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The Space of Functions with Tempered Increments on a Locally Compact and Countable at Infinity Metric Space

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Abstract: The aim of the paper is to introduce the Banach space consisting of real functions defined on a locally compact and countable at infinity metric space and having increments tempered by a modulus of continuity. We are going to provide a condition that is sufficient for the relative compactness in the Banach space in question. A few particular cases of that Banach space will be discussed.

Keywords: modulus of continuity; space of functions with tempered increments; locally compact metric space; metric space countable at infinity; relative compactness

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1. Introduction

In the present paper, we intend to consider the Banach space consisting of real functions that are defined on a metric space being locally compact and countable at infinity. We will discuss, further on, in detail some classes of metric spaces of such a type. Moreover, we will assume that the considered real functions have increments tempered by a given modulus of continuity. It turns out that the described function space can be normed by a suitable defined norm which is complete; i.e., the mentioned function space creates the Banach space under that norm.

Next, we will formulate a theorem containing a condition sufficient for relative compactness in the above-described Banach space.

We will discuss a few particular cases of the mentioned Banach space. Namely, we will consider the space consisting of functions satisfying the Lipschitz or the Hölder condition on a given locally compact and countable at infinity metric space. For example, as that metric space, we can consider the half-axis $\mathbb{R}_+$, the set of real numbers $\mathbb{R}$, or the Euclidean space $\mathbb{R}^k$.

It is worthwhile mentioning that in earlier papers, we considered the case when as the above-mentioned metric space we took the metric space $\mathbb{R}_+$.

Finally, let us pay attention to the fact that the Banach function space described above with the metric space taken as $\mathbb{R}_+$ finds a lot of applications in the theory of functional integral equations (cf. [1–5], among others). We expect that also the Banach space studied in this paper finds some applications in the theory of functional, differential, and integral equations.

Moreover, let us also pay attention to the fact that the results of the paper have some connections with recently published papers concerning fractional differential and integral equations (cf. [6,7], for example).

2. Locally Compact and Countable at Infinity Metric Spaces

In the theory of topological spaces, there are considered spaces being locally compact and which are called countable at infinite or $\sigma-$compact. Let us recall [8] that a topological space X is said to be countable at infinity if X can be represented as the union of a sequence of compact subsets of X.

On the other hand, we can encounter a topological space being countable at infinity but not locally compact (cf. [8]).

In this paper, we restrict ourselves to the case of metric spaces. In such a case, we can start with the following definition [9].

Definition 1. *Let X be a given metric space (with a metric d). We say that X satisfies the condition (z) if there exists a sequence (G_n) of open sets in X such that $X = \bigcup_{n=1}^{\infty} G_n$ and $\overline{G_n} \subset G_{n+1}$ for $n = 1, 2, \ldots$ (the symbol $\overline{A}$ denotes the closure of the set A). Moreover, $\overline{G}_n$ is a compact set for $n = 1, 2, \ldots$.*

It can be shown [9] that a metric space X satisfies condition (z) if and only if it is locally compact and separable.

Let us pay attention to the fact that the concept of a metric space satisfying condition (z) can be defined equivalently in the following way suggested by the definition of a space being locally compact and countable at infinity.

Indeed, we have the following well-known theorem [8].

Theorem 1. *Let X be a metric space that is locally compact and countable at infinity. Then, there exists a sequence (U_n) of relatively compact and open subsets of X such that $X = \bigcup_{n=1}^{\infty} U_n$ and $U_n \subset U_{n+1}$ for $n = 1, 2, \ldots$.*

As an immediate consequence of Theorem 1, we infer that any metric space that is locally compact and countable at infinity satisfies condition (z). Obviously, the converse implications is also true.

Thus, we have the following useful conclusion.

