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Generalized Summation Formulas for the Kampé de Fériet Function

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Abstract: By employing two well-known Euler's transformations for the hypergeometric function ${}_2F_1$, Liu and Wang established numerous general transformation and reduction formulas for the Kampé de Fériet function and deduced many new summation formulas for the Kampé de Fériet function with the aid of classical summation theorems for the ${}_2F_1$ due to Kummer, Gauss and Bailey. Here, by making a fundamental use of the above-mentioned reduction formulas, we aim to establish 32 general summation formulas for the Kampé de Fériet function with the help of generalizations of the above-referred summation formulas for the ${}_2F_1$ due to Kummer, Gauss and Bailey. Relevant connections of some particular cases of our main identities, among numerous ones, with those known formulas are explicitly indicated.

Keywords: Gamma function; Pochhammer symbol; Gauss's hypergeometric function ${}_2F_1$; generalized hypergeometric function ${}_pF_q$; Kampé de Fériet function; generalization of Kummer's summation theorem; generalization of Gauss' second summation theorem; generalization of Bailey's summation theorem

MSC: Primary 33B20; 33C20; Secondary 33B15; 33C05

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Citation: Choi, J.; Milovanović, G.V.; Rathie, A.K. Generalized Summation Formulas for the Kampé de Fériet Function. *Axioms* **2021**, *10*, 318. https://doi.org/10.3390/axioms10040318

Academic Editor: Clemente Cesarano

Received: 23 October 2021
Accepted: 21 November 2021
Published: 25 November 2021

1. Introduction and Preliminaries

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric series ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$ defined by (see, e.g., [1], ([2] p. 73) and ([3] pp. 71–75)):

$$ {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \tag{1} $$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see ([3], pp. 2 and 5)):

$$ (\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}), \end{cases} $$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here, an empty product is interpreted as 1, and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that each denominator parameter satisfies

$$ (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q). $$

In Wolfram's MATHEMATICA, the function ${}_pF_q$ is implemented as `HypergeometricPFQ` and is suitable for both symbolic and numerical calculation. For $p = q + 1$, it has a branch cut discontinuity in the complex z-plane running from 1 to ∞. If $p \leq q$ the series (1) converges for each $z \in \mathbb{C}$. For some recent results on this subject, especially on transformations, summations and some applications, see [4].

Here and elsewhere, let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively, and let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}.$$

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may be referred, for example, to [1–3,5–9].

It is worthy of note that whenever the generalized hypergeometirc function ${}_pF_q(z)$ (including ${}_2F_1(z)$) with its specified argument z (for example, $z = 1$ or $z = 1/2$) can be summed to be expressed in terms of the Gamma functions, the result may be very important from both theoretical and applicable points of view. Here, the classical summation theorems for the generalized hypergeometric series such as those of Gauss and Gauss second, Kummer, and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz summation theorems for the series ${}_3F_2$ and others play important roles in theory and application. During 1992–1996, in a series of works, Lavoie et al. [10–12] have generalized the above-mentioned classical summation theorems for ${}_3F_2$ of Watson, Dixon, and Whipple and presented a large number of special and limiting cases of their results, which have been further generalized and extended by Rakha and Rathie [13] and Kim et al. [14]. Those results have also been obtained and verified with the help of computer programs in MATHEMATICA and MAPLE.

The vast popularity and immense usefulness of the hypergeometric function and the generalized hypergeometric functions of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables. A serious, significant and systematic study of the hypergeometric functions of two variables was initiated by Appell [15] who presented the so-called Appell functions F_1, F_2, F_3 and F_4, which are generalizations of the Gauss' hypergeometric function. Here, we recall the Appell function F_3 (see, e.g., ([8] p. 23, Equation (4)))

$$\begin{aligned} F_3[a, a', b, b'; c; x, y] &= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \, {}_2F_1\left[\begin{matrix} a', b'; \\ c+m; \end{matrix} \; y\right] \frac{x^m}{m!}, \qquad \max\{|x|, |y|\} < 1. \end{aligned}$$

The confluent forms of the Appell functions were studied by Humbert [16]. A complete list of these functions can be seen in the standard literature, see, e.g., [5]. Later, the four Appell functions and their confluent forms were further generalized by Kampé de Fériet [17], who introduced more general hypergeometric functions of two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaudndy [18,19]. We recall here the definition of a more general double hypergeometric function (than one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda ([20] p. 423, Equation (26)). The convenient generalization of the Kampé de Fériet function is defined as follows:

$$\begin{aligned} &F^{H:A;B}_{G:C;D}\left[\begin{matrix} (h_H): & (a_A); & (b_B); \\ (g_G): & (c_C); & (d_D); \end{matrix} \; x, y\right] \\ &\quad = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((h_H))_{m+n} \, ((a_A))_m \, ((b_B))_n}{((g_G))_{m+n} \, ((c_C))_m \, ((d_D))_n} \frac{x^m}{m!} \frac{y^n}{n!}, \end{aligned} \tag{2}$$

where (h_H) denotes the sequence of parameters $(h_1, h_2, \ldots, h_H)$ and $((h_H))_n$ is defined by the following product of Pochhammer symbols:

$$((h_H))_n := (h_1)_n (h_2)_n \cdots (h_H)_n \quad (n \in \mathbb{N}_0),$$

where the product when $n = 0$ is to be accepted as unity. For more details about the function (2) including its convergence, the reader may be referred (for example) to ([8] pp. 26–33).

When some extensively generalized special functions like (2) were appeared, it has been an interesting and natural research subject to consider certain reducibilities of the functions. In this regard, many researchers have investigated the reducibility and transformation formulas of the Kampé de Fériet function. In fact, there are numerous reduction formulas and transformation formulas of the Kampé de Fériet function in the literature, see, e.g., [21–42]. In the above-cited references, most of the reduction formulae were related to both cases $H + A = 3$ and $G + C = 2$. In 2010, by using Euler's transformation formula for the ${}_2F_1$, Cvijović and Miller [26] established a reduction formula for the case $H + A = 2$ and $G + C = 1$. Motivated essentially by the work [26], recently, Liu and Wang [43] used Euler's first and second transformation formulas for the ${}_2F_1$ and the above-mentioned classical summation theorems for ${}_pF_q$ to present a number of very interesting reduction formulas and then deduced summation formulas for the Kampé de Fériet function. Indeed, only a few summation formulas for the Kampé de Fériet function are available in the literature.

In this paper, by choosing to make a basic use of 7 reduction formulas due to Liu and Wang [43], we aim to establish 32 general summation formulas for the Kampé de Fériet function, which are provided in 16 theorems, each one containing two formulas, with the help of generalizations of Kummer summation theorem, Gauss second summation theorem and Bailey summation theorem due to Rakha and Rathie [13]. The 32 general formulas afforded here are explicitly indicated to reduce to correspond with some special cases of the main results in Liu and Wang [43] and contain all of the main identities in Choi and Rathie [44].

