_1-1}(1-t)^{x_2-1} E_{r_1,r_2}\left(-r(t(1-t))^{-1}\right) dt.$

Then, apply the definition of 2-parameter Mittag-Leffler function (21).

$\frac{\partial}{\partial r} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \frac{\partial}{\partial r} \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \sum_{n=0}^{\infty} \frac{(-r)^n}{t^n (1-t)^n \Gamma(r_1 n + r_2)} dt.$

$\frac{\partial}{\partial r} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \left(\frac{\partial}{\partial r}\right) \sum_{n=0}^{\infty} \frac{(-r)^n}{t^n (1-t)^n \Gamma(r_1 n + r_2)} dt.$

By differentiating the above series with respect to the r term and then placing the value $\frac{\partial}{\partial r}(-r)^n = -n(-r)^{n-1}$ in the above equation and rearranging the terms, we obtain our desired result.

$\frac{\partial}{\partial r} B^{(r)}_{(r_1,r_2)}(x_1, x_2) = \left(\frac{1}{r}\right) \int_0^1 t^{x_1-1}(1-t)^{x_2-1} \sum_{n=0}^{\infty} \frac{(-r)^n n}{t^n (1-t)^n \Gamma(r_1 n + r_2)} dt.$

□

3. Another Representation of an Extended Beta Function

We can represent an extended beta function defined as (30) by replacing 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$ with well known functions such as the H-function, G-function, confluent hypergeometric function, Wright–Fox function and Bessel–Maitland function.

For integers r, n, t, u such that $0 \le r \le u$, $0 \le n \le t$ and for parameters $a_i, b_i \in \mathbb{C}$ and for parameters $A_i, B_j \in \mathbb{R}_+ = (0, \infty)$ $(i = 1, \cdots, t; j = 1, \cdots, u)$, the H-function is defined in terms of a Mellin–Barnes type integral in the following manner [17–19]:

$$\begin{aligned} H^{r,n}_{t,u}\left[z \middle| \begin{matrix}(a_i, A_i)_{1,t} \\ (b_j, B_j)_{1,u}\end{matrix}\right] &= H^{r,n}_{t,u}\left[z \middle| \begin{matrix}(a_1, A_1), \cdots, (a_t, A_t) \\ (b_1, B_1), \cdots, (b_u, B_u)\end{matrix}\right] \\ &= \frac{1}{2\pi i} \int_{\mathfrak{L}} \mathcal{H}^{r,n}_{t,u}(s) z^{-s} \mathrm{d}s, \end{aligned} \quad (47)$$

where

$$\mathcal{H}^{r,n}_{t,u}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s)}, \quad (48)$$

and the contour $\mathfrak{L}$ is suitably chosen and an empty product, if it occurs, is taken to be unity.

A Mellin–Barnes type integral of 2-parameter Mittag-Leffler function $E_{r_1,r_2}(z)$ is defined as follows [13].

$$E_{r_1,r_2}(z) = \frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(r_2 - r_1 s)} z^{-s} ds, r_1 \in R^+, \Re(r_2) > 0, |\arg(z)| < \pi. \tag{49}$$

From the above, we have a relationship between H-function and 2-parameter Mittag-Leffler function.

$$E_{r_1,r_2}(z) = H^{1,1}_{1,2}\left[-z\middle|\begin{matrix}(0,1)\\(0,1),(1-r_2,r_1)\end{matrix}\right]. \tag{50}$$

Using above relationship, we can replace the 2-parameter Mittag-Leffler function by the H-function, defined as follows (30).

$$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} H^{1,1}_{1,2}\left[\frac{r}{t(1-t)}\middle|\begin{matrix}(0,1)\\(0,1),((1-r_2),r_1)\end{matrix}\right]dt. \tag{51}$$

Similarly, we have relation between Meijer's G-function defined as [17] and 2-parameter Mittag-Leffler function defined as (21).

$$E_{r_1,r_2}(z) = G^{1,1}_{1,2}\left[-z\middle|\begin{matrix}0\\0,(1-r_2)\end{matrix}\right]. \tag{52}$$

Now we can replace the 2-parameter Mittag-Leffler function from (30) by the G-function.

$$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} G^{1,1}_{1,2}\left[\frac{r}{t(1-t)}\middle|\begin{matrix}0,\\0,(1-r_2)\end{matrix}\right]dt. \tag{53}$$

Similarly we have a relation between the Wright–Fox function defined as [17] and 2-parameter Mittag-Leffler function defined as (21).

$$E_{r_1,r_2}(z) = {}_1W_1\left[z\middle|\begin{matrix}(1,1)\\(r_2,r_1)\end{matrix}\right]. \tag{54}$$

Now, we can replace the 2-parameter Mittag-Leffler function from (30) by the Wright–Fox function.

$$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1} {}_1W_1\left[\frac{-r}{t(1-t)}\middle|\begin{matrix}(1,1)\\(r_2,r_1)\end{matrix}\right]dt. \tag{55}$$

Similarly, we have relation between the 2-parameter Mittag-Leffler function (21) and the confluent hypergeometric function ${}_1F_1$ (7) when $r_1 = 1$.

$$E_{1,r_2}(z) = \frac{{}_1F_1(1;r_2;z)}{\Gamma(r_2).} \tag{56}$$

Now, we can replace the 2-parameter Mittag-Leffler function from (30) with the confluent hypergeometric function.

