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Calculations on Matrix Transformations Involving an Infinite Tridiagonal Matrix

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Abstract: Given any sequence $z = (z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/z = (y_n/z_n)_{n\geq 1} \in E$; in particular, $\mathbf{s}_z^0$ denotes the set of all sequences y such that y/z tends to zero. Here, we consider the infinite tridiagonal matrix $\widetilde{B(r,s,t)}$, obtained from the triangle $B(r,s,t)$, by deleting its first row. Then we determine the sets of all positive sequences $a = (a_n)_{n\geq 1}$ such that $(E_a)_{\widetilde{B(r,s,t)}} \subset E_a$, where $E = \ell_\infty$, c_0, or c. These results extend some recent results.

Keywords: matrix transformations; BK space; (SSIE) with operator; triple band matrix

MSC: 40H05, 46A45

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1. Introduction

As usual, we denote by ω the set of all complex sequences $y = (y_n)_{n\geq 1}$ and by c_0, c and ℓ_∞ the subsets of all null, convergent and bounded sequences, respectively. Also let U^+ denote the set of all sequences $u = (u_n)_{n\geq 1}$ with $u_n > 0$ for all n. Given a sequence $a \in \omega$ and a subset E of ω, Wilansky [1], introduced the notation $a^{-1} * E = \{y \in \omega : ay = (a_n y_n)_{n\geq 1} \in E\}$. We write $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ for the sets $\left((1/a_n)_{n\geq 1}\right)^{-1} * E$ for any sequence $a \in U^+$ and $E \in \{\ell_\infty, c_0, c\}$. In [2], we gathered some results on the (SSIE) and the (SSE), defined as follows. The *sequence spaces inclusion equations (SSIE) and sequence spaces equations (SSE) with operators* are determined by an inclusion or identity each term of which is a *sum* or a *sum of products of sets of the form* $(\chi_a)_T$ *and* $\left(\chi_{f(x)}\right)_T$, where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, a is a given sequence in U^+, x is the unknown, f maps U^+ to itself and T is a triangle. In [2], we dealt with the class of (SSIE) of the form $F \subset E_a + F'_x$, where $F \in \{c_0, \ell^p, w_0, w_\infty\}$ and $E, F' \in \{c_0, c, \ell_\infty, \ell^p, w_0, w_\infty\}$, $(p \geq 1)$. In [3], Altay and Başar defined the *generalized operator of the first difference* defined by $B(r,s)_n y = r y_n + s y_{n-1}$ for all $n \geq 2$ and $B(r,s)_1 y_1 = r y_1$. Then, these authors dealt with the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c. In [4], Kirişçi and Başar gave characterizations of the classes $\left(E_{B(r,s)}, F\right)$ and $\left(E_{B(r,s)}, F_{B(r,s)}\right)$ where E is any of the spaces ℓ_∞, c, c_0, ℓ_p, or ℓ_1 and F is any of the spaces ℓ, c, c_0, or ℓ_1. In [5], the authors dealt with the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$. Then, in 2007 Furkan, Bilgic and Altay [6], dealt with the spectrum of the operator represented by the triangle

$$B(r,s,t) = \begin{pmatrix} r & & & \\ s & r & & 0 \\ t & s & r & \\ & \cdot & \cdot & \cdot \\ 0 & & \cdot & \cdot & \cdot \end{pmatrix}$$

over c_0 and c. In [7], Bilgic and Furkan dealt with the fine spectrum of $B(r,s,t)$ over the sequence spaces l_1 and bv. Finally, in 2010 Furkan, Bilgic and Başar [8], studied the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p.

In this paper, we extend some results stated in [9], and we consider the infinite matrix $\Lambda = \widetilde{B(r,s,t)}$ obtained from $B(r,s,t)$ by deleting its first row which is not a triangle, but an infinite tridiagonal matrix. The main results are stated in Sections 6 and 7, where we give some new characterizations of the inclusions $(\chi_a)_{\widetilde{B(r,s,t)}} \subset \chi_a$ where $\chi = \mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. We extend some results stated in [9] with the study of the cases (1) $s = 0$ and r, $t \neq 0$, (2) $r = 0$ and $s, t \neq 0$, and (3) $t = 0$ and $r, s \neq 0$. Then, we characterize the set of all positive sequences a such that $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_a^{(c)}$. So, we give some conditions, under which the condition $\lim_{n\to\infty}(ry_{n+1} + sy_n + ty_{n-1})/a_n = l$ implies $\lim_{n\to\infty} y_n/a_n = l'$, for all y and for some scalars l, l'.

This paper is organized as follows. In Section 2, we recall some results on AK and BK spaces and on the set $S_{a,b}$. In Section 3, we consider the operator $C(a)$, and recall the definitions and properties of the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C_1}$. Then we state some properties of the set $\widehat{c}$. In Section 4, we recall the inverse of $B(r,s,t)$. In Section 5, we state some characterizations of the sets of all positive sequences a such that $(\chi_a)_{\widetilde{B(r,s,t)}} \subset \chi_a$, where $\chi = \mathbf{s}$, or $\mathbf{s}^0$ in the general case. In Section 6, using the sets of the form $\widehat{C}_\alpha$, we give additional characterizations in each of the cases 1) $(\chi_a)_{\widetilde{B(r,s,t)}} \subset \chi_a$, with $\mathbf{\Delta} \geq 0$. 2) $(\chi_a)_{\widetilde{B(r,s,0)}} \subset \chi_a$ and $(\chi_a)_{\widetilde{B(0,s,t)}} \subset \chi_a$, and 3) $\mathbf{\Delta} < 0$. Finally, in Section 7, we extend the previous results to the set $\widehat{\mathcal{S}^{(c)}}$ of all positive sequences a such that $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_a^{(c)}$. Then, under some conditions we give simplifications of the previous set.

2. Notations and Preliminary Results

Let $A = (a_{nk})_{n,k\geq 1}$ be an infinite matrix and $y = (y_k)_{k\geq 1}$ be a sequence. Then, we write

$$A_n y = \sum_{k=1}^{\infty} a_{nk} y_k, \text{ for any integer } n \geq 1 \tag{1}$$

and $Ay = (A_n y)_{n\geq 1}$ provided all the series in (1) converge. Let E and F be any subsets of ω. Then, we write (E,F), (see for instance [10]), for the class of all infinite matrices A for which the series in (1) converge for all $y \in E$ and all n, and $Ay \in F$ for all $y \in E$. So, if $A \in (E,F)$ then we are led to the study of the operator $\Lambda = \Lambda_A : E \to F$ defined by $Ay = \Lambda y$ and we identify the operator Λ to the matrix A. A Banach space E of complex sequences is said to be a *BK space* if each projection $P_n : E \to \mathbb{C}$ defined by $P_n(y) = y_n$ for all $y = (y_n)_{n\geq 1} \in E$ is continuous. A BK space E is said to have *AK* if every sequence $y = (y_k)_{k\geq 1} \in E$ has a unique representation $y = \sum_{k=1}^{\infty} y_k e^{(k)}$ where $e^{(k)}$ is the sequence with 1 in the k-th position and 0 otherwise. To simplify the notations, we use the diagonal matrix D_a defined by $[D_a]_{nn} = a_n$ for all n, and write