2. Results Required

In order to make this paper self-contained, among numerous deduction formulas for the Kampé de Fériet function offered by Liu and Wang [43], we choose to recall 7 Formulas (3)–(9), which correspond, respectively, to ([43] Equations (2.11), (2.12), (2.14), (3.6), (4.2), (4.3) and (4.5)). We also recall classical summation formulas for ${}_2F_1$ due to Kummer, Gauss and Bailey, and their generalizations.

Here and throughout, restrictions of each formula involving the parameters and variables are omitted, which may be easily derived from the convergence conditions of the ${}_2F_1$ and the Kampé de Fériet function (see, e.g., ([3] p. 64) and ([8] p. 27); see also (50))

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{ll} \alpha: & \varepsilon;\ \beta-\varepsilon,\ \gamma; \\ \beta: & -;\ \gamma+\beta; \end{array} x,\ x\right] = (1-x)^{\beta-\varepsilon-\alpha}\,{}_2F_1[\beta-\varepsilon,\ \gamma+\beta-\alpha; \gamma+\beta;\ x]; \tag{3}$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{ll} \alpha: & \varepsilon;\ \beta-\varepsilon,\ \tfrac{1}{2}\alpha+1; \\ \beta: & -;\ \tfrac{1}{2}\alpha; \end{array} x,\ x\right] = (1-x)^{\beta-\varepsilon-\alpha}\,{}_2F_1\left[\beta-\varepsilon,\ 1+\frac{1}{2}\beta;\ \frac{1}{2}\beta;\ x\right]; \tag{4}$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{ll} \alpha: & \varepsilon;\ \beta-\varepsilon,\ 1+\tfrac{1}{2}\alpha,\ \tfrac{\alpha-\beta}{2}; \\ \beta: & -;\ \tfrac{1}{2}\alpha,\ 1+\tfrac{\alpha+\beta}{2}; \end{array} x,\ x\right] = (1-x)^{\beta-\varepsilon-\alpha}\,{}_2F_1\left[\beta-\varepsilon,\ \frac{\beta-\alpha}{2};\ 1+\frac{\alpha+\beta}{2};\ x\right]; \tag{5}$$

$$
\begin{aligned}
F^{0:2;2}_{1:0;0}\left[\begin{array}{r} -: \\ \beta: \end{array}\begin{array}{r} \alpha,\, \varepsilon; \\ -; \end{array}\begin{array}{r} \beta-\varepsilon,\, \gamma; \\ -; \end{array} x,\, \frac{x}{x-1}\right] &= F_3\left(\alpha,\, \beta-\varepsilon : \varepsilon,\, \gamma;\, \beta;\, x,\, \frac{x}{x-1}\right) \\
&= (1-x)^{-\alpha}\, {}_2F_1\left[\beta-\varepsilon,\, \alpha+\gamma;\, \beta;\, \frac{x}{x-1}\right];
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&F^{2:0;1}_{1:0;1}\left[\begin{array}{r} \alpha,\, \gamma: \\ \beta: \end{array}\begin{array}{r} -; \\ -; \end{array}\begin{array}{r} \varepsilon; \\ \beta+\varepsilon; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\, {}_2F_1\left[\alpha,\, \beta+\varepsilon-\gamma;\, \beta+\varepsilon;\, \frac{x}{x-1}\right];
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&F^{2:0;1}_{1:0;1}\left[\begin{array}{r} \alpha,\, \gamma: \\ \beta: \end{array}\begin{array}{r} -; \\ -; \end{array}\begin{array}{r} \frac{1}{2}\gamma+1; \\ \frac{1}{2}\gamma; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\, {}_2F_1\left[\alpha,\, 1+\frac{1}{2}\beta;\, \frac{1}{2}\beta;\, \frac{x}{x-1}\right];
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&F^{2:0;2}_{1:0;2}\left[\begin{array}{r} \alpha,\, \gamma: \\ \beta: \end{array}\begin{array}{r} -; \\ -; \end{array}\begin{array}{r} 1+\frac{1}{2}\gamma,\, \frac{\gamma-\beta}{2}; \\ \frac{1}{2}\gamma,\, 1+\frac{\gamma+\beta}{2}; \end{array} x,\, -x\right] \\
&\qquad = (1-x)^{-\alpha}\, {}_2F_1\left[\alpha,\, \frac{\beta-\gamma}{2};\, 1+\frac{\gamma+\beta}{2};\, \frac{x}{x-1}\right].
\end{aligned} \tag{9}
$$

In addition, we also recall the following generalizations of Kummer summation theorem, Gauss second summation theorem, and Bailey's summation theorem (see, e.g., [13]):

Generalizations of Kummer's summation theorem

$$
\begin{aligned}
{}_2F_1\left[\begin{array}{r} a,\, b; \\ 1+a-b+i; \end{array} -1\right] &= \frac{2^{i-2b}\, \Gamma(b-i)\, \Gamma(1+a-b+i)}{\Gamma(b)\, \Gamma(a-2b+i+1)} \\
&\quad \times \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{a+r+i+1}{2}-b\right)}{\Gamma\left(\frac{a+r-i+1}{2}\right)} \qquad (i \in \mathbb{N}_0)
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
{}_2F_1\left[\begin{array}{r} a,\, b; \\ 1+a-b-i; \end{array} -1\right] &= \frac{2^{-i-2b}\, \Gamma(1+a-b-i)}{\Gamma(a-2b-i+1)} \\
&\quad \times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{a+r-i+1}{2}-b\right)}{\Gamma\left(\frac{a+r-i+1}{2}\right)} \qquad (i \in \mathbb{N}_0).
\end{aligned} \tag{11}
$$

Generalizations of Gauss's second summation theorem

$$
\begin{aligned}
{}_2F_1\left[\begin{array}{r} a,\, b; \\ \frac{1}{2}(a+b+i+1); \end{array} \frac{1}{2}\right] &= \frac{2^{b-1}\, \Gamma\left(\frac{a+b+i+1}{2}\right) \Gamma\left(\frac{a-b-i+1}{2}\right)}{\Gamma(b)\, \Gamma\left(\frac{a-b+i+1}{2}\right)} \\
&\quad \times \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-i+1}{2}\right)} \qquad (i \in \mathbb{N}_0)
\end{aligned} \tag{12}
$$

and

$$ {}_2F_1\left[\begin{matrix} a,\, b\,; \\ \frac{1}{2}(a+b-i+1)\,; \end{matrix}\;\frac{1}{2}\right] = \frac{2^{b-1}\,\Gamma\left(\frac{a+b-i+1}{2}\right)}{\Gamma(b)} \times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-i+1}{2}\right)} \qquad (i\in\mathbb{N}_0). \tag{13} $$

Generalizations of Bailey's summation theorem

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a+i\,; \\ b\,; \end{matrix}\;\frac{1}{2}\right] = \frac{2^{i-a}\,\Gamma(a-i)\,\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)} \times \sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2}-i\right)} \qquad (i\in\mathbb{N}_0) \tag{14} $$

and

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a-i\,; \\ b\,; \end{matrix}\;\frac{1}{2}\right] = \frac{2^{-i-a}\,\Gamma(b)}{\Gamma(b-a)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2}\right)} \qquad (i\in\mathbb{N}_0), \tag{15} $$

which is a corrected version of ([13] Theorem 6) (see also ([45] Equation (20))).