Theorem 2. *A metric space (X, d) satisfies condition (z) if and only if it is locally compact and countable at infinity.*

Proof. First, let us assume that the metric space X satisfies condition (z). Then, in view of the above-mentioned result from [9], the space X is locally compact. Furthermore, keeping in mind that X satisfies condition (z), we infer that there exists a sequence of open subsets (G_n) of the space X such that $\overline{G}_n$ is compact and $X = \bigcup_{n=1}^{\infty} G_n$. Obviously, this implies that $X = \bigcup_{n=1}^{\infty} \overline{G}_n$; thus, the space X is countable at infinity.

Conversely, let us assume that X is locally compact and countable at infinity. Then, according to Theorem 1, we have that there exists a sequence (U_n) of open and relatively compact subsets of X such that $X = \bigcup_{n=1}^{\infty} U_n$. Hence, we deduce that $\overline{U}_n$ is compact. This allows us to conclude that the space X satisfies condition (z).

The proof is complete. □

For our further purposes, the following corollary will be very crucial.

Corollary 1. *Let (X, d) be a metric space that is locally compact and countable at infinity. Then, there exists an increasing sequence (K_n) (i.e., $K_n \subset K_{n+1}$ for $n = 1, 2, \ldots$) of compact subsets of X such that $X = \bigcup_{n=1}^{\infty} K_n$.*

In what follows, we will often base our considerations on Corollary 1.

3. The Space of Functions Defined on a Locally Compact and Countable at Infinity Metric Space with Increments Tempered by a Modulus of Continuity

This section is devoted to introducing and studying the Banach space consisting of real functions defined on a locally compact and countable at infinity metric space and having increments tempered by a given modulus of continuity.

To this end, consider a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, \infty)$) such that $\omega(0) = 0$, $\omega(\varepsilon) > 0$ for $\varepsilon > 0$. Moreover, we will assume that ω is nondecreasing on $\mathbb{R}_+$. Any such function will be called a modulus of continuity.

In what follows, we will also assume that the modulus of continuity is continuous at $\varepsilon = 0$; i.e., $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let us observe (cf. [1,3]) that the functions $\omega_L(\varepsilon) = \varepsilon$ and $\omega_\alpha(\varepsilon) = \varepsilon^\alpha$ ($\alpha \in (0, 1]$ is fixed) can serve as examples of moduli of continuity.

Now, let us take a metric space (X, d) which is locally compact and countable at infinity (cf. Section 2). For a given modulus of continuity $\omega = \omega(\varepsilon)$, consider the linear space $C_\omega(X)$ consisting of functions $x : X \to \mathbb{R}$ such that there exists a constant $k_x > 0$ (depending on the function x) such that

$$|x(u) - x(v)| \leqslant k_x \omega(d(u, v))$$

for all $u, v \in X$. In other words, we have that $x \in C_\omega(X)$ if and only if the quantity

$$\sup\left\{ \frac{|x(u) - x(v)|}{\omega(d(u, v))} : u, v \in X, u \neq v \right\}$$

is finite.

Obviously, the set $C_\omega(X)$ forms a linear space over the field of real numbers $\mathbb{R}$.

Let us notice that functions belonging to the linear space $C_\omega(X)$ are uniformly continuous on the metric space X. On the other hand, if we take a real function $x = x(u)$ which is defined and uniformly continuous on the metric space X, then for any number $\varepsilon > 0$, we can define the quantity $\nu(x, \varepsilon)$ by the following formula

$$\nu(x, \varepsilon) = \sup\{|x(u) - x(v)| : u, v \in X, d(u, v) \leqslant \varepsilon\}.$$

The function $\nu = \nu(x, \varepsilon)$ is well defined in view of assumption on uniform continuity of the function x and is said to be the modulus of continuity of the function x.

Let us notice that the function $x = x(u)$ is an element of the space $C_\omega(X)$ if and only if the modulus of continuity of x is majorized by the modulus of continuity $\omega(\varepsilon)$; i.e., there exists a constant $k_x > 0$ such that

$$\nu(x, \varepsilon) \leqslant k_x \omega(\varepsilon)$$

for any $\varepsilon > 0$.

