axioms

MDPI

Article

# On the Asymptotics and Distribution of Values of the Jacobi Theta Functions and the Estimate of the Type of the Weierstrass Sigma Functions

**Mykola Korenkov** and **Yurii Kharkevych** *

Faculty of Informations Technologies and Mathematics, Lesya Ukrainka Volyn National University, 43025 Lutsk, Ukraine; kalchuk_i@ukr.net
* Correspondence: kharkevich.juriy@gmail.com

**Abstract:** A refined asymptotics of the Jacobi theta functions and their logarithmic derivatives have been received. The asymptotics of the Nevanlinna characteristics of the indicated functions and the arbitrary elliptic function have been found. The estimation of the type of the Weierstrass sigma functions has been given.

**Keywords:** Weierstrass function; Jacobi theta functions; Nevanlinna characteristics; elliptic function; entire function; meromorphic function

**MSC:** 30D35, 30E15

check for updates

**Citation:** Korenkov, M.; Kharkevych, Y. On the Asymptotics and Distribution of Values of the Jacobi Theta Functions and the Estimate of the Type of the Weierstrass Sigma Functions. *Axioms* **2022**, *11*, 12. https://doi.org/10.3390/axioms11010012

Academic Editor: Andriy Zagorodnyuk

Received: 17 November 2021
Accepted: 23 December 2021
Published: 25 December 2021



## 1. Introduction

In this paper, we investigate issues concerning the refined asymptotics and the distribution of values of the well-known Jacobi theta functions $\vartheta_{j+1}(z), j = \overline{0,3}$ [1] (pp. 394–396) and the closely related Weierstrass functions $\sigma(z), \zeta(z), \wp(z)$ [1] (pp. 374, 372, 348). These functions play the important role in the elliptic functions theory [1]. We also consider issues concerning the Nevanlinna characteristics [2,3] of the arbitrary elliptic function, the type of the function $\sigma(z)$.

We have to recall some relations from [1] and the results of the well-known scientific works.

It is known that $\zeta(z) = \sigma'(z)/\sigma(z), \wp(z) = -\zeta'(z)$ [1] (pp. 348, 372) and the points $\Omega_{mn} = 2m\omega_1 + 2n\omega_3$, where $\operatorname{Im}(\omega_3/\omega_1) > 0 (n, m \in \mathbb{Z})$, are simple zeros of the function $\sigma(z)$ and the poles of the function $\zeta(z), \wp(z)$ of the first and second orders, respectively. We denote $\omega_2 = \omega_1 + \omega_3$ and note that the numbers $2\omega_1, 2\omega_3$ are the fundamental periods of the Weierstrass elliptic function $\wp(z)$.

A.A. Goldberg [4] investigated the asymptotics of the function $\sigma(z)$ and its Nevanlinna characteristics in the case of the rectangular grid of zeros of this function. The general case

$$\omega_1 = \frac{1}{2}, \ \omega_3 = \frac{\lambda}{2} e^{i\alpha} \ (0 < \lambda < +\infty, \ 0 < \alpha < \pi), \tag{1}$$

i.e.,

$$\Omega_{m,n} = m + n\lambda e^{i\alpha} \ (m \in \mathbb{Z}, m \in \mathbb{Z}), \tag{2}$$

has been considered in the work [5]. It has been shown that

$$\ln|\sigma(z)| = V(z) + o\left(|z|^2\right), \ z \to \infty, \ z \notin \Phi(d), \tag{3}$$

where

$$\Phi(d) = \bigcup_{m,n \in \mathbb{Z}} \{z \in \mathbb{C} : |z - \Omega_{mn}| < d\}, \ d = \text{const} > 0, \tag{4}$$

$V(z) = V(re^{i\varphi}) = H(\varphi)r^2$, and $H(\varphi)$ is the indicator of the function $\sigma(z)$ that had been introduced in [5]. The distribution of the values of the function $\sigma(z)$ had been investigated in the work [5]. Yu. I. Lyubarsky and M.L. Sodin [6] showed that the remainder in (3) can be estimated more accurately, that is, under the conditions (1) and (2), the following relation is true:

$$\ln|\sigma(z)| = V(z) + O(1),\ z \notin \Phi(d). \tag{5}$$

This result has been obtained on the basis of the double periodicity of the function $|\sigma(z)|\exp(-V(z))$. The asymptotics of the functions $\vartheta_{j+1}(z), \vartheta'_{j+1}(z)/\vartheta_{j+1}(z)(j = \overline{0,3})$ had been investigated in [7,8]. In particular, in [7], it had been shown that the following equalities are true under the conditions (1) and (2) ($z \to \infty$)