It is noted that the results (10), (12) and (14) are recorded earlier in [46,47]. Further, if we set $i = 0, 1, 2, 3, 4, 5$ in (10)–(15), we get the summation formulas obtained by Lavoie et al. [12] in compact forms.

It is also noted that the particular cases of (10) or (11), (12) or (13), (14) or (15) when $i = 0$ give, respectively, the following classical Kummer, Gauss second and Bailey summation theorems (see, e.g., [2]):

$$ {}_2F_1\left[\begin{matrix} a,\, b\,; \\ 1+a-b\,; \end{matrix}\;-1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)}, $$

$$ {}_2F_1\left[\begin{matrix} a,\, b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\;\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}, $$

$$ {}_2F_1\left[\begin{matrix} a,\, 1-a\,; \\ b\,; \end{matrix}\;\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)}. $$

3. General Summation Formulas for the Kampé de Fériet Function

In this section, we establish 32 general summation formulas for the Kampé de Fériet function, which are stated in Theorems 1–16. Each theorem includes two summation formulas. Additionally, some particular cases of the general summation formulas here are explicitly pointed out to correspond to those known identities in Remarks 1–16.

Theorem 1. *Let $i \in \mathbb{N}_0$. Then*

$$ F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \varepsilon; & \beta-\varepsilon,\, 1-\alpha-\varepsilon+i; \\ \beta: & -; & 1-\alpha-\varepsilon+\beta+i; \end{matrix}\;\frac{1}{2},\,\frac{1}{2}\right] = \frac{2^{-\alpha+i}\,\Gamma(\beta-\varepsilon-\alpha+i+1)\,\Gamma(\alpha-i)}{\Gamma(1-2\alpha-\varepsilon+\beta+i)\,\Gamma(\alpha)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{1-2\alpha-\varepsilon+\beta+i+r}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)} \tag{16} $$

and

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \varepsilon; & \beta-\varepsilon, 1-\alpha-\varepsilon-i; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1-\alpha-\varepsilon+\beta-i; & \end{matrix}\right] \\
&\qquad = \frac{2^{-\alpha-i}\,\Gamma(\beta-\varepsilon-\alpha-i+1)}{\Gamma(1-2\alpha-\varepsilon+\beta-i)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1-2\alpha-\varepsilon+\beta-i+r}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)}.
\end{aligned} \tag{17}
$$

Proof. Setting $x = 1/2$ and $\gamma = 1-\alpha-\varepsilon+i$ $(i \in \mathbb{N}_0)$ in (3), we get

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \varepsilon; & \beta-\varepsilon, 1-\alpha-\varepsilon+i; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1-\alpha-\varepsilon+\beta+i; & \end{matrix}\right] \\
&\qquad = 2^{\varepsilon+\alpha-\beta}\, {}_2F_1\left[\begin{matrix} \beta-\varepsilon, 1-2\alpha-\varepsilon+\beta+i; & \frac{1}{2} \\ 1-\alpha-\varepsilon+\beta+i; & \end{matrix}\right].
\end{aligned} \tag{18}
$$

Now, the ${}_2F_1$ in the right side of (18) can be evaluated with the help of the result (12) by taking $a = \beta-\varepsilon$ and $b = 1-2\alpha-\varepsilon+\beta+i$. After some simplification, we get the result (16).

The proof of the Formula (17) would run parallel with that of (16) by setting $x = 1/2$ and $\gamma = 1-\alpha-\varepsilon-i$ $(i \in \mathbb{N}_0)$ in (3) with the aid of the result (13). The details are omitted. □

Remark 1. *The particular case $i = 0$ in (16) or (17) corresponds with the known identity due to Lin and Wang ([43] The 2nd Equation, Corollary 5.1).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (16) and (17) yields 11 formulas, which are expressed in a single form in ([44] Theorem 3.1).

Theorem 2. *Let $i \in \mathbb{N}_0$. Then*

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \varepsilon; & \beta-\varepsilon, 1+\alpha-2\beta+\varepsilon+i; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1+\alpha-\beta+\varepsilon+i; & \end{matrix}\right] \\
&\qquad = \frac{2^{i+\alpha-2\beta+2\varepsilon}\,\Gamma(\beta-\varepsilon-i)\,\Gamma(1+\alpha-\beta+\varepsilon+i)}{\Gamma(\beta-\varepsilon)\,\Gamma(1+\alpha-2\beta+2\varepsilon+i)} \\
&\qquad\qquad \times \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\varepsilon-\beta+\frac{1+\alpha+i+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-i+r}{2}\right)}
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
&F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \varepsilon; & \beta-\varepsilon, 1+\alpha-2\beta+\varepsilon-i; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & 1+\alpha-\beta+\varepsilon-i; & \end{matrix}\right] \\
&\qquad = \frac{2^{-i+\alpha-2\beta+2\varepsilon}\,\Gamma(1+\alpha-\beta+\varepsilon-i)}{\Gamma(1+\alpha-2\beta+2\varepsilon-i)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\varepsilon-\beta+\frac{1+\alpha-i+r}{2}\right)}{\Gamma\left(\frac{1+\alpha-i+r}{2}\right)}.
\end{aligned} \tag{20}
$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = 1/2$ and $\gamma = 1+\alpha-2\beta+\varepsilon+i$ $(i \in \mathbb{N}_0)$ and (ii) $x = 1/2$ and $\gamma = 1+\alpha-2\beta+\varepsilon-i$ $(i \in \mathbb{N}_0)$ in (3) with the help of (14) and (15) yields, respectively, (19) and (20). We omit the details. □

Remark 2. *The particular case $i = 0$ in (19) or (20) corresponds to the known formula due to Lin and Wang ([43] The 3rd Equation, Corollary 5.1).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (19) and (20) produces 11 identities, which are expressed in a single form in ([44] Theorem 3.3).