$$B^{(r)}_{(r_1,r_2)}(x_1,x_2) = \frac{1}{\Gamma(r_2)}\int_0^1 t^{x_1-1}(1-t)^{x_2-1} {}_1F_1\left(1;r_2;\frac{-r}{t(1-t)}\right)dt. \tag{57}$$

Similarly, we have a relation between the 2-parameter Mittag-Leffler function (21) and Bessel-Maitland function defined as follows [17].

$$E_{r_1,r_2}(z) = n! J^{r_1}_{(r_2-1)}(-z). \tag{58}$$

Now, we can replace the 2-parameter Mittag-Leffler function from (30) by the Bessel–Maitland function.

$$B^{(r)}_{(r_1,r_2)}(x_1, x_2) = n! \int_0^1 t^{x_1-1}(1-t)^{x_2-1} J^{r_1}_{(r_2-1)}\left(\frac{r}{t(1-t)}\right) dt. \tag{59}$$

4. Conclusions

We conclude our investigation by remarking that the results presented in this paper are easily converted by using the interesting known extension of the beta function and other special functions. The results presented here are new and very useful for the extension of other special functions. We are also trying to find certain possible applications of those results presented here for other research areas.

Author Contributions: Conceptualization, R.G. and P.A.; methodology, P.A.; software, R.G.; validation, R.G., M.T.R., P.A. and S.M.; formal analysis, J.M.; investigation, J.M.; resources, P.A.; data writing—original draft preparation, P.A.; writing—review and editing, R.G., M.T.R., P.A.; visualization, P.A.; project administration, P.A.; funding acquisition, P.A. All authors have read and agreed to the published version of the manuscript.

Funding: Praveen Agarwal was very thankful to the SERB (project TAR/2018/000001), DST (project DST/INT/DAAD/P-21/2019, INT/RUS/RFBR/308) and NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support. Rahul Goyal also thankful for the NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support.

Acknowledgments: Praveen Agarwal was very thankful to the SERB (project TAR/2018/000001), DST (project DST/INT/DAAD/P-21/2019, INT/RUS/RFBR/308) and NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support. Rahul Goyal also thankful for the NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Agarwal, P.; Choi, J. Certain unified integrals associated with Bessel functions. *Bound. Value Probl.* **2013**, *1*, 1–9.
2. Srivastava, H.M.; Agarwal, P.; Jain, S. Generating functions for the generalized Gauss hypergeometric functions. *Commun. Appl. Math. Comput.* **2014**, *247*, 348–352. [CrossRef]
3. Luo, M.J.; Milovanovic, G.V.; Agarwal, P. Some results on the extended beta and extended hypergeometric functions. *Commun. Appl. Math. Comput.* **2014**, *248*, 631-651. [CrossRef]
4. Agarwal, P.; Al-Mdallal, Q.; Cho, Y.J.; Jain, S. Fractional differential equations for the generalized Mittag-Leffler function. *Adv. Differ. Equ.* **2018**, *1*, 1–8. [CrossRef]
5. Jain, S.; Agarwal, R.P.; Agarwal, P.; Singh, P. Certain Unified Integrals Involving a Multivariate Mittag–Leffler Function. *Axioms* **2021**, *10*, 81. [CrossRef]
6. Agarwal, P.; Choi, J.; Jain, S. Extended hypergeometric functions of two and three variables. *Commun. Korean Math. Soc.* **2015**, *30*, 403–414. [CrossRef]
7. Choi, J.; Agarwal, P. Certain fractional integral inequalities involving hypergeometric operators. *East Asian Math. J.* **2014**, *30*, 283–291. [CrossRef]
8. Çetinkaya, A.; Agarwal, P.; Kıymaz, I.O.; Agarwal, R.P. A comparative study on generating function relations for generalized hypergeometric functions via generalized fractional operators. *Adv. Differ. Equ.* **2018**, *1*, 1–11. [CrossRef]
9. Rainville, E.D. *Special Functions*; Macmillan: New York, NY, USA, 1960.
10. Chaudhry, M.A.; Zubair, S.M. Generalized incomplete gamma functions with Applications. *J. Comput. Appl. Math.* **1994**, *55*, 99–124. [CrossRef]
11. Chaudhry, M.A.; Qadir, A.; Rafique, M.; Zubair, S.M. Extension of Euler's Beta function. *J. Comput. Appl. Math.* **1997**, *78*, 19–32. [CrossRef]
12. Özergin, E.; Özarslan, M.A.; Altın, A. Extension Gamma, Beta and Hypergeometric Functions. *J. Comput. Appl. Math.* **2011**, *235*, 4601–4610. [CrossRef]
13. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittage Leffler Functions Related Topics and Application*, 2nd ed.; Springer: New York, NY, USA, 2020. [CrossRef]
14. Pucheta, P.I. A new extended beta function. *Int. J. Math. Appl.* **2017**, *5*, 255–260. [CrossRef]
15. Shadab, M.S.; Jabee, S.J.; Choi, J.C. An extended Beta function and its applications. *Far East J. Math. Sci. (FJMS)* **2018**, *103*, 235–251.
16. Atash, A.A.; Barahmah, S.S.; Kulib, M.A. On a new extensions of extended Gamma and Beta functions. *Int. J. Stat. Appl. Math.* **2018**, *3*, 4–18.

17. Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-Function Theory and Applications*; Springer: New York, NY, USA, 2009. [CrossRef]
18. Mittag-Leffler, G.M. Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci.* **1903**, *137*, 554–558.
19. Wiman, A. Über die Nullstellun der Funktionen $E_\alpha(x)$. *Acta Math.* **1905**, *29*, 217–234.