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Fixed Point Results for Frum-Ketkov Type Contractions in b-Metric Spaces

Cristian Chifu [1], Erdal Karapınar [2,3,4,*] and Gabriela Petrusel [1]

[1] Department of Business, Babeş-Bolyai University Cluj-Napoca, Horea Street, No. 7, 400000 Cluj-Napoca, Romania; cristian.chifu@ubbcluj.ro (C.C.); gabi.petrusel@ubbcluj.ro (G.P.)
[2] Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot City 75000, Vietnam
[3] Department of Mathematics, Çankaya University, Etimesgut 06790, Ankara, Turkey
[4] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
* Correspondence: erdalkarapinar@tdmu.edu.vn or erdalkarapinar@yahoo.com

Abstract: The purpose of this paper is to present some fixed point results for Frum-Ketkov type operators in complete b-metric spaces.

Keywords: b-metric space; Frum-Ketkov operators; φ-contractive mappings; weakly Picard operators

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1. Introduction and Preliminaries

In [1], Frum-Ketkov obtained a fixed point theorem, which was later generalized by Nussbaum [2] and Buley [3]. Later, Park and Kim [4] obtained other forms of the Frum-Ketkov theorem. Recently, Petrusel, Rus and Serban [5] gave sufficient conditions ensuring that a Frum-Ketkov operator is a weakly Picard operator and studied also some generalized Frum-Ketkov operators, see also [6].

The purpose of this paper is to obtain similar results for generalized Frum-Ketkov operators in the context of b-metric spaces.

We start by recalling the definition of Frum-Ketkov operators and some notions given in [5].

Let (M, d) be a metric space. We denote by $P(M)$ the family of all nonempty subsets of M, by $P_{cl}(M)$ the family of all nonempty closed subsets of M and by $P_{cp}(M)$ the family of all nonempty compact subsets of M.

The ω-limit set of $x \in M$ under the self-mapping f is defined as

$$\omega_f(x) = \bigcap_{n=0}^{+\infty} \overline{\{f^k(x) : k \geq n\}},$$

where f^k is the iterate of order k of f.

Remark 1. *Ref. [5]* $\omega_f(x) = \{x^* \in M :$ *there exists* n_k *such that* $f^{n_k}(x) \to x^*\}$.

Definition 1. *Ref. [5] Let (M, d) be a metric space. A self-mapping $f : M \to M$ is called:*

1. *l-contraction if $l \in (0, 1)$ and $d(f(x), f(y)) \leq l d(x, y)$, for every $x, y \in M$;*
2. *Contractive if $d(f(x), f(y)) < d(x, y)$, for every $x, y \in M$ with $x \neq y$;*
3. *Nonexpansive if $d(f(x), f(y)) \leq d(x, y)$, for every $x, y \in M$;*
4. *Quasinonexpansive if $F_f \neq \emptyset$ and, if $x^* \in F_f$ then $d(f(x), x^*) \leq d(x, x^*)$, for every $x \in M$, where F_f is the set of fixed point of the mapping f;*
5. *Asymptotical regular in a point $x \in M$, if $d(f^n(x), f^{n+1}(x)) \to 0$, as $n \to +\infty$.*

Definition 2. *Ref. [7] Let $X \in P_{cl}(M)$ and $f : X \to X$. f is called weakly Picard operator (WPO) if the sequence of successive approximation $\left\{f^k(x)\right\}_{n\in\mathbb{N}}$ converges for all $x \in X$ and its limit* *(which in general depends on x) is a fixed point of f. If f is a WPO with a unique fixed point, then f is called Picard operator (PO).*

Definition 3. *Ref. [5] Let (M, d) be a metric space, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. A continuous operator $f : X \to X$ is said to be a Frum-Ketkov (l, K)-operator if $l \in (0, 1)$ and*

$$d(f(x), K) \leq l d(x, K), \text{ for every } x \in X,$$

where

$$d(x, K) = \inf\{d(x, z) : z \in K\}.$$

In what follows, we recollect the definition of b-metric that was considered by several authors, including Bakhtin [8] and Czerwik [9].