$$D_a * E = (1/a)^{-1} * E = \{(y_n)_{n\geq 1} \in \omega : (y_n/a_n)_{n\geq 1} \in E\},$$

for any $a \in U^+$ and any $E \subset \omega$. We may also write the identity $E_a = D_a * E$. Then, we define $\mathbf{s}_a = D_a * \ell_\infty$, $\mathbf{s}_a^0 = D_a * c_0$ and $\mathbf{s}_a^{(c)} = D_a * c$. Each of the spaces $D_\alpha * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|y\|_{\mathbf{s}_a} = \sup_{n\geq 1}(|y_n|/a_n)$ and $\mathbf{s}_a^0$ has AK. Now, let $a = (a_n)_{n\geq 1}, b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of all infinite matrices $\Lambda = (\lambda_{nk})_{n,k\geq 1}$ such that $\|\Lambda\|_{S_{a,b}} = \sup_{n\geq 1}\left(b_n^{-1}\sum_{k=1}^{\infty}|\lambda_{nk}|a_k\right) < \infty$. It is well known that $\Lambda \in (\mathbf{s}_a, \mathbf{s}_b)$ if and only if $\Lambda \in S_{a,b}$. So, we can write $(\mathbf{s}_a, \mathbf{s}_b) = S_{a,b}$. When $\mathbf{s}_a = \mathbf{s}_b$ we obtain the *Banach algebra with identity* $S_{a,b} = S_a$, (cf. [2]), normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$. We also have $\Lambda \in (\mathbf{s}_a, \mathbf{s}_a)$ if and only if $\Lambda \in S_a$. If $a = (r^n)_{n\geq 1}$, the sets S_a, $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ are denoted by S_r, $\mathbf{s}_r$, $\mathbf{s}_r^0$ and $\mathbf{s}_r^{(c)}$, respectively. When $r = 1$, we obtain $\mathbf{s}_1 = \ell_\infty$, $\mathbf{s}_1^0 = c_0$ and $\mathbf{s}_1^{(c)} = c$, and writing $e = (1, 1, ...)$ we have $S_1 = S_e$. It is well known that $(\mathbf{s}_1, \mathbf{s}_1) = (c_0, \mathbf{s}_1) = (c, \mathbf{s}_1) = S_1$ (see, for instance, [1]). We also have $\Lambda \in (c_0, c_0)$ if and only if $\Lambda \in S_1$ and $\lim_{n\to\infty}\lambda_{nk} = 0$ for $k = 1, 2, ...$; and $\Lambda \in (c, c)$ if and only if $\Lambda \in S_1$, $\lim_{n\to\infty}\sum_{k=1}^{\infty}\lambda_{nk} = l$ and $\lim_{n\to\infty}\lambda_{nk} = l_k$, for some scalars l and l_k, $k = 1, 2, \ldots$ In the sequel, we use the next property. Let χ and χ' be any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$, then the condition $\Lambda \in (\chi_a, \chi'_b)$ and $D_{1/b}\Lambda D_a \in (\chi_e, \chi'_e)$ are equivalent. For any subset E of ω, we put $\Lambda E = \{\eta \in \omega : \eta = \Lambda y \text{ for some } y \in E\}$. If F is a subset of ω, then we write $F(\Lambda) = F_\Lambda = \{y \in \omega : \Lambda y \in F\}$ for the matrix domain of Λ *in* F.

3. The Operators $C(a)$, $\mathbf{\Delta}(a)$ and the Sets $\widehat{\Gamma}$, $\widehat{C}$, Γ, $\widehat{C_1}$ and $\widehat{c}$

An infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. Now, let U be the set of all sequences $(u_n)_{n\geq 1} \in \omega$ with $u_n \neq 0$ for all n. If $a = (a_n)_{n\geq 1} \in U$, we define by $C(a)$ the triangle defined by $[C(a)]_{nk} = 1/a_n$ for $k \leq n$, (see, for instance, [2] (p. 166)). It is easy to see that the triangle $\mathbf{\Delta}(a)$, whose the nonzero entries are defined by $[\mathbf{\Delta}(a)]_{nn} = a_n$ and $[\mathbf{\Delta}(a)]_{n,n-1} = a_{n-1}$, is the inverse of $C(a)$, that is, $C(a)(\mathbf{\Delta}(a)y) = \mathbf{\Delta}(a)(C(a)y) = y$ for all $y \in \omega$. If $a = e$ then we obtain $\mathbf{\Delta}(e) = \mathbf{\Delta}$, where $\mathbf{\Delta}$ is the well–known operator of the first difference defined by $\mathbf{\Delta}_n y = y_n - y_{n-1}$ for all $y \in \omega$ and all $n \geq 1$, with the convention $y_0 = 0$. It is usual to write $\Sigma = C(e)$. We note that $\mathbf{\Delta}$ and Σ are inverse to one another, and $\mathbf{\Delta}, \Sigma \in S_R$ for any $R > 1$.

To simplify notation, for $a \in U^+$, we write $c_n(a) = a_n^{-1}\sum_{k=1}^{n} a_k$, for all n. We also consider the sets $\widehat{C}$ and $\widehat{C_1}$ of all positive sequences $a = (a_n)_{n\geq 1}$ such that $(c_n(a))_{n\geq 1} \in c$, $\sup_n c_n(a) < \infty$, respectively. It is known that, $\lim_{n\to\infty} a_n^\bullet = 1 - 1/l$ holds if and only if $\lim_{n\to\infty} a_n^{-1}\sum_{k=1}^{n} a_k = l$, for some scalar $l > 0$. In all that follows, we associate with any positive sequence a the sequence a^- defined by $[a^-]_n = a_{n-1}$ for all $n \geq 1$ with the convention $[a^-]_1 = a_0 = 1$. We write $a^\bullet = (a_n^\bullet)_{n\geq 1}$, where $a_n^\bullet = [a^-]_n/a_n$ and we let $\widehat{c} = \{a \in U^+ : a^\bullet \in c\}$. We define by $\widehat{\Gamma}$ and Γ the sets of all positive sequences such that $\lim_{n\to\infty} a_n^\bullet < 1$ and $\limsup_{n\to\infty} a_n^\bullet < 1$. Finally, by G_1 we define the set of all positive sequences such that $a_n \geq C\gamma^n$, for all n, and for some $C > 0$ and $\gamma > 1$. Note that, if a and $b \in \widehat{C_1}$, then we have $a + b$ and $ab \in \widehat{C_1}$. It can easily be seen that $(R^n)_{n\geq 1} \in \widehat{C_1}$ if and only if $R > 1$, and there is no real number α for which the sequence $(n^\alpha)_{n\geq 1}$ belongs to $\widehat{C_1}$. It is known that $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1} \subset G_1$, (cf. [2]). Now, we need the following lemmas.

