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# $(\rho, \eta, \mu)$-Interpolative Kannan Contractions I

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**Abstract:** We point out a vital error in the paper of Gaba et al. (2019), showing that a $(\rho,\eta,\mu)$ interpolative Kannan contraction in a complete metric space need not have a fixed point. Then we give an appropriate restriction on a $(\rho,\eta,\mu)$-interpolative Kannan contraction that guarantees the existence of a fixed point and provide an equivalent formulation. Moreover, we show that this formulation can be extended to the interpolative Reich-Rus-Ćirić type contraction.

**Keywords:** $(\rho,\eta,\mu)$-interpolative Kannan contraction; fixed point; metric space.

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## 1. Introduction and Preliminaries

A mapping $T$ on a metric space $(X, d)$ is called Kannan if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X.$$

Kannan [1] proved that if $X$ is complete, then a Kannan mapping admits a fixed point. Please note that this well-known Kannan contraction that does not require a continuous mapping. Recently, Karapinar [2] proposed a new Kannan-type contractive mapping via the notion of interpolation and proved a fixed point theorem over metric space. The interpolative method has been used by several researchers to obtain generalizations of other forms of contractions [3–5]. This notion of interpolative contractions gives directions to investigate whether existing contraction inequalities can be redefined in this way or not. The purpose of this paper is to revisit the approach to attain a more general and less restrictive formultion of Karapinar's result [2]. Some examples are given to illustrate the new approach.

Throughout this manuscript, we denote an interpolative Kannan contraction by IKC and a $(\rho, \eta, \mu)$-interpolative Kannan contraction by $(\rho, \eta, \mu)$-IKC. The main result of Karapinar [2] is as follows:

**Theorem 1.** *([2] Theorem 2.2)*
*Let $(X, d)$ be a complete metric space and $T : X \to X$ be an interpolative Kannan type contraction, i.e., a self-map such that there are $\rho \in [0, 1)$ and $\eta \in (0, 1)$ so that*

$$d(Ta, Tb) \leq \rho d(a, Ta)^{\eta} d(b, Tb)^{1-\eta} \tag{1}$$

*for all $a, b \in X$ with $a \neq Ta$. Then $T$ has a unique fixed point in $X$.*

This theorem has been generalized in 2019 by Gaba et al. [6], where they initiated the notion of $(\rho, \eta, \mu)$-IKCs. In [6], the authors have defined $(\rho, \eta, \mu)$-IKC and proved a fixed point theorem for such mappings. The definition of that mapping is given as follows:

**Definition 1.** *(See [6]) Let $(X, d)$ a metric space and $T : X \to X$ be a self-map. We shall call $T$ a $(\rho, \eta, \mu)$-IKC or GIK (Gaba Interpolative Kannan) contraction, if there exist $0 \leq \rho < 1$ and $0 < \eta, \mu < 1$ with $\eta + \mu < 1$ such that*

$$d(Ta, Tb) \leq \rho\, d(a, Ta)^{\eta} d(b, Tb)^{\mu} \tag{2}$$

*whenever $a \neq Ta$ and $b \neq Tb$.*

**Theorem 2.** *(See [6]) Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $(\rho, \eta, \mu)$-IKC. Then $T$ has a fixed point in $X$.*

The interpolative strategy has been successfully applied to a variant types of contractions (see [7,8]). One of our goals in this paper is to show that Theorem 2 has a gap by giving an illustrated example. We will also give its proof correctly.

## 2. An Error in the Fixed Point Theorem for GIK Contractions

Theorem 2 is not true in general. The next example proves our assertion.