Further on, let us fix an arbitrary element $u_0 \in X$. Next, for an arbitrary function $x \in C_\omega(X)$, we define the quantity $||x||_\omega$ by the following formula

$$||x||_\omega = |x(u_0)| + \sup\left\{ \frac{|x(u) - x(v)|}{\omega(d(u, v))} : u, v \in X, u \neq v \right\}. \tag{1}$$

Observe that $||x||_\omega < \infty$ for any $x \in C_\omega(X)$. We can also show that $|| \cdot ||_\omega$ is a norm in the space $C_\omega(X)$; i.e., $C_\omega(X)$ forms a real normed space with the norm defined by (1).

Remark 1. *Notice that up to now, we have not utilized the assumption on the local compactness and the countability at infinity of the metric space (X, d). Indeed, in the definition of the normed space $C_\omega(X)$, we can dispense with the mentioned assumption. Nevertheless, this assumption plays an essential role in the forthcoming theorem.*

Theorem 3. *Let (X, d) be a metric space that is locally compact and countable at infinity. Then, the space $C_\omega(X)$ is the Banach space with the norm defined by (1).*

Proof. Let us take a Cauchy sequence (x_n) in the space $C_\omega(X)$. This means that the following condition is satisfied:

$$\forall_{\varepsilon>0} \exists_{n_0\in\mathbb{N}} \forall_{\substack{n,m\in\mathbb{N}\\ n,m\geqslant n_0}} \Big[|x_n(u_0) - x_m(u_0)| + \sup\left\{ \frac{|[x_n(u) - x_m(u)] - [x_n(v) - x_m(v)]|}{\omega(d(u,v))} : u, v \in X, u \neq v \right\} \Big] \leqslant \varepsilon. \tag{2}$$

Hence, in particular, we obtain

$$\forall_{\varepsilon>0} \exists_{n_0\in\mathbb{N}} \forall_{\substack{n\in\mathbb{N}\\ n\geqslant n_0}} \sup\left\{ \frac{|[x_n(u) - x_n(v)] - [x_{n_0}(u) - x_{n_0}(v)]|}{\omega(d(u,v))} : u, v \in X, u \neq v \right\} \leqslant \varepsilon$$

or, equivalently

$$\forall_{\varepsilon>0} \exists_{n_0\in\mathbb{N}} \forall_{\substack{n\in\mathbb{N}\\ n\geqslant n_0}} \forall_{\substack{u,v\in X\\ u\neq v}} \frac{|[x_n(u) - x_n(v)] - [x_{n_0}(u) - x_{n_0}(v)]|}{\omega(d(u,v))} \leqslant \varepsilon.$$

The above established fact implies that for an arbitrarily fixed $\varepsilon > 0$, there exists a natural number n_0 such that for any number $n \in \mathbb{N}$, $n \geqslant n_0$ and for $u, v \in X$, $u \neq v$, the following inequality is satisfied

$$|[x_n(u) - x_n(v)] - [x_{n_0}(u) - x_{n_0}(v)]| \leqslant \varepsilon\omega(d(u,v)).$$

Hence, we obtain

$$|x_n(u) - x_n(v)| \leqslant |x_{n_0}(u) - x_{n_0}(v)| + \varepsilon\omega(d(u,v)), \tag{3}$$

for $u, v \in X$, $n \in \mathbb{N}$, and $n \geqslant n_0$.

Now, keeping in mind Corollary 1, we can find an increasing sequence (K_p) of compact subsets of the metric space X such that $X = \bigcup_{p=1}^{\infty} K_p$.

Thus, let us fix a number $p \in \mathbb{N}$; i.e., let us fix a compact set K_p of the above established sequence (K_p). On the base of inequality (3), we deduce that the functions from the set $\{x_n : n \geqslant n_0\}$ are equicontinuous on the set K_p. Obviously, this implies that the functions of the sequence (x_n) are equicontinuous on the set K_p.