Theorem 3. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \beta-2-i; & 2+i,\, \frac{1}{2}\alpha+1; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} -1,\, -1\right] \\ &= \frac{2^{-2-\alpha}\,\Gamma\left(\frac{1}{2}\beta\right)}{(i+1)!\,\Gamma\left(\frac{1}{2}\beta-i-2\right)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta-1+\frac{r-i}{2}\right)}{\Gamma\left(\frac{1}{4}\beta+1+\frac{r-i}{2}\right)} \end{aligned} \tag{21}$$

and

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \beta-2+i; & 2-i,\, \frac{1}{2}\alpha+1; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} -1,\, -1\right] \\ &= \frac{2^{-2-\alpha}\,\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{2}\beta+i-2\right)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta-1+\frac{r+i}{2}\right)}{\Gamma\left(\frac{1}{4}\beta+1+\frac{r-i}{2}\right)}. \end{aligned} \tag{22}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\varepsilon=\beta-2-i$ $(i\in\mathbb{N}_0)$ and (ii) $x=-1$ and $\varepsilon=\beta-2+i$ $(i\in\mathbb{N}_0)$ in (4) with the help of (10) and (11) offers, respectively, (21) and (22). The details are omitted. □

Remark 3. *The particular case $i=0$ in (21) or (22) corresponds with the known identity due to Lin and Wang ([43] The 1st Equation, Corollary 5.2).*

Additionally, setting $i=0,1,2,3,4,5$ in (21) and (22) gives 11 formulas, which are expressed in a single form in ([44] Theorem 3.5).

Theorem 4. *Let $i \in \mathbb{N}_0$. Then*

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \frac{1}{2}\beta+2+i; & \frac{1}{2}\beta-2-i,\, \frac{1}{2}\alpha+1; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} \frac{1}{2},\, \frac{1}{2}\right] \\ &= \frac{(-1)^i\,2^{\alpha+3+i}}{\beta\,(i+1)!} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta+\frac{r+1}{2}\right)}{\Gamma\left(\frac{1}{4}\beta-i+\frac{r-1}{2}\right)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \frac{1}{2}\beta+2-i; & \frac{1}{2}\beta-2+i,\, \frac{1}{2}\alpha+1; \\ \beta: & -; & \frac{1}{2}\alpha; \end{array} \frac{1}{2},\, \frac{1}{2}\right] \\ &= \frac{2^{\alpha+3-i}}{\beta} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta+\frac{r+1}{2}\right)}{\Gamma\left(\frac{1}{4}\beta+\frac{r-1}{2}\right)}. \end{aligned} \tag{24}$$

Proof. For (23), setting $x=1/2$ and $\varepsilon=\beta/2+2+i$ $(i\in\mathbb{N}_0)$ in (4), we obtain

$$\begin{aligned} \mathcal{L}_{23} &:= F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \frac{\beta}{2}+2+i; & \frac{\beta}{2}-2-i,\, \frac{\alpha}{2}+1; \\ \beta: & -; & \frac{\alpha}{2}; \end{array} \frac{1}{2},\, \frac{1}{2}\right] \\ &= 2^{-\frac{\beta}{2}+2+i+\alpha}\,{}_2F_1\left[\frac{\beta}{2}-2-i,\, 1+\frac{\beta}{2};\, \frac{\beta}{2};\, \frac{1}{2}\right]. \end{aligned}$$

Using (12) with $a=\beta/2-2-i$ and $b=1+\beta/2$ in the above ${}_2F_1[1/2]$, with the aid of $\Gamma(z+1)=z\,\Gamma(z)$, we get

$$\mathcal{L}_{23} = \frac{2^{\alpha+3+i}}{\beta}\cdot\frac{\Gamma(-i-1)}{\Gamma(-1)}\cdot\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{\beta}{4}+\frac{r+1}{2}\right)}{\Gamma\left(\frac{\beta}{4}-i+\frac{r-1}{2}\right)}. \tag{25}$$

Now, consider

$$\frac{\Gamma(-i-1)}{\Gamma(-1)} = \lim_{\eta \to 0} \frac{\Gamma(-i-1+\eta)}{\Gamma(-1+\eta)}.$$

Here, employing the following well-known identity (see, e.g., ([3] p. 3, Equation (12))):

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}),$$

we find

$$\begin{aligned}\frac{\Gamma(-i-1)}{\Gamma(-1)} &= \lim_{\eta \to 0} \frac{\Gamma(2-\eta)}{\Gamma(2+i-\eta)} \cdot \frac{\sin(\pi(-1+\eta))}{\sin(\pi(-i-1+\eta))} \\ &= (-1)^i \lim_{\eta \to 0} \frac{\Gamma(2-\eta)}{\Gamma(2+i-\eta)} = (-1)^i \frac{\Gamma(2)}{\Gamma(2+i)} = \frac{(-1)^i}{(i+1)!}.\end{aligned}$$

Finally, using the last identity in (25) yields the desired identity (23).

For (24), setting $x = 1/2$ and $\varepsilon = \beta/2 + 2 - i$ $(i \in \mathbb{N}_0)$ in (4) with the help of (13), similarly, we can obtain the identity (24). We omit the details. □

Remark 4. *The particular case $i = 0$ in (23) or (24) corresponds to the known identity due to Lin and Wang ([43] The 2nd Equation, Corollary 5.2).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (23) and (24) provides 11 formulas, which are expressed in a single form in ([44] Theorem 3.8).

Theorem 5. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned}&F^{1:1;3}_{1:0;2}\left[\begin{matrix}\alpha: & \alpha+\beta-i; & i-\alpha,\, 1+\frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\alpha,\, 1+\frac{\alpha+\beta}{2};\end{matrix}\; -1,\, -1\right] \\ &\quad = \frac{\Gamma(-\alpha)\,\Gamma\left(1+\frac{\alpha+\beta}{2}\right)}{\Gamma(i-\alpha)\,\Gamma\left(\frac{1}{2}\beta+\frac{3}{2}\alpha-i+1\right)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\alpha+\frac{r+1-i}{2}\right)}{\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\alpha+\frac{r+1-i}{2}\right)}\end{aligned} \tag{26}$$

and

$$\begin{aligned}&F^{1:1;3}_{1:0;2}\left[\begin{matrix}\alpha: & \alpha+\beta+i; & -\alpha-i,\, 1+\frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; \\ \beta: & -; & \frac{1}{2}\alpha,\, 1+\frac{\alpha+\beta}{2};\end{matrix}\; -1,\, -1\right] \\ &\quad = \frac{\Gamma\left(1+\frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\beta+\frac{3}{2}\alpha+i+1\right)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{1}{4}\beta+\frac{3}{4}\alpha+\frac{r+1+i}{2}\right)}{\Gamma\left(\frac{1}{4}\beta-\frac{1}{4}\alpha+\frac{r+1-i}{2}\right)}.\end{aligned} \tag{27}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = -1$ and $\varepsilon = \alpha + \beta - i$ $(i \in \mathbb{N}_0)$ and (ii) $x = -1$ and $\varepsilon = \alpha + \beta + i$ $(i \in \mathbb{N}_0)$ in (5) with the help of (10) and (11) yields, respectively, (26) and (27). The details are omitted. □

Remark 5. *The particular case $i = 0$ in (26) or (27) corresponds with the known formula due to Lin and Wang ([43] The 1st Equation, Corollary 5.3).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (26) and (27) presents 11 formulas, which are expressed in a single form in ([44] Theorem 3.10).