**Example 1.** *Let $X = \left\{\frac{1}{4}, \frac{1}{6}\right\}$ be endowed with the usual metric and $T : X \to X$ be given as*

$$T\left(\frac{1}{4}\right) = \frac{1}{6};\ T\left(\frac{1}{6}\right) = \frac{1}{4}.$$

*We have:*

$$0.0833 = \left|\frac{1}{4} - \frac{1}{6}\right| \leq \frac{3}{5} \cdot \left|\frac{1}{4} - \frac{1}{6}\right|^{(1/3)} \cdot \left|\frac{1}{6} - \frac{1}{4}\right|^{(1/3)} = 0.1144.$$

*Hence, $T$ is a GIK contraction with $\rho = \frac{3}{5}$ and $\eta = \mu = \frac{1}{3}$. Here, $X$ is complete, but $T$ has no fixed point in $X$.*

In the proof of Theorem 2 proposed by Gaba et al. in [6], the vital error emanated from the fact that the inequality, for the real numbers $a, \eta, \mu$ such that $0 < \eta \leq \mu$:

$$a^{\eta} \leq a^{\mu}$$

holds if and only if $a \geq 1, \eta \leq \mu$.

## 3. Revisiting the GIK Contraction Fixed Point Theorem

We provide an alternative formulation to the existence of $(\rho, \eta, \mu)$-IKCs.

**Theorem 3.** *(GIK fixed point revisited) Let $(X, d)$ be a complete metric space such that $d(a, b) \geq 1$ for $a \neq b$ and $T : X \to X$ be a GIK contraction. Then $T$ has a fixed point in $X$.*

**Proof.** Following the steps of the proof of [2] (Theorem 2.2), we build the sequence $(a_n)_{n \geq 1}$ of iterates $a_n = T^n a_0$, where $a_0 \in X$ is an arbitrary starting point. Without loss of generality, making the hypothesis that $a_{n+1} \neq a_n$ for each nonnegative integer $n$, we observe that

$$d(a_n, a_{n+1}) = d(Ta_{n-1}, Ta_n) \leq \rho\, d(a_{n-1}, a_n)^{\eta}\, d(a_n, a_{n+1})^{\mu},$$

i.e.,

$$d(a_n, a_{n+1})^{1-\mu} \le \rho\, d(a_{n-1}, a_n)^{\eta} \le \rho\, d(a_{n-1}, a_n)^{1-\mu}$$

since $\eta < 1 - \mu$ and $d(a_n, a_{n+1}) \ge 1$.

Similar to the proof of [2] (Theorem 2.2), the usual strategy ensures that there is a unique fixed point $a^* \in X$. □

**Example 2.** *(See [6] Example 1.) Take $X = \{a, b, z, w\}$. We equip with metric:*

| | $a$ | $b$ | $z$ | $w$ |
|---|---|---|---|---|
| $a$ | 0 | 5/2 | 2 | 5/2 |
| $b$ | 5/2 | 0 | 3/2 | 1 |
| $z$ | 4 | 3/2 | 0 | 3/2 |
| $w$ | 5/2 | 1 | 3/2 | 0 |

*Consider on $X$ the self-map $T$ given as $Ta = a$, $Tb = w$, $Tz = a$ and $Tw = b$.*
*We observed that the inequality:*

$$d(Ta, Tb) \le \rho d(a, Ta)^{\eta} d(b, Tb)^{\mu}$$

*is satisfied for:*

$$\eta = \frac{1}{8},\ \mu = \frac{3}{4},\ \rho = \frac{8}{9} \le \frac{9}{10};$$
$$\eta = \frac{1}{9},\ \mu = \frac{3}{4},\ \rho = \frac{8}{9} \le \frac{9}{10};$$
$$\eta = \frac{1}{8},\ \mu = \frac{4}{5},\ \rho = \frac{8}{9} \le \frac{9}{10}.$$

*In all above cases, $\eta + \mu < 1$, i.e., $\mu < 1 - \eta$ and the hypotheses of Theorem 3 are satisfied. Moreover, the map clearly possesses a unique fixed point.*