In the similar way, putting in (3) $v = u_0$ (or utilizing inequality (2)), we infer that

$$|x_n(u)| \leqslant |x_n(u_0)| + |x_{n_0}(u) - x_{n_0}(u_0)| + \varepsilon\omega(d(u,u_0)).$$

Hence, in virtue of (2) we get

$$|x_n(u)| \leqslant |x_{n_0}(u_0)| + k_{n_0}\omega(d(u,u_0)) + \varepsilon(1 + \omega(d(u,u_0))).$$

From the above inequality, we deduce that the functions of the set $\{x_n : n \geqslant n_0\}$ are equibounded on the set K_p. Obviously, this allows us to infer that the functions from the sequence (x_n) are equibounded on the set K_p.

The above established properties of the functions of the sequence (x_n) and the Ascoli–Arzelá theorem allows us to conclude that the sequence (x_n) is relatively compact on the set K_p for any $p \in \mathbb{N}$. Thus, applying the diagonal procedure, we can select from the sequence (x_n) a subsequence (x_{k_n}), which converges nearly uniformly on the metric space X. This means that the subsequence (x_{k_n}) is uniformly convergent on each set K_p to a function $x = x(u)$ defined on X.

Next, let us take into account inequality (3) being valid for $n \in \mathbb{N}$, $n \geqslant n_0$ and for arbitrary $u, v \in X$. Fixing u and v and passing with $n \to \infty$, from that inequality, we get

$$|x(u) - x(v)| \leqslant |x_{n_0}(u) - x_{n_0}(v)| + \varepsilon\omega(d(u,v)). \tag{4}$$

On the other hand, keeping in mind that $x_{n_0} \in C_\omega(X)$, we deduce that there exists a constant $k_{n_0} > 0$ such that

$$|x_{n_0}(u) - x_{n_0}(v)| \leqslant k_{n_0}\omega(d(u,v)) \tag{5}$$

for $u, v \in X$.

Now, joining (4) and (5), we derive the following estimate

$$|x(u) - x(v)| \leqslant (k_{n_0} + \varepsilon)\omega(d(u,v))$$

for $u, v \in X$. This shows that $x \in C_\omega(X)$.

In what follows, taking into account inequality (2), for an arbitrarily fixed $\varepsilon > 0$ and for a natural number n_0 chosen according to (2), we have that

$$\begin{gathered} |x_n(u_0) - x_m(u_0)| \\ + \sup\left\{\frac{|[x_n(u) - x_m(u)] - [x_n(v) - x_m(v)]|}{\omega(d(u,v))} : u, v \in X, u \neq v\right\} \leqslant \varepsilon, \end{gathered}$$

for $n, m \in \mathbb{N}$, $n \geqslant n_0$, $m \geqslant n_0$.

Letting in the above inequality with $m \to \infty$, we obtain the following estimate

$$\begin{gathered} |x_n(u_0) - x(u_0)| \\ + \sup\left\{\frac{|[x_n(u) - x(u)] - [x_n(v) - x(v)]|}{\omega(d(u,v))} : u, v \in X, u \neq v\right\} \leqslant \varepsilon. \end{gathered}$$

This shows that $\lim_{n\to\infty} ||x_n - x||_\omega = 0$.

Thus, the function $x = x(u)$ is the limit of the function sequence (x_n) with respect to the norm $|| \cdot ||_\omega$ of the space $C_\omega(X)$ defined by (1). The proof is complete. $\square$

In what follows, let us mention that as the modulus of continuity $\omega = \omega(\varepsilon)$ indicated in our earlier considerations conducted in this paper, we can take the function $\omega_L(\varepsilon) = \varepsilon$. In this case, the function space $C_{\omega_L}(X)$ represents the space consisting of functions $x : X \to \mathbb{R}$ satisfying the Lipschitz condition on the metric space X (locally compact and countable at infinity). Thus, $x \in C_{\omega_L}(X)$ if and only if there exists a constant $k_x > 0$ such that

$$|x(u) - x(v)| \leqslant k_x d(u,v)$$

for arbitrary $u, v \in X$.