Lemma 1. *We have* $\widehat{C_1} \cap \widehat{c} = \widehat{\Gamma}$.

Proof. The inclusion $\widehat{\Gamma} \subset \widehat{C_1} \cap \widehat{c}$ is immediate. So, we only need to show the inclusion $\widehat{C_1} \cap \widehat{c} \subset \widehat{\Gamma}$. For this, we assume $a \notin \widehat{\Gamma}$, under the condition $a \in \widehat{c}$. Then we have $\lim_{n\to\infty} a_n^\bullet \geq 1$. So, for any given $\varepsilon > 0$ there is an integer $q > 0$ such that $a_n^\bullet \geq 1 - \varepsilon$ for all $n \geq q + 1$ and

$$\begin{aligned} c_{2q}(a) &= \frac{1}{a_{2q}}\sum_{k=1}^{2q} a_k \geq \frac{1}{a_{2q}}\left(\sum_{k=q}^{2q} a_k\right) \geq \sum_{k=q}^{2q-1}\left(\frac{a_k}{a_{k+1}}\cdots\frac{a_{2q-1}}{a_{2q}}\right) + 1 \\ &\geq (1-\varepsilon)^q + \ldots + (1-\varepsilon) + 1 = \frac{1-(1-\varepsilon)^{q+1}}{\varepsilon}. \end{aligned}$$

Then we have

$$\frac{1-(1-\varepsilon)^{q+1}}{\varepsilon} \sim \frac{1-[1-(q+1)\varepsilon]}{\varepsilon} \sim q+1 \ (\varepsilon \to 0)$$

and $\left([C(a)a]_{2q}\right)_q \notin \ell_\infty$ which implies $a \notin \widehat{C_1}$. So, we have shown $\widehat{C_1} \cap \widehat{c} \subset \widehat{\Gamma}$ and Part (ii) holds. This completes the proof. □

Lemma 2 ([2], Theorem 4.2, p.172)**.** *for each $a \in \omega$ we have $a \in \widehat{\Gamma}$ if and only if $\left(\mathbf{s}_a^{(c)}\right)_{\boldsymbol{\Delta}} = \mathbf{s}_a^{(c)}$.*

Lemma 3. *Let $a \in U^+$. Then we have: $\lim_{n\to\infty} a_n^\bullet < 1$ implies*

$$\lim_{n\to\infty} a_n^{-1} \sum_{k=2}^{n} (n-k+1)a_{k-1} = l, \text{ for some scalar } l.$$

Proof. By Lemma 2, the condition $a \in \widehat{\Gamma}$ implies $\left(s_a^{(c)}\right)_{\boldsymbol{\Delta}} = s_a^{(c)}$ and $\left(s_a^{(c)}\right)_{\boldsymbol{\Delta}^2} = \left(\left(s_a^{(c)}\right)_{\boldsymbol{\Delta}}\right)_{\boldsymbol{\Delta}} = \left(s_a^{(c)}\right)_{\boldsymbol{\Delta}} \subset s_a^{(c)}$. Since $\left(s_a^{(c)}\right)_{\boldsymbol{\Delta}^2} = \Sigma^2 s_a^{(c)}$, the condition $a \in \widehat{\Gamma}$ implies $D_{1/a}\Sigma^2 D_a \in (c,c)$. Now, the matrix $D_{1/a}\Sigma^2 D_a$ is the triangle defined by $\left[D_{1/a}\Sigma^2 D_a\right]_{nk} = a_n^{-1}(n-k+1)a_k$ for $k \leq n$, and we conclude that $a \in \widehat{\Gamma}$ implies

$$\lim_{n\to\infty} a_n^{-1} \sum_{k=1}^{n} (n-k+1)a_k = l.$$

Finally, from the inclusion $\widehat{\Gamma} \subset \widehat{C_1}$, we obtain

$$\begin{aligned} \frac{1}{a_n}\sum_{k=2}^{n}(n-k+1)a_{k-1} &= \frac{1}{a_n}\sum_{j=1}^{n}(n-j)a_j \\ &= \frac{1}{a_n}\sum_{j=1}^{n-1}(n-j+1)a_j - \frac{1}{a_n}\sum_{j=1}^{n-1}a_j = O(1) \ (n \to \infty). \end{aligned}$$

This concludes the proof. □

4. The Inverse of the Triangle $B(r,s,t)$

In the following, we use the triangle $B(r,s,t)$ which can be considered as the operator defined by $(B(r,s,t)y)_1 = ry_1$, $(B(r,s,t)y)_2 = ry_2 + sy_1$ and $(B(r,s,t)y)_n = ry_n + sy_{n-1} + ty_{n-2}$ for all $n \geq 3$, where r, s, t are real numbers. Throughout this paper, we assume that two reals among the reals r, s, t are nonzero. We associate with the matrix $B(r,s,t)$ the equation

$$b(u) = tu^2 + su + r = 0. \tag{2}$$

We denote by u_1 and u_2 the roots of (2). In the case $r, t \neq 0$ the roots of (2) are distinct from zero. We have the following result, where we let $\boldsymbol{\Delta}= s^2 - 4tr$, which was stated in [6], and rewritten in [9].

Lemma 4 ([9])**.** *Let r, s, t be reals with $r, t \neq 0$. Then, the inverse of $B(r,s,t)$ is a triangle whose the nonzero entries are defined for $k \leq n$, in the following way.*

(i) *If $\boldsymbol{\Delta}\neq 0$ then $u_1 = \left(-s-\sqrt{\boldsymbol{\Delta}}\right)/2t$ and $u_2 = \left(-s+\sqrt{\boldsymbol{\Delta}}\right)/2t$ are the real or complex roots of (2) and we have:*

(a) $\left([B(r,s,t)]^{-1}\right)_{nk} = -\left(u_2^{k-n-1} - u_1^{k-n-1}\right)/\sqrt{\boldsymbol{\Delta}}$, *for* $\boldsymbol{\Delta} > 0$.

(b) $\left([B(r,s,t)]^{-1}\right)_{nk} = i\left(u_2^{k-n-1} - u_1^{k-n-1}\right)/\sqrt{-\boldsymbol{\Delta}}$, *for* $\boldsymbol{\Delta} < 0$.

(ii) *If* $\Delta = 0$ *then* $u_1 = -s/2t$, *is the double root of* (2) *and the non-zero entries of the inverse of* $B(r,s,t)$ *are defined by* $\left([B(r,s,t)]^{-1}\right)_{nk} = r^{-1}(n-k+1)u_1^{k-n}$.