*On the other hand, when a metric $d$ is such that $d(a, b) \ge 1$ whenever $a \neq y$, the inequality*

$$d(Ta, Tb) \le \rho d(a, Ta)^{\eta} d(b, Tb)^{1-\eta}$$

*could just be replaced by the existence of two reals $\eta, \mu$ so that $\eta + \mu < 1$,*

$$d(Ta, Tb) \le \rho d(a, Ta)^{\eta} d(b, Tb)^{\mu}.$$

## 4. Equivalent GIK Formulations

Let $(X, d)$ be a metric space. Denote by $\Gamma(GIK)$ the set of all GIK contractions on $X$. For a mapping $T : X \to X$, $T$ is an $s$-GIK contraction if there are $0 \le \rho < 1, 0 < \eta, \mu < 1$ with $\eta + \mu < 1$ so that

$$d(Ta, Tb) \le \rho\, [d(a, Ta) d(b, Tb)]^{\frac{\eta+\mu}{2}}$$

whenever $a \neq Ta, b \neq Tb$.

Let us denote by $\tilde{\Gamma}(GIK)$ the set of all $s$-GIK contractions on $X$.

**Theorem 4.** *In a metric space $(X, d)$, such that $d(a, b) \ge 1$ for $a \neq b$, we have the equality*

$$\Gamma(GIK) = \tilde{\Gamma}(GIK).$$

**Proof.** Clearly,

$$\tilde{\Gamma}(GIK) \subset \Gamma(GIK)$$

since for any $s$-GIK contraction $T$, one has

$$d(Ta, Tb) \leq \rho \, [d(a, Ta)d(b, Tb)]^{\frac{\eta+\mu}{2}} \iff d(Ta, Tb) \leq \rho \, d(a, Ta)^{\frac{\eta}{2}} \, d(b, Tb)^{\frac{\mu}{2}}$$

and

$$\frac{\eta}{2} + \frac{\mu}{2} = \frac{\eta + \mu}{2} < \eta + \mu < 1.$$

Now, let $T \in \Gamma(GIK)$, so there are $0 \leq \rho < 1$ and $0 < \eta, \mu < 1$ with $\eta + \mu < 1$ so that

$$d(Ta, Tb) \leq \rho d(a, Ta)^{\eta} d(b, Tb)^{\mu} \tag{3}$$

whenever $a \neq Ta, b \neq Tb$.

Additionally, due to symmetry,

$$d(Ta, Tb) = d(Tb, Ta) \leq \rho d(b, Tb)^{\eta} d(a, Ta)^{\mu}. \tag{4}$$

Multiplying the inequalities (3) and (4), it follows that

$$d(Ta, Tb) \leq \rho \, [d(a, Ta)d(b, Tb)]^{\frac{\eta+\mu}{2}}. \tag{5}$$

□

So far, in our discussions regarding GIK contractions, we overlooked the case where $\eta + \mu = 1$. This case is actually central in the present investigation. Indeed, in the definition of a $(\rho, \eta, \mu)$-IKC, if we allow the sum $\eta + \mu$ to attain 1, one can see that the IKC in the sense of Karapinar [2] is a particular case of a GIK. In particular, we have:

**Definition 2.** *Let $(X, d)$ a metric space and $T : X \to X$ be a self-map. $T$ is called an extended $(\rho, \eta, \mu)$-IKC or extended GIK contraction, if there are $0 \leq \rho < 1$ and $0 < \eta, \mu < 1$ with $\eta + \mu \leq 1$ so that*

$$d(Ta, Tb) \leq \rho \, d(a, Ta)^{\eta} d(b, Tb)^{\mu} \tag{6}$$

*whenever $a \neq Ta$ and $b \neq Tb$.*

For a metric space $(X, d)$, let's denote by $e$-$\Gamma(GIK)$ the set of all extended GIK contractions on $X$. Moreover, if $\Gamma(IK)$ denotes the set of all interpolative Kannan type contractions, it is clear that:

**Corollary 1.** *In a metric space $(X, d)$, such that $d(a, b) \geq 1$ for $a \neq y$, we have*

$$\Gamma(IK) \subset e\text{-}\Gamma(GIK).$$

For a mapping $T : X \to X$, $T$ is an $s$-GIK contraction if there are $0 \leq \rho < 1$ and $0 < \eta, \mu < 1$ with $\eta + \mu < 1$ so that

$$d(Ta, Tb) \leq \rho \, [d(a, Ta)d(b, Tb)]^{\frac{\eta+\mu}{2}}$$

whenever $a \neq Ta$ and $b \neq Tb$.