Theorem 6. *Let $i \in \mathbb{N}_0$. Then,*

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lccc} \alpha: & \frac{1}{2}\beta - \frac{3}{2}\alpha - 1 + i; & \frac{1}{2}\beta + \frac{3}{2}\alpha + 1 - i,\, 1 + \frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & \frac{1}{2}\alpha,\, 1 + \frac{\alpha+\beta}{2}; & \end{array}\right]$$
$$= \frac{2^{i-\alpha-2}\,\Gamma(\alpha - i + 1)\,\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma(\alpha+1)\,\Gamma\left(\frac{\beta-\alpha}{2}\right)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\beta-\alpha}{4} + \frac{1}{2}r\right)}{\Gamma\left(\frac{\beta+3\alpha}{4} + 1 - i + \frac{1}{2}r\right)} \tag{28}$$

and

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lccc} \alpha: & \frac{1}{2}\beta - \frac{3}{2}\alpha - i - 1; & \frac{1}{2}\beta + \frac{3}{2}\alpha + 1 + i,\, 1 + \frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & \frac{1}{2}\alpha,\, 1 + \frac{\alpha+\beta}{2}; & \end{array}\right]$$
$$= \frac{2^{-i-\alpha-2}\,\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{\beta-\alpha}{2}\right)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{\beta-\alpha}{4} + \frac{1}{2}r\right)}{\Gamma\left(\frac{\beta+3\alpha}{4} + 1 + \frac{1}{2}r\right)}. \tag{29}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = 1/2$ and $\varepsilon = \beta/2 - 3\alpha/2 - 1 + i$ $(i \in \mathbb{N}_0)$ and (ii) $x = 1/2$ and $\varepsilon = \beta/2 - 3\alpha/2 - 1 - i$ $(i \in \mathbb{N}_0)$ in (5) with the help of (12) and (13) produces, respectively, (28) and (29). We omit the details. □

Remark 6. *The particular case $i = 0$ in (28) or (29) corresponds to the known identity due to Lin and Wang ([43] The 2nd Equation, Corollary 5.3).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (28) and (29) offers 11 formulas, which are expressed in a single form in ([44] Theorem 3.12).

Theorem 7. *Let $i \in \mathbb{N}_0$. Then,*

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lccc} \alpha: & \frac{3}{2}\beta - \frac{1}{2}\alpha - 1 - i; & \frac{\alpha-\beta}{2} + 1 + i,\, 1 + \frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & \frac{1}{2}\alpha,\, 1 + \frac{\alpha+\beta}{2}; & \end{array}\right]$$
$$= \frac{2^{\alpha-1}\,\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\alpha - i\right)\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\beta - \frac{1}{2}\alpha\right)\Gamma(1+\alpha)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\alpha+r+1}{2}\right)}{\Gamma\left(\frac{\beta+r+1}{2} - i\right)} \tag{30}$$

and

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lccc} \alpha: & \frac{3}{2}\beta - \frac{1}{2}\alpha - 1 + i; & \frac{\alpha-\beta}{2} + 1 - i,\, 1 + \frac{1}{2}\alpha,\, \frac{\alpha-\beta}{2}; & \frac{1}{2}, \frac{1}{2} \\ \beta: & -; & \frac{1}{2}\alpha,\, 1 + \frac{\alpha+\beta}{2}; & \end{array}\right]$$
$$= \frac{2^{\alpha-1}\,\Gamma\left(1 + \frac{\alpha+\beta}{2}\right)}{\Gamma(1+\alpha)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{\alpha+r+1}{2}\right)}{\Gamma\left(\frac{\beta+r+1}{2}\right)}. \tag{31}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = 1/2$ and $\varepsilon = 3\beta/2 - \alpha/2 - 1 - i$ $(i \in \mathbb{N}_0)$ and (ii) $x = 1/2$ and $\varepsilon = 3\beta/2 - \alpha/2 - 1 + i$ $(i \in \mathbb{N}_0)$ in (5) with the help of (14) and (15) affords, respectively, (30) and (31). The details are omitted. □

Remark 7. *The particular case $i = 0$ in (30) or (31) corresponds to the known formula due to Lin and Wang ([43] The 3rd Equation, Corollary 5.3).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (30) and (31) provides 11 formulas, which are expressed in a single form in ([44] Theorem 3.14).

Theorem 8. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -: \quad \alpha,\, \varepsilon; \quad \beta-\varepsilon,\, 1-\alpha-\varepsilon+i;\ \frac{1}{2},\, -1 \\ \beta: \quad -; \qquad\qquad\qquad\qquad -; \end{array}\right] \\
&= F_3\left(\alpha,\, \beta-\varepsilon\,:\,\varepsilon,\, 1-\alpha-\varepsilon+i;\, \beta;\, \frac{1}{2},\, -1\right) \\
&= \frac{2^{\alpha-i+2\varepsilon-2}\,\Gamma(1-\varepsilon)\,\Gamma(\beta)}{\Gamma(1-\varepsilon+i)\,\Gamma(\beta+\varepsilon-i-1)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\beta+\varepsilon-i+r-1}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)}
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -: \quad \alpha,\, \varepsilon; \quad \beta-\varepsilon,\, 1-\alpha-\varepsilon-i;\ \frac{1}{2},\, -1 \\ \beta: \quad -; \qquad\qquad\qquad\qquad -; \end{array}\right] \\
&= F_3\left(\alpha,\, \beta-\varepsilon\,:\,\varepsilon,\, 1-\alpha-\varepsilon-i;\, \beta;\, \frac{1}{2},\, -1\right) \\
&= \frac{2^{\alpha+i+2\varepsilon-2}\,\Gamma(\beta)}{\Gamma(\beta+\varepsilon+i-1)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{\beta+\varepsilon+i+r-1}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)}.
\end{aligned} \tag{33}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = 1/2$ and $\gamma = 1-\alpha-\varepsilon+i$ $(i \in \mathbb{N}_0)$ and (ii) $x = 1/2$ and $\gamma = 1-\alpha-\varepsilon-i$ $(i \in \mathbb{N}_0)$ in (6) with the help of (10) and (11) gives, respectively, (32) and (33). We omit the details. □

Remark 8. *The particular case $i = 0$ in (32) or (33) corresponds to the known formula due to Lin and Wang ([43] The 1st Equation, Corollary 5.4).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (32) and (33) presents 11 formulas, which are expressed in a single form in ([44] Theorem 3.16).