(iii) *Assume* $\Delta < 0$, *and let* $u_1 = \rho e^{i\theta}$, *be a root of* (2). *Then, the inverse of* $B(r,s,t)$ *is the triangle whose the entries are given by*

$$\left([B(r,s,t)]^{-1}\right)_{nk} = \frac{1}{r}\frac{\sin(n-k+1)\theta}{\rho^{n-k}\sin\theta} \text{ for } k \leq n.$$

5. On the Sets $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$

In the following, we use the infinite tridiagonal matrix $B\widetilde{(r,s,t)}$ obtained from $B(r,s,t)$ by deleting its first row. For $r = 0$ the matrix $B\widetilde{(r,s,t)}$ is the double band matrix denoted by $B(s,t)$. In this section, we recall the characterizations of the set of all $a \in U^+$ such that $(\chi_a)_{B\widetilde{(r,s,t)}} \subset \chi_a$, with $\chi = s$, or $\mathbf{s}^0$ in the case $\Delta \neq 0$. Let

$$\widehat{\mathcal{S}} = \left\{a \in U^+ : (\mathbf{s}_a)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_a\right\} \text{ and } \widehat{\mathcal{S}^0} = \left\{a \in U^+ : \left(\mathbf{s}_a^0\right)_{B\widetilde{(r,s,t)}} \subset \mathbf{s}_a^0\right\}.$$

Then we have $a \in \widehat{\mathcal{S}}$ if and only if the condition $|ry_{n+1} + sy_n + ty_{n-1}|/a_n \leq K_1$ implies $|y_n|/a_n \leq K_2$ for all y, for all n and for some K_1 and $K_2 > 0$. Similarly, we have $a \in \widehat{\mathcal{S}^0}$ if and only if the condition $(ry_{n+1} + sy_n + ty_{n-1})/a_n \to 0$ implies $y_n/a_n \to 0$ $(n \to \infty)$ for all y. In the following, we recall some results on the characterizations of $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$ stated in [9]. We begin with the characterizations of $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$ in the case $\Delta \neq 0$ and we consider the conditions:

$$\sup_n\left(\frac{1}{a_n}\sum_{k=1}^{n}\left|u_2^{k-n-1} - u_1^{k-n-1}\right|a_{k-1}\right) < \infty \tag{3}$$

and

$$\lim_{n\to\infty}\frac{1}{a_n}\left(u_2^{k-n-1} - u_1^{k-n-1}\right)a_{k-1} = 0 \ (n \to \infty) \text{ for } k = 1,2,\dots. \tag{4}$$

Using the identity $(\chi_a)_{B\widetilde{(r,s,t)}} = (\chi_{a^-})_{B(r,s,t)}$ for $\chi = s$, or $\mathbf{s}^0$, (cf. [9]), we obtain the following proposition, where we assume $\Delta \neq 0$, the case $\Delta = 0$ is studied in Part (ii) of Theorem 1.

Proposition 1 ([9]). *Let* r, s, t *be reals with* $r, t \neq 0$. *Assume* $\Delta \neq 0$ *and let* u_1 *and* u_2 *be the roots of* (2). *Then we have: (i)* $a \in \widehat{\mathcal{S}}$ *if and only if* (3) *holds. (ii)* $a \in \widehat{\mathcal{S}^0}$ *if and only if* (3) *and* (4) *hold.*

6. New Characterizations of the Sets $\widehat{\mathcal{S}}$, or $\widehat{\mathcal{S}^0}$

In the following, we extend some results on the characterizations of $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$ stated in [9]. For this, we let $\chi = s$, or $\mathbf{s}^0$, and we simplify these characterizations using the sets of the form $\widehat{C}_\alpha = \left[\widehat{C_1}\right]_{|\alpha|}$, for $\alpha \neq 0$, in each of the cases (1) $(\chi_a)_{B\widetilde{(r,s,t)}} \subset \chi_a$, with $\Delta \geq 0$, (2) $(\chi_a)_{B\widetilde{(r,s,0)}} \subset \chi_a$ and $(\chi_a)_{B\widetilde{(0,s,t)}} \subset \chi_a$, and (3) $\Delta < 0$.

6.1. Characterizations of $(\chi_a)_{B\widetilde{(r,s,t)}} \subset \chi_a$ *Where* $\chi = s$, *or* $\mathbf{s}^0$ *for* $\Delta \geq 0$.

For any nonzero real number α, we write

$$\widehat{C}_\alpha = D_{\left(|\alpha|^n\right)_{n\geq 1}} * \widehat{C}_1 = \left\{a \in U^+ : \sup_{n\geq 1}\left(\frac{|\alpha|^n}{a_n}\sum_{k=1}^{n}\frac{a_k}{|\alpha|^k}\right) < \infty\right\}.$$

Note that $\widehat{C}_{\alpha} = \widehat{C}_{|\alpha|}$. It is trivial that, if a and $a' \in \widehat{C}_{\alpha}$ then we have $a + a' \in \widehat{C}_{\alpha}$. We obtain the following extension of the results stated in [9], since we only dealt with the sets $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$, for $\boldsymbol{\Delta} > 0$, in the case $s \neq 0$.

Theorem 1. *Let $r, t \neq 0$. Then we have:*

(i) *Assume $\boldsymbol{\Delta} > 0$ and let $u_1 \neq u_2$ be the roots of (2). Then*

$$\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}.$$

(ii) *Assume $\boldsymbol{\Delta} = 0$, and let u_1 be the double root of (2). Then $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/|u_1|}$.*

Proof. Statement (i) with $s \neq 0$ and (ii) were shown in [9]. It remains to study the case $s = 0$ for $\boldsymbol{\Delta} > 0$, where the polynomial associated with the matrix $B(r, 0, t)$ is $b(u) = tu^2 + r$. The equation $tu^2 + r = 0$ has two roots u_1 and $-u_1$ where $u_1 = \sqrt{-r/t}$ if $rt < 0$ and $u_1 = i\sqrt{r/t}$ if $rt > 0$. Then we have $a \in \widehat{\mathcal{S}}$ if and only if (3) holds, and the condition in (3) is equivalent to

$$\frac{1}{a_n} \sum_{k=1}^{n} \left| u_1^{k-n-1} \right| \left| 1 - (-1)^{k-n-1} \right| a_{k-1} = O(1) \ (n \to \infty). \tag{5}$$

The sequence $\left| 1 - (-1)^{k-n-1} \right|$ is nonzero only if $n - k$ is even, that is, $n - k = 2i$, and we have $\left| 1 - (-1)^{k-n-1} \right| = 2$. So, the condition in (5) is equivalent to

$$\frac{1}{a_n} \sum_{i=0}^{E\left(\frac{n-1}{2}\right)} \left| u_1^{-2i-1} \right| a_{n-2i-1} = O(1) \ (n \to \infty).$$

Now, if we let $j = n - 2i - 1$ we obtain

$$\frac{1}{a_n} \sum_{j=0}^{n-1} \left| u_1^{j-n} \right| a_j = \frac{1}{a_n \left| u_1^n \right|} \sum_{j=0}^{n-1} \left| u_1^j \right| a_j = O(1) \ (n \to \infty).$$

This last condition means $\left(a_n \left| u_1^n \right| \right)_{n \geq 1} \in \widehat{C}_1$ and $a \in \widehat{C}_{\sqrt{|t/r|}}$. Then, the identity $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$, follows from the inclusion $\widehat{C_1} \subset G_1$. So, the condition $\left(a_n \left| u_1^n \right| \right)_{n \geq 1} \in \widehat{C}_1$ implies there are $K > 0$ and $\gamma > 1$ such that $a_n \left| u_1^n \right| \geq K\gamma^n$. This completes the proof. $\square$

Example 1. *Assume $r = 2$, $t = 1$ and $s = -3$. Then we have $u_1 = 1$ and $u_2 = 2$, and by Theorem 1, we obtain $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C_1}$. Moreover if $\overline{\lim}_{n \to \infty} a_n^\bullet < 1$, then $x \in \widehat{\mathcal{S}}$.*

Example 2. *We obtain a similar result, for $r = -t = 1$ and $s = 0$.*

In the following we need the next remark.