If we take as the modulus of continuity, the modulus generated by the Hölder condition—i.e., if we take the function $\omega_\alpha(\varepsilon) = \varepsilon^\alpha$ (where α is a fixed number such that $\alpha \in (0,1]$)—then the suitable space $C_{\omega_\alpha}(X)$ consists of functions $x : X \to \mathbb{R}$ such that there exists a constant $k_x > 0$ (depending on the function x) such that

$$|x(u) - x(v)| \leqslant k_x(d(u,v))^\alpha$$

for arbitrary $u, v \in X$.

Now, let us pay attention to some particular cases of the metric space (X,d) being locally compact and countable at infinity. The simple example of such a metric space is the set $\mathbb{R}_+$ with the natural metric $d(t,s) = |t-s|$. Such a case was considered in paper [3].

Further, let us observe that if we take a metric space (X,d) with the metric d generated by a norm i.e., if X is a normed space, then the assumption on the local compactness of X implies that X is finite dimensional. In the real case, we get that X is isometric to the Euclidean space $\mathbb{R}^k$. Hence, we conclude that the most natural metric space X being locally compact and countable at infinity seems to be the Euclidean space $\mathbb{R}^k$.

Thus, it is natural to consider as the most representable Banach space $C_\omega(X)$ the space $C_\omega(\mathbb{R}^k)$, where the metric in $\mathbb{R}^k$ can be considered as the Euclidean metric or a metric equivalent to that metric.

In our further consideration, it is reasonable to consider as the Banach space $C_\omega(X)$ the space $C_\omega(\mathbb{R}^k)$, where $\omega(\varepsilon) = \varepsilon$ or $\omega(\varepsilon) = \varepsilon^\alpha$ for $\alpha \in (0,1)$.

4. A Sufficient Condition for Relative Compactness in the Space $C_\omega(\mathbb{R}^k)$

In this section, we are going to describe a criterion being a sufficient condition for relative compactness in the Banach space $C_\omega(X)$, where X (with a metric d) is a metric space being locally compact and countable at infinity. The space $C_\omega(X)$ of such a type was described in detail in Section 3.

In view of Corollary 1, we will assume in this section that (K_n) is a sequence of nonempty and compact subsets of X such that the sequence (K_n) is increasing (i.e., $K_n \subset K_{n+1}$ for $n = 1,2,\ldots$) and $X = \bigcup_{n=1}^{\infty} K_n$.

Taking into account the practical utility of the space $C_\omega(X)$, we restrict ourselves to the case when $X = \mathbb{R}^k$, since only in such a case we are in a position to apply such a criterion for relative compactness in a concrete situation (cf. Section 3). Thus, in this section, we will consider the Banach space $C_\omega(\mathbb{R}^k)$ described in the previous section. Obviously, in this case, instead of the sequence (K_n) of compact subset of $\mathbb{R}^k$ such that $K_n \subset K_{n+1}$ for $n = 1,2,\ldots$ and $\mathbb{R}^k = \bigcup_{n=1}^{\infty} K_n$, it will be convenient to take the family of balls $\{B_T\}_{T>0}$ centered at the zero point $\theta = (0,0,\ldots,0) \in \mathbb{R}^k$ and with radius $T > 0$. To fix our attention, we will consider in the Euclidean space $\mathbb{R}^k$ the classical maximum metric; i.e., if $u = (u_i)$, $v = (v_i) \in \mathbb{R}^k$, then we take $d_m(u,v) = \max\{|u_i - v_i| : i = 1,2,\ldots,k\}$.

Now, we are prepared to formulate the announced sufficient condition.

Theorem 4. *Let A be a bounded subset of the space $C_\omega(\mathbb{R}^k)$ satisfying the following to conditions:*

(i)

$$\forall_{T>0}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in A}\ \forall_{\substack{u,v\in B_T \\ u\neq v}} \left[d_m(u,v) \leqslant \delta \implies \frac{|x(u)-x(v)|}{\omega(d_m(u,v))} \leqslant \varepsilon \right],$$

(ii)

$$\forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{x\in A}\ \forall_{\substack{u,v\in \mathbb{R}^k\setminus B_T \\ u\neq v}} \frac{|x(u)-x(v)|}{\omega(d_m(u,v))} \leqslant \varepsilon.$$

Then, the set A is relatively compact in the space $C_\omega(\mathbb{R}^k)$.