Furthermore, if we plug $\eta + \mu = 1$ in (5), we achieve

$$d(Ta, Tb) \leq \rho \, [d(a, Ta)d(b, Tb)]^{\frac{1}{2}}, \tag{7}$$

which naturally leads to

**Corollary 2.**

$$T : X \to X \in \Gamma(IK)$$

$$\Updownarrow$$

*$T : X \to X$ and there is $0 \leq \rho < 1$ so that*

$$d(Ta, Tb) \leq \rho \, [d(a, Ta) d(b, Tb)]^{\frac{1}{2}}$$

*whenever $a \neq Ta$ and $b \neq Tb$.*

## 5. GI-RRC Contractions

As an extension of interpolative Kannan-type contractive mappings, Karapinar et al. introduced Interpolative Reich-Rus-Ćirić type contractions (see [9]). The definition is given below:

**Definition 3.** *([9]) In a metric space $(X, m)$, a mapping $T : X \to X$ is called an interpolative Reich-Rus-Ćirić type contraction if it satisfies*

$$m(Ta, Tb) \leq \rho [d(a, y)]^{\mu} [m(a, Ta)]^{\eta} [m(b, Tb)]^{1-\eta-\mu}$$

*for all $a, b \in X \setminus Fix(T) = \{\sigma \in X : T\sigma = \sigma\}$ for some $\rho \in [0, 1)$ and for $\eta, \mu \in (0, 1)$.*

**Theorem 5.** *([9]) Let $(X, d)$ be a complete metric space and $T : X \to X$ be an interpolative Reich-Rus-Ćirić type contraction mapping. Then $T$ has a fixed point in $X$.*

In the present paper, we introduce the concept of $(\rho, \eta, \mu, \gamma)$-interpolative Reich-Rus-Ćirić type contractions, which we also call GI-RRC contractions.

**Definition 4.** *Let $(X, d)$ a metric space and $T : X \to X$ be a self-map. $T$ is named a $(\rho, \eta, \mu)$-Reich-Rus-Ćirić contraction or GI-RRC (Gaba Interpolative Reich-Rus-Ćirić) contraction, if there exist $0 \leq \rho < 1, 0 < \eta, \mu, \mu < 1$ with $\eta + \mu + \gamma < 1$ such that*

$$d(Ta, Tb) \leq \rho [d(a, y)]^{\eta} [d(a, Ta)]^{\mu} [d(b, Tb)]^{\gamma} \tag{8}$$

*for all $a, b \in X \setminus Fix(T)$.*

Let us denote by $\Gamma(GI - RRC)$ the set of all GI-RRC contractions on $X$. A mapping $T : X \to X$, $T$ is an $s$-GI-RRC contraction if there exist $0 \leq \rho < 1, 0 < \eta, \mu, \gamma < 1$ with $\eta + \mu + \gamma < 1$ such that

$$d(Ta, Tb) \leq \rho \, d(a, b)^{\eta} [d(a, Ta) d(b, Tb)]^{\frac{\mu+\gamma}{2}}$$

whenever $a \neq Ta$ and $b \neq Tb$.

Let us denote by $\tilde{\Gamma}(GI - RRC)$ the set of all $s$-GI-RRC contractions on $X$.

**Theorem 6.** *In a metric space $(X, d)$, such that $d(a, b) \geq 1$ for $a \neq b$, we have the equality*

$$\Gamma(GI - RRC) = \tilde{\Gamma}(GI - RRC).$$

**Proof.** Clearly,

$$\tilde{\Gamma}(GI - RRC) \subset \Gamma(GI - RRC)$$

since for any $s$-GI-RRC contraction $T$, one has

$$d(Ta, Tb) \leq \rho \, d(a, b)^{\eta} [d(a, Ta) d(b, Tb)]^{\frac{\mu+\gamma}{2}} \iff d(Ta, Tb) \leq \rho d(a, b)^{\eta} \, d(a, Ta)^{\frac{\eta}{2}} \, d(b, Tb)^{\frac{\mu}{2}}$$

and

$$\frac{\mu}{2} + \frac{\gamma}{2} = \frac{\mu + \gamma}{2} < \mu + \gamma < 1.$$