Remark 1. *By Theorem 1, we can state the following result. Let $r, t \neq 0$ and assume $\boldsymbol{\Delta} > 0$. Then, the condition $\overline{\lim}_{n \to \infty} a_n^\bullet < \min(|u_1|, |u_2|)$ implies $a \in \widehat{\mathcal{S}}$. Then, if $\boldsymbol{\Delta} = 0$ then $u_1 = u_2 = -s/2t$ and the condition $\overline{\lim}_{n \to \infty} a_n^\bullet < |u_1|$ implies $a \in \widehat{\mathcal{S}}$.*

Theorem 1 may be rewritten in the following way.

Corollary 1. *Let $r, t \neq 0$ and assume $a^\bullet \in c$. Then we have:*

(i) *Assume $\boldsymbol{\Delta} > 0$ and let $u_1 \neq u_2$ be the roots of (2). Then $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ and $a \in \widehat{\mathcal{S}}$ if and only if $\lim_{n \to \infty} a_n^\bullet < \min(|u_1|, |u_2|)$.*

(ii) *Assume* $\mathbf{\Delta}= 0$ *and let* $u_1 = u_2 = -s/2t$ *be the double root of* (2). *Then we have* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ *and* $a \in \widehat{\mathcal{S}}$ *if and only if* $\lim_{n\to\infty} a_n^\bullet < |u_1|$.

Proof. The identity $\widehat{\mathcal{S}^0} = \widehat{\mathcal{S}}$ follows from Theorem 1. (i) We only study the case $s \neq 0$, since the proof of the case $s = 0$ is similar. By Remark 1, we have $\lim_{n\to\infty} a_n^\bullet < \min(|u_1|, |u_2|)$ implies $a \in \widehat{\mathcal{S}}$. Conversely, let $a \in \widehat{\mathcal{S}}$. By Theorem 1, we have $a \in \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$. If $|u_1| < |u_2|$, then we have $\left(a_n |u_1|^n\right)_{n\geq 1} \in \widehat{C_1}$ and since $a^\bullet \in c$, by Lemma 1 we obtain $\lim_{n\to\infty} a_n^\bullet < |u_1|$. Similarly, if $|u_2| < |u_1|$, then we have $\left(a_n |u_2|^n\right)_{n\geq 1} \in \widehat{C_1}$ and by Lemma 1, we have $\lim_{n\to\infty} a_n^\bullet < |u_2|$. So, the condition $a \in \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$ implies $\lim_{n\to\infty} a_n^\bullet < \min(|u_1|, |u_2|)$ and we have shown that $a \in \widehat{\mathcal{S}}$ implies $\lim_{n\to\infty} a_n^\bullet < \min(|u_1|, |u_2|)$. This concludes the proof of Part (i). (ii) can be shown using similar arguments as those used above. This completes the proof. $\square$

As a direct consequence of Theorem 1, we state a result which is an extension of Corollary 1.

Corollary 2. *Let* $a^\bullet \in c$. *If* $s = 0$ *and* $\mathbf{\Delta}= -rt < 0$, *then* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ *and the condition* $a \in \widehat{\mathcal{S}}$ *holds if and only if* $\lim_{n\to\infty} a_n^\bullet < \sqrt{r/t}$.

By Corollary 1, we obtain the following result stated in [9].

Corollary 3. *Assume* $\mathbf{\Delta}> 0$ *with* $r, t \neq 0$. *The condition* $\min(|u_1|, |u_2|) > 1$ *is equivalent to the statement:*

$$\lim_{n\to\infty} (ry_{n+1} + sy_n + ty_{n-1}) = 0 \Longrightarrow \lim_{n\to\infty} y_n = 0 \text{ for all } y.$$

We may illustrate Corollary 3 with the next examples.

Example 3. *Since the absolute values of the roots of the equation* $u^2 - u - 3 = 0$ *are strictly upper than* 1, *we have*

$$\lim_{n\to\infty} (3y_{n+1} + y_n - y_{n-1}) = 0 \Longrightarrow \lim_{n\to\infty} y_n = 0 \text{ for all } y.$$

Example 4. *The condition* $|\alpha| > 1$ *is equivalent to the statement*

$$\lim_{n\to\infty} [\alpha y_{n+1} - (\alpha + 1)y_n + y_{n-1}] = 0 \Longrightarrow \lim_{n\to\infty} y_n = 0 \text{ for all } y.$$

6.2. Characterizations of the Inclusions $(\chi_a)_{\widetilde{B(r,s,0)}} \subset \chi_a$ *and* $(\chi_a)_{\widetilde{B(0,s,t)}} \subset \chi_a$

Using the equivalence of the conditions $B^{-1}(r,s) \in (\chi_a, \chi'_b)$ and $D_{1/b}B^{-1}(r,s)D_a \in (\chi_e, \chi'_e)$, we obtain the following known result on the inclusions $(\chi_a)_{B(r,s)} \subset \chi_a$, with $\chi = \mathbf{s}$, or $\mathbf{s}^0$.

Lemma 5. *Let* $r, s \neq 0$, $\alpha = -s/r$ *and let* $a \in U^+$. *Then, the following statements are equivalent, where* $\chi = s$, *or* $\mathbf{s}^0$, *(i)* $(\chi_a)_{B(r,s)} \subset \chi_a$, *(ii)* $(\chi_a)_{B(r,s)} = \chi_a$, *(iii)* $B(r,s) \in (\chi_a, \chi_a)$ *is surjective, (iv)* $B(r,s) \in (\chi_a, \chi_a)$ *is bijective, (v)* $a \in \widehat{C}_\alpha$.

Using Lemma 5, we may extend the results stated in Corollary 1 and determine the sets $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$ when either r, or t is equal to zero.

Proposition 2. *Let* $r, s, t \in \mathbb{R}$. *Let* u_0 *be the root of the equation* $tu + s = 0$ *if* $s, t \neq 0$, *and let* u'_0 *be the root of the equation* $su + r = 0$ *if* $r, s \neq 0$. *Then we have:*

(i) *(a) If* $r = 0$ *and* $s, t \neq 0$, *then we have:* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/u_0}$. *(b) If* $t = 0$ *and* $r, s \neq 0$, *then* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/u'_0}$.