Proof. Let us fix arbitrarily a number $\varepsilon > 0$. Further, choose a number $T > 0$ according to condition (ii). Next, keeping in mind condition (i), we can find a number $\delta > 0$. Consider the set $A_{|B_T} = \{x_{|B_T} : x \in A\}$, where the symbol $x_{|B_T}$ denotes the restriction of the function x to the set B_T. Taking into account the fact that the set $A_{|B_T}$ satisfies condition (i), in view of Theorem 4 in [1], we infer that the set $A_{|B_T}$ is relatively compact in the space $C_\omega(B_T)$. This means that there exists a finite $\frac{\varepsilon}{2}-$net of this set in the space $C_\omega(B_T)$ which consists of functions $\overline{z_1}, \overline{z_2}, \ldots, \overline{z_m}$ being restrictions of functions $z_1, z_2, \ldots, z_m$ belonging to the space $C_\omega(\mathbb{R}^k)$ i.e., $\overline{z_i}(u) = z_{i|B_T}(u)$ for $u \in B_T$ and for $i = 1, 2, \ldots, m$. This implies (cf. Lemma 2.6 in [3]) that there exists a finite $\varepsilon-$net of the set A that consists of functions $\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m}$; i.e., there exist functions $\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m}$ being restrictions of functions $y_1, y_2, \ldots, y_m$ belonging to the space $C_\omega(\mathbb{R}^k)$, which forms an $\varepsilon-$net of the set A.

Now, let us take an arbitrary function $x(u) = x \in A$. Then, taking into account the above reasonings, we infer that there exists a number $j \in \{1, 2, \ldots, m\}$ such that

$$||x - y_j||_{C_\omega(B_T)} \leqslant \varepsilon.$$

This means that the following inequality is satisfied

$$\begin{aligned} &|x(u_0) - y_j(u_0)| \\ &+ \sup\left\{\frac{|[x(u) - y_j(u)] - [x(v) - y_j(v)]|}{\omega(d_m(u,v))} : u, v \in B_T, u \neq v\right\} < \varepsilon. \end{aligned} \tag{6}$$

Next, let us choose a number $T > 0$ to the number $\varepsilon > 0$ according to condition (ii). Then, for arbitrary $u, v \in \mathbb{R}^k \setminus B_T,\ u \neq v$, we have the following inequality

$$\frac{|x(u) - x(v)|}{\omega(d_m(u,v))} \leqslant \varepsilon, \tag{7}$$

which is satisfied for any function $x \in A$.

Now, taking the function $y_j = y_j(u)$, which was chosen previously to the function $x = x(u)$, for $u, v \in \mathbb{R}^k \setminus B_T,\ u \neq v$, we get:

$$\begin{aligned} \frac{|[x(u) - y_j(u)] - [x(v) - y_j(v)]|}{\omega(d_m(u,v))} &= \frac{|[x(u) - x(v)] - [y_j(u) - y_j(v)]|}{\omega(d_m(u,v))} \\ &\leqslant \frac{|x(u) - x(v)|}{\omega(d_m(u,v))} + \frac{|y_j(u) - y_j(v)|}{\omega(d_m(u,v))}. \end{aligned}$$

Hence, taking into account (7) and the fact that $y_j \in A$ we obtain

$$\frac{|[x(u) - y_j(u)] - [x(v) - y_j(v)]|}{\omega(d_m(u,v))} \leqslant 2\varepsilon. \tag{8}$$