Theorem 9. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -: \quad \alpha,\, \varepsilon; \quad \beta-\varepsilon,\, \beta+\varepsilon-\alpha-i-1;\ -1,\, \frac{1}{2} \\ \beta: \quad -; \qquad\qquad\qquad\qquad -; \end{array}\right] \\
&= F_3\left(\alpha,\, \beta-\varepsilon\,:\,\varepsilon,\, \beta+\varepsilon-\alpha-i-1;\, \beta;\, -1,\, \frac{1}{2}\right) \\
&= \frac{2^{\beta+\varepsilon-\alpha-i-2}\,\Gamma(1-\varepsilon)\,\Gamma(\beta)}{\Gamma(1-\varepsilon+i)\,\Gamma(\beta+\varepsilon-i-1)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\beta+\varepsilon-i+r-1}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)}
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -: \quad \alpha,\, \varepsilon; \quad \beta-\varepsilon,\, \beta+\varepsilon-\alpha+i-1;\ -1,\, \frac{1}{2} \\ \beta: \quad -; \qquad\qquad\qquad\qquad -; \end{array}\right] \\
&= F_3\left(\alpha,\, \beta-\varepsilon\,:\,\varepsilon,\, \beta+\varepsilon-\alpha+i-1;\, \beta;\, -1,\, \frac{1}{2}\right) \\
&= \frac{2^{\beta+\varepsilon-\alpha+i-2}\,\Gamma(\beta)}{\Gamma(\beta+\varepsilon+i-1)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{\beta+\varepsilon+i+r-1}{2}\right)}{\Gamma\left(\frac{\beta-\varepsilon-i+r+1}{2}\right)}.
\end{aligned} \tag{35}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = -1$ and $\gamma = \beta + \varepsilon - \alpha - i - 1$ $(i \in \mathbb{N}_0)$ and (ii) $x = -1$ and $\gamma = \beta + \varepsilon - \alpha + i - 1$ $(i \in \mathbb{N}_0)$ in (6) with the help of (12) and (13) yields, respectively, (34) and (35). The details are omitted. $\square$

Remark 9. *The particular case $i = 0$ in (34) or (35) corresponds with the known identity due to Lin and Wang ([43] The 2nd Equation, Corollary 5.4).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (34) and (35) produces 11 formulas, which are expressed in a single form in ([44] Theorem 3.18).

Theorem 10. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned} &F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\varepsilon; & \beta-\varepsilon, 1-\alpha-\beta+\varepsilon+i; \\ \beta: & -; & -; \end{matrix}\; -1, \frac{1}{2}\right] \\ &= F_3\left(\alpha, \beta-\varepsilon : \varepsilon, 1-\alpha-\beta+\varepsilon+i; \beta; -1, \frac{1}{2}\right) \\ &= \frac{2^{\varepsilon-\alpha-\beta+i}\,\Gamma(\beta-\varepsilon+i)\,\Gamma(\beta)}{\Gamma(\varepsilon)\,\Gamma(\beta-\varepsilon)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\varepsilon+r}{2}\right)}{\Gamma\left(\beta-i+\frac{r-\varepsilon}{2}\right)} \end{aligned} \tag{36}$$

and

$$\begin{aligned} &F^{0:2;2}_{1:0;0}\left[\begin{matrix} -: & \alpha,\varepsilon; & \beta-\varepsilon, 1-\alpha-\beta+\varepsilon-i; \\ \beta: & -; & -; \end{matrix}\; -1, \frac{1}{2}\right] \\ &= F_3\left(\alpha, \beta-\varepsilon : \varepsilon, 1-\alpha-\beta+\varepsilon-i; \beta; -1, \frac{1}{2}\right) \\ &= \frac{2^{\varepsilon-\alpha-\beta-i}\,\Gamma(\beta)}{\Gamma(\varepsilon)} \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma\left(\frac{\varepsilon+r}{2}\right)}{\Gamma\left(\beta+\frac{r-\varepsilon}{2}\right)}. \end{aligned} \tag{37}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x = -1$ and $\gamma = 1 - \alpha - \beta + \varepsilon + i$ $(i \in \mathbb{N}_0)$ and (ii) $x = -1$ and $\gamma = 1 - \alpha - \beta + \varepsilon - i$ $(i \in \mathbb{N}_0)$ in (6) with the help of (14) and (15) offers, respectively, (36) and (37). The details are omitted. $\square$

Remark 10. *The particular case $i = 0$ in (36) or (37) corresponds with the known formula due to Lin and Wang ([43] The 3rd Equation, Corollary 5.4).*

Additionally, setting $i = 0, 1, 2, 3, 4, 5$ in (36) and (37) gives 11 formulas, which are expressed in a single form in ([44] Theorem 3.20).

Theorem 11. *Let $i \in \mathbb{N}_0$. Then,*

$$\begin{aligned} &F^{2:0;1}_{1:0;1}\left[\begin{matrix} \alpha,\gamma: & -; & \alpha-\beta-\gamma+1+i; \\ \beta: & -; & \alpha-\gamma+1+i; \end{matrix}\; -1, 1\right]r \\ &= \frac{2^{i-2\gamma}\,\Gamma(1+\alpha-\gamma+i)\,\Gamma(\gamma-i)}{\Gamma(\gamma)\,\Gamma(1+\alpha-2\gamma+i)} \sum_{r=0}^{i} (-1)^r \binom{i}{r} \frac{\Gamma\left(\frac{\alpha+i+r+1}{2}-\gamma\right)}{\Gamma\left(\frac{\alpha-i+r+1}{2}\right)} \end{aligned} \tag{38}$$

and

$$
{}_{}F^{2:0;1}_{1:0;1}\left[\begin{array}{rl} \alpha,\gamma: & -;\ \ \alpha-\beta-\gamma+1-i; \\ \beta: & -;\ \ \alpha-\gamma+1-i; \end{array} -1,1\right] = \frac{2^{-i-2\gamma}\,\Gamma(1+\alpha-\gamma-i)}{\Gamma(1+\alpha-2\gamma-i)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{\alpha-i+r+1}{2}-\gamma\right)}{\Gamma\left(\frac{\alpha-i+r+1}{2}\right)}. \tag{39}
$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\varepsilon=\alpha-\beta-\gamma+1+i$ $(i\in\mathbb{N}_0)$ and (ii) $x=-1$ and $\varepsilon=\alpha-\beta-\gamma+1-i$ $(i\in\mathbb{N}_0)$ in (7) with the help of (12) and (13) affords, respectively, (38) and (39). We omit the details. $\square$

Remark 11. *The particular case $i=0$ in (38) or (39) corresponds to the known identity due to Lin and Wang ([43] The 2nd Equation, Corollary 5.7).*

Additionally, setting $i=0,1,2,3,4,5$ in (38) and (39) provides 11 formulas, which are expressed in a single form in ([44] Theorem 3.22).