(ii) *Let $a^{\bullet} \in c$. Then we have: (a) If $r = 0$ and $s, t \neq 0$, then $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ and $a \in \widehat{\mathcal{S}}$ if and only if $\lim_{n\to\infty} a_n^{\bullet} < |u_0|$. (b) If $t = 0$ and $r, s \neq 0$, then $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0}$ and $a \in \widehat{\mathcal{S}}$ if and only if $\lim_{n\to\infty} a_n^{\bullet} < |u_0'|$.*

Proof. (i) (a) Case $r = 0$ and $s, t \neq 0$. Then, the matrix $\widetilde{B(0,s,t)}$ is the triangle denoted by $B(s,t)$ and by Lemma 5, we have $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/u_0}$, where u_0 is the root of the equation $tu + s = 0$. (i) (b) We have $(\chi_a)_{\widetilde{B(r,s,0)}} = (\chi_{a^-})_{B(r,s)}$, for $\chi = s$, or s^0 and as above, by Lemma 5 we obtain $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{1/u_0'}$. (ii) follows from Lemma 1. This means that, under the condition $a^{\bullet} \in c$, we have $a \in \widehat{C_1}$ if and only if $a \in \widehat{\Gamma}$. □

6.3. Case $\boldsymbol{\Delta} < 0$ with $r, s, t \neq 0$

Here, we obtain interesting results on the characterizations of $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}^0}$ stated in Part (i) of Proposition 1, with $\boldsymbol{\Delta} < 0$. We have $u_1 = \rho e^{i\theta}$ with $\rho > 0$ and $u_2 = \overline{u}_1$ are the roots of Equation (2). Consider the conditions,

$$\sup_n \left(\frac{1}{\rho^n a_n} \sum_{k=1}^{n} |\sin(n-k+1)\theta| \rho^k a_{k-1} \right) < \infty, \tag{6}$$

$$\lim_{n\to\infty} \frac{\sin(n-k+1)\theta}{\rho^n a_n} = 0 \text{ for } k = 1, 2, \ldots \tag{7}$$

and

$$\overline{\lim_{n\to\infty}} a_n^{\bullet} < \rho. \tag{8}$$

We obtain the following results.

Proposition 3 ([9])**.** *Assume $\boldsymbol{\Delta} < 0$ and let $u_1 = \rho e^{i\theta}$ be a root of Equation (2). We have: (i) α) $a \in \widehat{\mathcal{S}}$ if and only if condition (6) holds. β) $a \in \widehat{\mathcal{S}^0}$ if and only if conditions (6) and (7) hold. (ii) $\widehat{C}_{1/\rho} \subset \widehat{\mathcal{S}^0} \subset \widehat{\mathcal{S}}$. (iii) The condition in (8) implies $a \in \widehat{\mathcal{S}^0}$.*

By Proposition 3 we obtain the following corollary.

Corollary 4 ([9])**.** *Assume $\boldsymbol{\Delta} < 0$ and let $u_1 = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \neq m\pi$ for $m \in \mathbb{Z}$, be a root of Equation (2).*

(i) *Let $(a_n \rho^n)_{n\geq 1} \in \widehat{C}_1$. Then, for every y:*

$$\lim_{n\to\infty} \left[\rho^2 y_{n+1} - 2(\rho\cos\theta) y_n + y_{n-1} \right] / a_n = 0 \Rightarrow \lim_{n\to\infty} y_n / a_n = 0.$$

(ii) *For any $\rho > 1$, we have $\lim_{n\to\infty} \left[\rho^2 y_{n+1} - 2(\rho\cos\theta) y_n + y_{n-1} \right] = 0$ implies $\lim_{n\to\infty} y_n = 0$ for all y.*

Finally, we state an elementary example.

Example 5. *If $\overline{\lim}_{n\to\infty} a_n^{\bullet} < 1$, then we have $\lim_{n\to\infty} (y_{n+1} + y_n + y_{n-1}) / a_n = 0$ implies $\lim_{n\to\infty} y_n / a_n = 0$, for all y. This result follows from the fact that (8) implies $a \in \widehat{C}_{1/\rho}$ and from Corollary 4, where $u_1 = e^{2i\pi/3}$, is a root of the equation $u^2 + u + 1 = 0$.*

7. Characterization of the Set $\widehat{\mathcal{S}^{(c)}}$

In this section, we deal with the set $\widehat{\mathcal{S}^{(c)}} = \left\{ a \in U^+ : \left(\mathbf{s}_a^{(c)} \right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_a^{(c)} \right\}$. This study consists in determining the set of all $a \in U^+$ for which

$$\lim_{n\to\infty} \frac{ry_{n+1} + sy_n + ty_{n-1}}{a_n} = l \Longrightarrow \lim_{n\to\infty} \frac{y_n}{a_n} = l',$$

for all y and for some scalars l, l'. We state some general results on $\widehat{S^{(c)}}$ and give interesting simplifications of this set. In this way, we confine our study to the case $\mathbf{\Delta} \geq 0$ and we assume $st < 0 < rt$ if $\mathbf{\Delta} > 0$.

7.1. General Case

In this part, we use the identity $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,s,t)}} = \left(\mathbf{s}_{a^-}^{(c)}\right)_{B(r,s,t)}$, which is a direct consequence the identity $\left(\widetilde{B(r,s,t)}y\right)_{n-1} = (B(r,s,t)y)_n$, for all $n \geq 2$ and for all y, and we consider the following statements.

(i) For $\mathbf{\Delta} \neq 0$, we use the condition in (3) and the conditions

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \left(u_2^{k-n-1} - u_1^{k-n-1}\right) a_{k-1} = L \text{ for some scalar } L, \tag{9}$$

and

$$\lim_{n\to\infty} \frac{1}{a_n} \left(u_2^{k-n-1} - u_1^{k-n-1}\right) a_{k-1} = l_k \text{ for some scalars } l_k \text{ with } k = 2, 3, \dots \tag{10}$$

(ii) For $\mathbf{\Delta} = 0$, we have $u_1 = -s/2t$ and we consider the conditions

$$\sup_{n\geq 1} \left(\frac{1}{|u_1^n| a_n} \sum_{k=1}^{n} (n-k+1) \left|u_1^k\right| a_{k-1} \right) < \infty, \tag{11}$$

$$\lim_{n\to\infty} \frac{1}{u_1^n a_n} \sum_{k=1}^{n} (n-k+1) u_1^k a_{k-1} = l \text{ for some scalar } l \tag{12}$$

and

$$\lim_{n\to\infty} \frac{1}{u_1^n a_n} (n-k+1) u_1^k a_{k-1} = l_k \text{ for some scalar } l_k,\ k = 2, 3, \dots \tag{13}$$

We can state the following result.

Proposition 4. *Let $a \in U^+$ and let $r, t \neq 0$. Then we have:*

(i) If $\mathbf{\Delta} \neq 0$, then $a \in \widehat{S^{(c)}}$ if and only if (3), (9) and (10) hold.