Definition 4. *Let M be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : M \times M \to [0, +\infty)$ is said to be a b-metric with constant s, if*

1. *d is symmetric, that is, $d(x, y) = d(y, x)$ for all x, y,*
2. *d is self-distance, that is, $d(x, y) = 0$ if and only if $x = y$,*
3. *d provides s-weighted triangle inequality, that is*

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in M.$$

In this case the triple (M, d, s) is called a b-metric space with constant $s \geq 1$.

It is evident that the notions of b-metric and standard metric coincide in case of $s = 1$. For more details on b-metric spaces see, e.g., [10–12] and corresponding references therein.

Example 1. *Let $M = [0, +\infty)$ and $d : M \times M \to [0, +\infty)$ such that $d(x, y) = |x - y|^p$, $p > 1$. It's easy to see that d is a b-metric with $s = 2^{p-1}$, but is not a metric.*

Definition 5. *A mapping $\varphi : [0, +\infty) \to [0, +\infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$, as $n \to +\infty$, for any $t \in [0, +\infty)$.*

Lemma 1. *Ref. [11] If $\varphi : [0, +\infty) \to [0, +\infty)$ is a comparison function, then:*

1. *Each iterate φ^k of φ, $k \geq 1$, is also a comparison function;*
2. *φ is continuous at 0;*
3. *$\varphi(t) < t$, for any $t > 0$.*

Definition 6. *A function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be a c-comparison function if*

1. *φ is increasing;*
2. *There exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{+\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, +\infty)$.*

In order to give some fixed point results to the class of b-metric spaces, the notion of c-comparison function was extended to b-comparison function by V. Berinde [12].

Definition 7. *Ref. [12] Let $s \geq 1$ be a real number. A mapping $\varphi : [0, +\infty) \to [0, +\infty)$ is called a b-comparison function if the following conditions are fulfilled*

1. *φ is monotone increasing;*
2. *There exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{+\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, +\infty)$.*

The following lemma is very important in the proof of our results.

Lemma 2. *Ref. [12] If $\varphi : [0, +\infty) \to [0, +\infty)$ is a b-comparison function, then we have the following conclusions:*

1. *The series $\sum_{k=0}^{+\infty} s^k \varphi^k(t)$ converges for any $t \in [0, +\infty)$;*
2. *The function $S_b : [0, +\infty) \to [0, +\infty)$ defined by $S_b(t) = \sum_{k=0}^{+\infty} s^k \varphi^k(t)$, $t \in [0, +\infty)$, is increasing and continuous at 0.*

Remark 2. *Due to the Lemma 1.2, any b-comparison function is a comparison function.*

2. Frum-Ketkov Operators in b-Metric Spaces

Definition 8. *Let (M, d) be a b-metric space with constant $s \geq 1$, $X \in P_{cl}(M)$ and $K \in P_{cp}(M)$. A continuous function $f : X \to X$ is said to be a Frum-Ketkov (φ, K)-operator if there exists $\varphi : [0, +\infty) \to [0, +\infty)$ a b-comparison function such that*

$$d(f(x), K) \leq \varphi(d(x, K)), \text{ for every } x \in X.$$

Example 2. *Let $M = [0, +\infty), d : M \times M \to [0, +\infty), d(x, y) = (x - y)^2, s = 2$. From Example 1.1. we have that (M, d) is a b-metric space. Let $X = [0, 1], K = \{0\}, f : X \to X$, $f(x) = \frac{x}{x+2}, \varphi : [0, +\infty) \to [0, +\infty), \varphi(t) = \frac{t}{t+4}$. f is Frum-Ketkov operator.*

Theorem 1. *Let (M, d) be a b-metric space with constant $s \geq 1$, $X \in P_{cl}(M)$, $K \in P_{cp}(M)$ and $f : X \to X$ a Frum-Ketkov (φ, K)-operator. Then the following conclusion hold:*

(i) $\omega_f(x) \neq \varnothing$ *and* $\omega_f(x) \subset X \cap K$, *for every* $x \in X$;
(ii) $F_f \subset X \cap K$;
(iii) $f(X \cap K) \subset X \cap K$;
(iv) *If f is asymptotically regular, then $\varpi_f(x) \subset F_f$, for every $x \in X$. If, in addition, f is quasinonexpansive, then f is WPO.*

Proof. (i) Let $x \in X$ arbitrary. Because $K \in P_{cp}(M)$, there exists (y_n) such that $d(f(x), K) = d(f(x), y_n)$

$$d(f(x), y_n) \leq \varphi(d(x, y_n))$$
$$d(f^2(x), y_n) \leq \varphi(d(f(x), y_n)) \leq \varphi^2(d(x, y_n))$$

Inductively, we obtain

$$d(f^n(x), y_n) \leq \varphi^n(d(x, y_n)) \to 0, \text{ as } n \to +\infty.$$

Hence, $d(f^n(x), y_n) \to 0$, as $n \to +\infty$.