Further, let us assume that $u, v \in \mathbb{R}^k$ are such that $u \in \mathring{B}_T$ (the symbol $\mathring{B}_T$ denotes the interior of the ball B_T), $v \in \mathbb{R}^k \setminus B_T$, and $d_m(u,v) < \delta$. Let us consider the segment $\overline{uv}$ joining the points u and v. Denote by w_T the intersection of the segment $\overline{uv}$ with the sphere $S_T = \{z \in B_T : d_m(z, \theta) = T\}$. Then ,

$$d_m(u,v) = d_m(u, w_T) + d_m(w_T, v).$$

Consequently, we have

$$\frac{1}{\omega(d_m(u,v))} \leqslant \frac{1}{\omega(d_m(u,w_T))}, \tag{9}$$
$$\frac{1}{\omega(d_m(u,v))} \leqslant \frac{1}{\omega(d_m(v,w_T))}. \tag{10}$$

Then, keeping in mind (9) and (10), for a fixed function $x \in A$, we obtain:

$$\begin{aligned}\frac{|x(u)-x(v)|}{\omega(d_m(u,v))} &\leqslant \frac{|x(u)-x(w_T)|}{\omega(d_m(u,v))} + \frac{|x(w_T)-x(v)|}{\omega(d_m(u,v))}\\ &\leqslant \frac{|x(u)-x(w_T)|}{\omega(d_m(u,w_T))} + \frac{|x(w_T)-x(v)|}{\omega(d_m(v,w_T))}.\end{aligned}$$

Consequently, in view of condition (i) and inequality (7), we deduce the following estimate

$$\frac{|x(u)-x(v)|}{\omega(d_m(u,v))} \leqslant 2\varepsilon. \tag{11}$$

Next, taking into account the fact that $y_j \in A$ for $j \in \{1,2,\dots,m\}$, we arrive at the following inequality

$$\frac{|y_j(u)-y_j(v)|}{\omega(d_m(u,v))} \leqslant \varepsilon \tag{12}$$

for $u \in \mathring{B}_T$ and $v \in \mathbb{R}^k \setminus B_T$.

Further, in virtue of (11) and (12), we get

$$\begin{aligned}\frac{|[x(u)-y_j(u)]-[x(v)-y_j(v)]|}{\omega(d_m(u,v))} &\leqslant \frac{|x(u)-x(v)|}{\omega(d_m(u,v))}\\ &+\frac{|y_j(u)-y_j(v)|}{\omega(d_m(u,v))} \leqslant 2\varepsilon+2\varepsilon = 4\varepsilon.\end{aligned} \tag{13}$$

Finally, combining (6), (8), and (13), we conclude that for any $x \in A$ there exists a function y_j $(j \in \{1,2,\dots,m\})$ such that the following inequality holds

$$||x-y_j||_{C_\omega(\mathbb{R}^k)} \leqslant 5\varepsilon.$$

This means that the functions $y_1, y_2, \dots, y_m$ form a finite $5\varepsilon-$net of the set A in the space $C_\omega(\mathbb{R}^k)$. Thus, the set A is relatively compact in the space $C_\omega(\mathbb{R}^k)$.

The proof is complete. □

5. Conclusions and Final Remarks

The results obtained in the paper can be treated as the platform for further study concerning the construction of suitable measures of noncompatness in the space $C_\omega(\mathbb{R}^k)$. Particularly, we can look both for criteria for relative compactness in the space $C_\omega(\mathbb{R}^k)$ and measures of noncompactness constructed on the basic of such criteria which will be more convenient in applications to the theory of integral equations as well as the theory of partial differential equations considered in the space $C_\omega(\mathbb{R}^k)$. In paper [3], we investigated an integral equation in the space $C_\omega(\mathbb{R}^k)$, where $\omega(\varepsilon) = \varepsilon^\alpha$ was the Hölder modulus of continuity. It is worthwhile mentioning that the integral equation considered in [3] has a rather very simple form. Moreover, the integral equation investigated in paper [5] has also rather simple form.

Therefore, the present paper and the result contained in Theorem 4 create the challenge for further investigations, which can be conducted in spaces of the type $C_\omega(\mathbb{R}^k)$ and which can be applied to the theory of integral equations of several variables and to the theory of partial differential equations.

The authors of this paper obtained a few tentative results in the indicated direction, which will be published in forthcoming papers.

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