(ii) If $\mathbf{\Delta} = 0$, then $a \in \widehat{S^{(c)}}$ if and only if the conditions in (11) and (12) hold.

Proof.

(i) We have $a \in \widehat{S^{(c)}}$ if and only if $\left(\mathbf{s}_{a^-}^{(c)}\right)_{B(r,s,t)} \subset \mathbf{s}_a^{(c)}$ and

$$D_{1/a}[B(r,s,t)]^{-1} D_{a^-} \in (c, c). \tag{14}$$

Since $\mathbf{\Delta} \neq 0$, from the characterization of (c, c) and Lemma 4, the condition in (14) is equivalent to (3), (9) and (10).

(ii) Here, the condition in (14) is equivalent to (11)–(13). Then, the condition in (11) implies $\left(a_n \left|u_1^n\right|\right)_{n\geq 1} \in \widehat{C_1}$. Since $a^\bullet \in c$, by Lemma 1 we deduce that, there are $K > 0$ and $\gamma > 1$ such that $a_n \left|u_1^n\right| \geq K\gamma^n$, for all n and the condition in (13) holds with $l_k = 0$ for all k. This concludes the proof.

□

7.2. Characterizations of the Set $\widehat{\mathcal{S}^{(c)}}$ under the Conditions $\mathbf{\Delta} \geq 0$

In this part, we give interesting characterizations of the set $\widehat{\mathcal{S}^{(c)}}$ in special cases. We obtain the following theorem.

Theorem 2. *Let $a^{\bullet} \in c$ and assume $r, s, t \neq 0$. Then we have:*

(i) *Case $\mathbf{\Delta} > 0$ with $st < 0 < rt$. Then, the roots of (2) are positive, we have $a \in \widehat{\mathcal{S}^{(c)}}$ if and only if $\lim_{n\to\infty} a_n^{\bullet} < \min(u_1, u_2)$.*

(ii) *Case $\mathbf{\Delta} = 0$. We have: (α) If $u_1 = -s/2t > 0$ then $a \in \widehat{\mathcal{S}^{(c)}}$ if and only if $\lim_{n\to\infty} a_n^{\bullet} < u_1$. ($\beta$) If $u_1 = -s/2t < 0$, then the condition $a \in \widehat{\mathcal{S}^{(c)}}$ implies $\lim_{n\to\infty} a_n^{\bullet} < -u_1$.*

Proof.

(i) We show that, $\lim_{n\to\infty} a_n^{\bullet} < \min(u_1, u_2)$ implies $a \in \widehat{\mathcal{S}^{(c)}}$. The condition $\lim_{n\to\infty} a_n^{\bullet} < \min(u_1, u_2)$, means $(a_n \alpha^n)_{n\geq 1} \in \widehat{\Gamma}$, where α is either u_1, or u_2. Since $\widehat{\Gamma} = \widehat{C}$ we obtain

$$\left(\frac{1}{a_n}\sum_{k=1}^{n} u_2^{k-n-1} a_{k-1}\right)_{n\geq 1} \text{ and } \left(\frac{1}{a_n}\sum_{k=1}^{n} u_1^{k-n-1} a_{k-1}\right)_{n\geq 1} \in c, \tag{15}$$

and (9) holds. Since $\lim_{n\to\infty} a_n^{\bullet} < \min(u_1, u_2)$, we have $a \in \widehat{\mathcal{S}}$ and (3) holds. Then, the condition $(a_n \alpha^n)_{n\geq 1} \in \widehat{\Gamma}$ implies $a_n \alpha^n \to \infty$ $(n \to \infty)$. So, we obtain

$$\begin{aligned}
\kappa_{nk} &= a_n^{-1}\left(u_2^{k-n-1} - u_1^{k-n-1}\right) a_{k-1} \\
&= \left(a_n u_2^n\right)^{-1} u_2^{k-1} a_{k-1} - \left(a_n u_1^n\right)^{-1} u_1^{k-1} a_{k-1} \\
&= o(1) \ (n \to \infty) \text{ for all } k.
\end{aligned}$$

This shows that the condition in (10) also holds. Conversely, assume $a \in \widehat{\mathcal{S}^{(c)}}$. Then we have $a \in \widehat{\mathcal{S}}$ and by Theorem 1, we have $a \in \widehat{C}_{\max(1/u_1, 1/u_2)}$. So, we have $\left(a_n u_1^n\right)_{n\geq 1} \in \widehat{C}_1$ if $u_1 < u_2$ and $\left(a_n u_2^n\right)_{n\geq 1} \in \widehat{C}_1$ if $u_1 > u_2$. Since $a^{\bullet} \in c$ we have $\left(a_n u_i^n\right)_{n\geq 1} \in c$ with $i = 1, 2$ and by Lemma 1, we conclude $\lim_{n\to\infty} a_n^{\bullet} < \min(u_1, u_2)$. This completes the proof.

(ii) α) It can easily be seen that $\widehat{\mathcal{S}^{(c)}} \subset \widehat{\mathcal{S}}$ and since $\widehat{\mathcal{S}} = \widehat{C}_{1/u_1}$ and $a^{\bullet} \in c$, we deduce that the condition $a \in \widehat{\mathcal{S}^{(c)}}$ implies $\lim_{n\to\infty} a_n^{\bullet} < u_1$. So, we have shown the necessity. Conversely, assume $\lim_{n\to\infty} a_n^{\bullet} < u_1$. Then we have $\left(a_n u_1^n\right)_{n\geq 1} \in \Gamma$ and by Lemma 3, this condition implies (12). Since $u_1 > 0$ the condition in (12) implies (11) and by Part (ii) of Proposition 4, we have shown that the condition $\lim_{n\to\infty} a_n^{\bullet} < u_1$ implies $a \in \widehat{\mathcal{S}^{(c)}}$. This concludes the proof of α). As we have just seen, the statement in Part (ii) β), follows from the inclusion $\widehat{\mathcal{S}^{(c)}} \subset \widehat{\mathcal{S}}$, where $\widehat{\mathcal{S}} = \widehat{C}_{1/u_1}$. This concludes the proof of Part (ii).

□

Remark 2. *Under the conditions of Theorem 2, where $\mathbf{\Delta} \geq 0$, it can easily be seen that $\widehat{\mathcal{S}^{(c)}} \cap \widehat{c} = \widehat{\mathcal{S}} \cap \widehat{c} = \widehat{\mathcal{S}^{0}} \cap \widehat{c}$.*

As a direct consequence of Part (i) of Theorem 2, with $a = e$, we obtain the next tauberian result which can be stated as follows.