Theorem 12. *Let $i\in\mathbb{N}_0$. Then,*

$$
F^{2:0;1}_{1:0;1}\left[\begin{array}{rl} \alpha,\gamma: & -;\ \ 1-\alpha-\beta+\gamma+i; \\ \beta: & -;\ \ 1-\alpha+\gamma+i; \end{array} -1,1\right] = \frac{2^{i-2\alpha}\,\Gamma(\alpha-i)\,\Gamma(1-\alpha+\gamma+i)}{\Gamma(\alpha)\,\Gamma(1+\gamma-2\alpha+i)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{1+\gamma+i+r}{2}-\alpha\right)}{\Gamma\left(\frac{1+\gamma-i+r}{2}\right)} \tag{40}
$$

and

$$
F^{2:0;1}_{1:0;1}\left[\begin{array}{rl} \alpha,\gamma: & -;\ \ 1-\alpha-\beta+\gamma-i; \\ \beta: & -;\ \ 1-\alpha+\gamma-i; \end{array} -1,1\right] = \frac{2^{-i-2\alpha}\,\Gamma(1-\alpha+\gamma-i)}{\Gamma(1+\gamma-2\alpha-i)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{1+\gamma-i+r}{2}-\alpha\right)}{\Gamma\left(\frac{1+\gamma-i+r}{2}\right)}. \tag{41}
$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\varepsilon=1-\alpha-\beta+\gamma+i$ $(i\in\mathbb{N}_0)$ and (ii) $x=-1$ and $\varepsilon=1-\alpha-\beta+\gamma-i$ $(i\in\mathbb{N}_0)$ in (7) with the help of (14) and (15) presents, respectively, (40) and (41). The details are omitted. $\square$

Remark 12. *The particular case $i=0$ in (40) or (41) corresponds with the known identity due to Lin and Wang ([43] The 3rd Equation, Corollary 5.7).*

Additionally, setting $i=0,1,2,3,4,5$ in (40) and (41) offers 11 formulas, which are expressed in a single form in ([44] Theorem 3.24).

Theorem 13. *Let $i\in\mathbb{N}_0$. Then,*

$$
F^{2:0;1}_{1:0;1}\left[\begin{array}{rl} \alpha,\gamma: & -;\ \ \frac{1}{2}\gamma+1; \\ 2\alpha+4+2i: & -;\ \ \frac{1}{2}\gamma; \end{array} -1,1\right] = \frac{(-1)^i\,2^{i+2}}{(\alpha+2+i)\,(i+1)!}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{\alpha+i+r+3}{2}\right)}{\Gamma\left(\frac{\alpha-i+r+1}{2}\right)} \tag{42}
$$

and

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{r} \alpha,\,\gamma: \\ 2\alpha+4-2i: \end{array}\begin{array}{c} -; \\ -; \end{array}\begin{array}{c} \frac{1}{2}\gamma+1; \\ \frac{1}{2}\gamma; \end{array}\ -1,\,1\right]=\frac{2^{-i+2}}{\alpha+2-i}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{\alpha-i+r+3}{2}\right)}{\Gamma\left(\frac{\alpha-i+r+1}{2}\right)}. \tag{43}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\beta=2\alpha+4+2i\ \ (i\in\mathbb{N}_0)$ and (ii) $x=-1$ and $\beta=2\alpha+4-2i\ \ (i\in\mathbb{N}_0)$ in (8) with the help of (12) and (13) produces, respectively, (42) and (43). We omit the details. □

Remark 13. *The particular case $i=0$ in (42) or (43) corresponds to the known formula due to Lin and Wang ([43] Corollary 5.8).*

Additionally, setting $i=0,1,2,3,4,5$ in (42) and (43) affords 11 formulas, which are expressed in a single form in ([44] Theorem 3.26).

Theorem 14. *Let $i\in\mathbb{N}_0$. Then*

$$\begin{aligned} &F^{2:0;2}_{1:0;2}\left[\begin{array}{r} \alpha,\,i-\alpha: \\ \beta: \end{array}\begin{array}{c} -; \\ -; \end{array}\begin{array}{c} 1-\frac{1}{2}\alpha+\frac{1}{2}i,\,\frac{i-\alpha-\beta}{2}; \\ -\frac{1}{2}\alpha+\frac{1}{2}i,\,1+\frac{i-\alpha+\beta}{2}; \end{array}\ \frac{1}{2},\,-\frac{1}{2}\right] \\ &\quad=\frac{2^{i-\alpha}\,\Gamma(\alpha-i)\,\Gamma\left(1+\frac{\beta-\alpha+i}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(1+\frac{\beta-3\alpha+i}{2}\right)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{\beta-3\alpha+i}{4}+\frac{r+1}{2}\right)}{\Gamma\left(\frac{\beta+\alpha-3i}{4}+\frac{r+1}{2}\right)} \end{aligned} \tag{44}$$

and

$$\begin{aligned} &F^{2:0;2}_{1:0;2}\left[\begin{array}{r} \alpha,\,-\alpha-i: \\ \beta: \end{array}\begin{array}{c} -; \\ -; \end{array}\begin{array}{c} 1-\frac{1}{2}\alpha-\frac{1}{2}i,\,\frac{-i-\alpha-\beta}{2}; \\ -\frac{1}{2}\alpha-\frac{1}{2}i,\,1+\frac{\beta-\alpha-i}{2}; \end{array}\ \frac{1}{2},\,-\frac{1}{2}\right] \\ &\quad=\frac{2^{-i-\alpha}\,\Gamma\left(1+\frac{\beta-\alpha-i}{2}\right)}{\Gamma\left(1+\frac{\beta-3\alpha-i}{2}\right)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{\beta-3\alpha-i}{4}+\frac{r+1}{2}\right)}{\Gamma\left(\frac{\beta+\alpha-i}{4}+\frac{r+1}{2}\right)}. \end{aligned} \tag{45}$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=1/2$ and $\gamma=-\alpha+i\ \ (i\in\mathbb{N}_0)$ and (ii) $x=1/2$ and $\gamma=-\alpha-i\ \ (i\in\mathbb{N}_0)$ in (9) with the help of (10) and (11) produces, respectively, (44) and (45). The details are omitted. □

Remark 14. *The particular case $i=0$ in (44) or (45) corresponds with the known identity due to Lin and Wang ([43] The 1st Equation, Corollary 5.9).*

Additionally, setting $i=0,1,2,3,4,5$ in (44) and (45) presents 11 formulas, which are expressed in a single form in ([44] Theorem 3.28).