Now, let $T \in \Gamma(GI - RRC)$, so there are $0 \leq \rho < 1, 0 < \eta, \mu, \gamma < 1$ with $\eta + \mu + \gamma < 1$ so that

$$d(Ta, Tb) \leq \rho d(a, b)^{\eta} d(a, Ta)^{\mu} d(b, Tb)^{\gamma} \tag{9}$$

whenever $a \neq Ta, b \neq Tb$.

Additionally, due to symmetry,

$$d(Ta, Tb) = d(Tb, Ta) \leq \rho d(b, a)^{\eta} d(b, Tb)^{\mu} d(a, Ta)^{\gamma}. \tag{10}$$

Multiplying the inequalities (9) and (10), it follows that

$$d(Ta, Tb) \leq \rho\, d(a, b)^{\eta} [d(a, Ta) d(b, Tb)]^{\frac{\mu+\gamma}{2}}. \tag{11}$$

□

To include the interpolative Reich-Rus-Ćirić type contraction in our study, we allow $\eta + \mu + \gamma = 1$ in the following definition:

**Definition 5.** *Let $(X, d)$ be a metric space and $T : X \to X$ be a self-map. $T$ is named an extended $(\rho, \eta, \mu)$-Reich-Rus-Ćirić contraction or extended GI-RRC contraction, if there exist $0 \leq \rho < 1, 0 < \eta, \mu, \mu < 1$ with $\eta + \mu + \gamma \leq 1$ such that*

$$d(Ta, Tb) \leq \rho [d(a, y)]^{\eta} [d(a, Ta)]^{\mu} [d(b, Tb)]^{\gamma} \tag{12}$$

*for all $a, b \in X \setminus Fix(T)$.*

For a metric space $(X, d)$, let's denote by $e$-$\Gamma(GI - RRC)$ the set of all extended GI-RRC contractions on $X$. Moreover, if $\Gamma(GI - RRC)$ denotes the set of all interpolative Kannan type contractions, then it is clear that:

**Corollary 3.** *In a metric space $(X, d)$ so that $d(a, b) \geq 1$ for $a \neq b$, we have*

$$\Gamma(GI - RRC) \subset e\text{-}\Gamma(GI - RRC).$$

For a mapping $T : X \to X$, $T$ is an $s$-GI-RRC contraction if $0 \leq \rho < 1, 0 < \eta, \mu, \gamma < 1$ with $\eta + \mu + \gamma < 1$ such that

$$d(Ta, Tb) \leq \rho\, d(a, b)^{\eta} [d(a, Ta) d(b, Tb)]^{\frac{\mu+\gamma}{2}}$$

whenever $a \neq Ta, b \neq Tb$.

Furthermore, if we plug $\eta + \mu + \gamma = 1$ in (11), we achieve

$$d(Ta, Tb) \leq \rho\, [d(a, Ta) d(b, Tb)]^{\frac{1}{2}}, \tag{13}$$

which naturally leads to:

**Corollary 4.**

$$T : X \to X \in \Gamma(GI - RRC)$$

$$\Updownarrow$$

*$T : X \to X$ and there exists $0 \leq \rho < 1$ such that*

$$d(Ta, Tb) \leq \rho d(a, b)^{\eta} [d(a, Ta) d(b, Tb)]^{\frac{1-\eta}{2}}$$

*whenever $a \neq Ta, b \neq Tb$.*

## 6. Conclusions

In this paper, we provided conditions under which a $(\rho, \eta, \mu)$-IKC on a complete metric space can lead a fixed point. Moreover, we show how this new formulation can be extended to the interpolative Reich-Rus-Ćirić type contraction. The authors' plan is, in another manuscript (part 2 of the present manuscript), to enlarge the scope of this new formulation to the frame of different type of interpolative contractions.

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