Corollary 5. *Let $r, s \in \mathbb{R}$ with $r < 0$, $s > 2$, and assume $r + s - 1 < 0$ and $s^2 + 4r > 0$. Then, for every $y \in \omega$, the condition $\lim_{n\to\infty}(r y_{n+1} + s y_n - y_{n-1}) = l$ implies $\lim_{n\to\infty} y_n = l'$ for some scalars l and l'.*

Proof. The proof is elementary and follows from Theorem 2, where $a = e$, $t = -1$, $\mathbf{\Delta} > 0$ and $\min(u_1, u_2) = u_2 = \left(s - \sqrt{\mathbf{\Delta}}\right)/2 > 1$. □

We can state the next applications where $a^{\bullet} \in c$.

Example 6. *The condition* $\lim_{n\to\infty} a_n^{\bullet} < 1$ *is equivalent to the following statement: for every* $y \in \omega$, *we have*

$$\lim_{n\to\infty} \frac{2y_{n+1} - 3y_n + y_{n-1}}{a_n} = l \Longrightarrow \lim_{n\to\infty} \frac{y_n}{a_n} = l',$$

for some scalars l *and* l'. *This result follows from Part (i) of Theorem 2, where* $b(u) = u^2 - 3u + 2 = 0$.

Example 7. *By Corollary 5, with* $t = -1$, $r = -6$ *and* $s = 5$ *we obtain the following result. For every* $y \in \omega$, *the condition*

$$\lim_{n\to\infty} (6y_{n+1} - 5y_n + y_{n-1}) = l$$

implies $\lim_{n\to\infty} y_n = l'$, *for some scalars* l *and* l'.

In the case $\mathbf{\Delta} = 0$ we obtain the following examples.

Example 8. *Let* $\alpha > 0$. *Then, the condition* $\lim_{n\to\infty} a_n^{\bullet} < \alpha$ *is equivalent to the following statement: for every* $y \in \omega$ *we have*

$$\lim_{n\to\infty} \frac{\alpha^2 y_{n+1} - 2\alpha y_n + y_{n-1}}{a_n} = l \Longrightarrow \lim_{n\to\infty} \frac{y_n}{a_n} = l',$$

for some scalars l *and* l'. *This result follows from Part (ii) of Theorem 2, where* $b(u) = (u - \alpha)^2$.

Example 9. *As a direct consequence of Example 8, for any given* $\alpha > 1$ *we obtain the following statement. For every* $y \in \omega$ *there are scalars* l *and* l' *such that the condition* $\lim_{n\to\infty}(\alpha^2 y_{n+1} - 2\alpha y_n + y_{n-1}) = l$ *implies* $\lim_{n\to\infty} y_n = l'$.

We may state some characterizations of the set $\widehat{\mathcal{S}^{(c)}}$ when either r, or t is equal to zero. Then, $B(r,s,t)$ is reduced to a double band matrix and we obtain the following result, whose the elementary proof is left to the reader.

Proposition 5. *Let* $a^{\bullet} \in c$ *and let* $r, s, t \in \mathbb{R}$. *Then we have:*

(i) *Assume* $r = 0$ *and* $st < 0$. *Let* $u_0 > 0$ *be the root of the equation* $tu + s = 0$. *Then we have* $a \in \widehat{\mathcal{S}^{(c)}}$ *if and only if* $\lim_{n\to\infty} a_n^{\bullet} < u_0$.

(ii) *Assume* $t = 0$ *and* $rs < 0$. *Then, the matrix* $\widetilde{B(r,s,0)} = B(s,r)^T$, *is upper triangular and if* $u_0' > 0$ *is the root of the equation* $su + r = 0$ *then we have* $a \in \widehat{\mathcal{S}^{(c)}}$ *if and only if* $\lim_{n\to\infty} a_n^{\bullet} < u_0'$.

We are led to state the next remark, on the similar spaces associated with the double band matrix $B(r,s)$.

Remark 3. *We have seen in Theorem 1, that the sets* $\widehat{\mathcal{S}}$ *and* $\widehat{\mathcal{S}^0}$ *with* $\mathbf{\Delta} > 0$ *and* $r, s, t \neq 0$, *are determined by* $\widehat{\mathcal{S}} = \widehat{\mathcal{S}^0} = \widehat{C}_{\max(|1/u_1|,|1/u_2|)}$, *where* $u_1 \neq u_2$ *are the roots of (2). In a similar way, let* $r, s \neq 0$, *and define by* $\mathcal{S}$, $\mathcal{S}^0$ *and* $\mathcal{S}^{(c)}$, *the sets of all positive sequences* a *such that* $(\chi_a)_{B(r,s)} \subset \chi_a$, *where* χ *is any of the symbols* $\mathbf{s}$, $\mathbf{s}^0$, *or* $\mathbf{s}^{(c)}$. *Using Proposition 15, we have* $\mathcal{S} = \mathcal{S}^0 = \widehat{C}_{1/u_0'}$, *where* u_0' *is the root of the equation* $su + r = 0$. *Concerning the sets* $\widehat{\mathcal{S}^{(c)}}$ *and* $\mathcal{S}^{(c)}$, *we can state the following results, for* $a^{\bullet} \in c$. *If* $\mathbf{\Delta} > 0$ *with* $st < 0 < rt$, *then the roots of (2)* *are positive, and we have* $a \in \widehat{\mathcal{S}^{(c)}}$ *if and only if* $\lim_{n\to\infty} a_n^\bullet < \min(u_1, u_2)$. *Then, if* $rs < 0$, *it can easily be shown that* $a \in \mathcal{S}^{(c)}$ *if and only if* $\lim_{n\to\infty} a_n^\bullet < u_0'$.

8. Conclusions

In this article, we have extended some results stated in [9], where we determined each of the sets of all $a \in U^+$ such that $(\chi_a)_{\widetilde{B(r,s,t)}} \subset \chi_a$, where χ is any of the symbols $\mathbf{s}$, or $\mathbf{s}^0$. Then, we have determined the sets of all $a \in U^+$, that satisfy each of the next inclusions, (1) $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,0,t)}} \subset \mathbf{s}_a^{(c)}$ and $r, t \neq 0$, (2) $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(0,s,t)}} \subset \mathbf{s}_a^{(c)}$ and $s, t \neq 0$, and (3) $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,s,0)}} \subset \mathbf{s}_a^{(c)}$ and $r, s \neq 0$. In this way, we have stated some characterizations of the set of all positive sequences a, such that $\left(\mathbf{s}_a^{(c)}\right)_{\widetilde{B(r,s,t)}} \subset \mathbf{s}_a^{(c)}$. In future, it should be interesting to extend these results, using the set ℓ^p of all sequences of $p-$absolute type, with $p \geq 1$, and determine each of the sets of all positive sequences a such that $\left(\ell_a^1\right)_{\widetilde{B(r,s,t)}} \subset \ell_a^p$ for $p \geq 1$, and $\left(\ell_a^p\right)_{\widetilde{B(r,s,t)}} \subset \chi_a$, where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. These results can also lead to a connection between the fine spectrum theory and the solvability of some (SSIE) of the form $\chi_{B(r,s,t)-\lambda I} \subset \chi_x$, for $\lambda \in \mathbb{C}$, where χ is a linear space of sequences.

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