As $K \in P_{cp}(M)$, there exists a subsequence (y_{n_k}) of (y_n), such that $y_{n_k} \to y^*(x) \in K$, $n_k \to +\infty$.

Since $d(f^n(x), y_n) \to 0$, then $d(f^{n_k}(x), y^*(x)) \to 0$ and hence $f^{n_k}(x) \to y^*(x), n_k \to +\infty$, and thus $y^*(x) \in \omega_f(x)$.

In this way $\omega_f(x) \neq \varnothing$ and $\omega_f(x) \subset X \cap K$, for every $x \in X$.

(ii) Let $x \in F_f$. Suppose $d(x, K) \neq 0$.

$$d(x, K) = d(f(x), K) \leq \varphi(d(x, K)) < d(x, K),$$

which is a contradiction.

Hence, $d(x, K) = 0$ which implies $x \in K$ and thus $F_f \subset X \cap K$.

(iii) Let $x \in X \cap K$

$$d(f(x), K) \leq \varphi(d(x, K)) = \varphi(0) = 0.$$

Hence, $f(x) \in K$.

(iv) From (i) we have that $\omega_f(x) \neq \varnothing$, for every $x \in X$. Let $x^*(x) \in \omega_f(x)$. There exists n_k such that $f^{n_k}(x) \to x^*(x)$ as $n_k \to +\infty$.

$$\begin{aligned} d(x^*, f(x^*)) &\leq sd(x^*, f^{n_k}(x^*)) + sd(f^{n_k}(x^*), f(x^*)) \\ &\leq sd(x^*, f^{n_k}(x^*)) + s^2 d\big(f^{n_k}(x^*), f^{n_k+1}(x^*)\big) + s^2 d\big(f^{n_k+1}(x^*), f(x^*)\big) \end{aligned} \tag{1}$$

From (i) and (iii) since $x^*(x) \in \omega_f(x)$ we have that

$$d\Big(f^2(x^*), f(x^*)\Big) \leq \varphi(d(x^*, f(x^*))).$$

Inductively, we obtain

$$d\Big(f^{n_k}(x^*), f^{n_k+1}(x^*)\Big) \leq \varphi^{n_k}(d(x^*, f(x^*))).$$

Now, if in (1) we consider $n_k \to +\infty$, then we obtain $d(x^*, f(x^*))$, which implies that $x^* \in F_f$ and thus $\varpi_f(x) \subset F_f$.

Consider now that, in addition, f is quasinonexpansive and let $x \in X$ and $f^{n_k}(x) \to y^*(x), n_k \to +\infty$ (see (i)). Because f is asymptotically regular, $y^*(x) \in F_f$.

$$\begin{aligned} &d(f(x), y^*) \leq \varphi(d(x, y^*)) \\ &d\big(f^2(x), y^*\big) \leq \varphi(d(f(x), y^*)) < d(f(x), y^*). \end{aligned}$$

Hence the sequence $(d(f^n(x), y^*))$ is decreasing and since $(d(f^{n_k}(x), y^*)) \to 0$ as $n_k \to +\infty$, we obtain $d(f^n(x), y^*) \to 0$ as $n \to +\infty$ and thus f is WPO. □

3. Conclusions

Frum-Ketkov type contractions are an interesting topic that has been overlooked and has not attracted anyone's attention for many years. The very attractive recent publication of Petrusel–Rus–Serban [5] is the one that brought this shadowy concept to light. In this paper, we consider the Frum-Ketkov type contractions in the framework of b-metric space. For this reason, this paper should be considered as an initial paper that opens a new trend in metric fixed point theory.

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