Theorem 15. *Let $i\in\mathbb{N}_0$. Then,*

$$\begin{aligned} &F^{2:0;2}_{1:0;2}\left[\begin{array}{r} \frac{1}{2}\beta+\frac{3}{2}\gamma+1-i,\,\gamma: \\ \beta: \end{array}\begin{array}{c} -; \\ -; \end{array}\begin{array}{c} 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2}; \\ \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}; \end{array}\ -1,\,1\right] \\ &\quad=\frac{2^{i-2-2\gamma}\,\Gamma\left(1+\frac{\beta+\gamma}{2}\right)\Gamma(\gamma-i+1)}{\Gamma\left(\frac{\beta-\gamma}{2}\right)\Gamma(\gamma+1)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{\beta-\gamma}{4}+\frac{r}{2}\right)}{\Gamma\left(\frac{\beta+3\gamma}{4}+1-i+\frac{r}{2}\right)} \end{aligned} \tag{46}$$

and

$$
\begin{aligned}
&{}F^{2:0;2}_{1:0;2}\left[\begin{array}{r} \frac{1}{2}\beta+\frac{3}{2}\gamma+1+i,\,\gamma:\ \ -;\ \ 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2};\\ \beta:\ \ -;\ \ \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}; \end{array}\ -1,\,1\right]\\
&\quad=\frac{2^{-i-2-2\gamma}\,\Gamma\left(1+\frac{\beta+\gamma}{2}\right)}{\Gamma\left(\frac{\beta-\gamma}{2}\right)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{\beta-\gamma}{4}+\frac{r}{2}\right)}{\Gamma\left(\frac{\beta+3\gamma}{4}+1+\frac{r}{2}\right)}.
\end{aligned}
\tag{47}
$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\alpha=\beta/2+3\gamma/2+1-i\ \ (i\in\mathbb{N}_0)$ and (ii)$x=-1$ and $\alpha=\beta/2+3\gamma/2+1+i$ $(i\in\mathbb{N}_0)$ in (9) with the help of (12) and (13) offers, respectively, (46) and (47). We omit the details. □

Remark 15. *The particular case $i=0$ in (46) or (47) corresponds to the known result due to Lin and Wang ([43] The 2nd Equation, Corollary 5.9).*

Additionally, setting $i=0,1,2,3,4,5$ in (46) and (47) affords 11 formulas, which are expressed in a single form in ([44] Theorem 3.30).

Theorem 16. *Let $i\in\mathbb{N}_0$. Then,*

$$
\begin{aligned}
&{}F^{2:0;2}_{1:0;2}\left[\begin{array}{r} 1-\frac{1}{2}\beta+\frac{1}{2}\gamma+i,\,\gamma:\ \ -;\ \ 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2};\\ \beta:\ \ -;\ \ \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}; \end{array}\ -1,\,1\right]\\
&\quad=\frac{\Gamma\left(\frac{\beta-\gamma}{2}-i\right)\Gamma\left(1+\frac{\beta+\gamma}{2}\right)}{2\,\Gamma\left(\frac{\beta-\gamma}{2}\right)\Gamma(1+\gamma)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(\frac{\gamma+r+1}{2}\right)}{\Gamma\left(\frac{\beta+r+1}{2}-i\right)}
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
&{}F^{2:0;2}_{1:0;2}\left[\begin{array}{r} 1-\frac{1}{2}\beta+\frac{1}{2}\gamma-i,\,\gamma:\ \ -;\ \ 1+\frac{1}{2}\gamma,\,\frac{\gamma-\beta}{2};\\ \beta:\ \ -;\ \ \frac{1}{2}\gamma,\,1+\frac{\gamma+\beta}{2}; \end{array}\ -1,\,1\right]\\
&\quad=\frac{\Gamma\left(1+\frac{\beta+\gamma}{2}\right)}{2\,\Gamma(1+\gamma)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma\left(\frac{\gamma+r+1}{2}\right)}{\Gamma\left(\frac{\beta+r+1}{2}\right)}.
\end{aligned}
\tag{49}
$$

Proof. A similar process of the proof of Theorem 1 can establish the results here. Setting (i) $x=-1$ and $\alpha=1-\beta/2+\gamma/2+i\ \ (i\in\mathbb{N}_0)$ and (ii) $x=-1$ and $\alpha=1-\beta/2+\gamma/2-i$ $(i\in\mathbb{N}_0)$ in (9) with the help of (14) and (15) yields, respectively, (48) and (49). The details are omitted. □

Remark 16. *The particular case $i=0$ in (48) or (49) corresponds with known identity due to Lin and Wang ([43] The 3rd Equation, Corollary 5.9).*

Additionally, setting $i=0,1,2,3,4,5$ in (48) and (49) provides 11 formulas, which are expressed in a single form in ([44] Theorem 3.32).

4. Concluding Remarks

We chose to make an essential use of 7 reduction formulas for the Kampé de Fériet function due to Liu and Wang [43], with the help of generalizations of Kummer summation theorem, Gauss second summation theorem and Bailey summation theorem due to Rakha and Rathie [13], to present 32 general summation formulas for the Kampé de Fériet function.

With the aid of other general summation formulas for ${}_2F_1$ (if any), a similar method used in this paper is available to provide the corresponding summation formulas for the Kampé de Fériet function, which remains for future investigation.

As commented in the beginning of Section 2, constraints of each formula in this paper are omitted. The restrictions of Formula (3) is demonstrated to provide as follows:

$$x,\ \alpha,\ \beta,\ \gamma,\ \varepsilon \in \mathbb{C} \ \text{ such that } \ |x| < 1,\ \ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \gamma + \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{50}$$

Author Contributions: Conceptualization, J.C.; methodology, J.C., G.V.M. and A.K.R.; formal analysis, J.C. and A.K.R.; investigation, J.C., G.V.M. and A.K.R.; resources, J.C. and A.K.R.; writing—original draft preparation, J.C.; writing—review and editing, J.C., G.V.M. and A.K.R.; visualization, J.C., G.V.M. and A.K.R.; supervision, G.V.M.; project administration, G.V.M.; funding acquisition, G.V.M. All authors have read and agreed to the published version of the manuscript.

Funding: The work of the second author was supported in part by the Serbian Academy of Sciences and Arts (Φ-96).

Acknowledgments: The authors are quite appreciative of the helpful and encouraging remarks made by anonymous reviewers, which helped to enhance this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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