$$\ln|\vartheta_1(z)| = U(z) + o\left(|z|^2\right),\ z \notin \Phi(d), \tag{6}$$

$$\vartheta'_1(z)/\vartheta_1(z) = -\frac{2\pi i}{\lambda \sin\alpha}\mathrm{Im}\, z + o(z),\ z \notin \Phi(d), \tag{7}$$

where $U(z) = U(re^{i\varphi}) = S(\varphi)r^2$ and $S(\varphi)$ is the indicator of the function $\vartheta_1(z)$ that had been given in [7]. Similar formulas have been received in the case $j = \overline{1,3}$. In the works [8,9], it has been revealed that exceptional sets (outside which Formulas (5)–(7) are true) can be significantly narrowed but due to less accurate estimate of their remainder. This is true for the functions $\vartheta_{j+1}(z), \vartheta'_{j+1}(z)/\vartheta_{j+1}(z)(j = \overline{1,3})$. In the work [10], the Julia rays [1] (pp. 572–573) of the function $\sigma(z)$ have been examined on the basis of Formula (2) from [9]. The papers [11,12] have been devoted to various issues related to the application of the Nevanlinna theory of meromorphic functions values distribution.

In this paper, we have proved that the following formulas are true under the conditions (1) and (2)

$$\ln|\vartheta_1(z)| = U(z) + O(1),\ z \notin \Phi(d),$$

$$\ln\left|\vartheta_{j+1}(z)\right| = U(z) + O(|z|),\ z \notin \Phi_j(d)\ (j = \overline{1,3}),$$

$$\vartheta'_1(z)/\vartheta_1(z) = -\frac{2\pi i}{\lambda \sin\alpha}\mathrm{Im}\, z + O(1),\ z \notin \Phi_j(d)\ (j = \overline{0,3}),$$

where

$$U(z) = \frac{\pi(|z|^2 - \mathrm{Re}z^2)}{2\lambda \sin\alpha}, \tag{8}$$

$$\Phi_j(d) = \bigcup_{m,n\in\mathbb{Z}} \left\{z \in \mathbb{C} : \left|z - \Omega_{mn} + \omega_j\right| < d\right\}\ (j = \overline{0,3}), \tag{9}$$

$d$ is an arbitrary constant, $d > 0$, $\omega_0 = 0$, $\Phi_0(d) = \Phi(d)$ and $\Phi(d)$ is given by the equality (4). We found the more accurate estimates of the remainders in the above mentioned asymptotic formulas than in the similar formulas in the works [7,8]. We have shown that the following equalities are true under the conditions (1) and (2):

$$T(r, \vartheta_{j+1}) = \frac{\pi r^2}{2\lambda \sin\alpha} + o(r^2),\quad T(r, \vartheta'_{j+1}/\vartheta_{j+1}) = \frac{\pi r^2}{2\lambda \sin\alpha} + O(r)\quad (j = \overline{0,3}),$$

where the Nevanlinna characteristics of the corresponding functions are on the left-hand sides of the equalities. The similar formula has been obtained for the characteristic $T(r, f)$ of the arbitrary elliptic function $f$, $f \neq$ const. We have also found the estimation of the type of the function $\sigma(z)$ and proved that none of the numbers $a, a \in \mathbb{C}$, is the exceptional value for the functions $\vartheta_{j+1}(z), \vartheta'_{j+1}(z)/\vartheta_{j+1}(z)(j = \overline{0,3})$ and for the arbitrary elliptic functions $f$, $f \neq$ const in Nevanlinna's sense. We have obtained the formula

$$\delta(0, \sigma) = 1 - \frac{\pi^2}{\lambda \sin\alpha}\left(\int_0^{2\pi} H^+(\varphi)d\varphi\right)^{-1} \tag{10}$$

for the Nevanlinna defect $\delta(0,\sigma)$ of the function $\sigma(z)$.

Concerning a possible continuation of research and an application of the obtained results, let us indicate the following. It would be good, based on the Formula (10) and the formula for the indicator $H(\varphi)$, write down the defect $\delta(0,\sigma)$ in an explicit form via the parameters $\lambda, \alpha$. One can also investigate the question if the number $a = 0$ is an exceptional value of the function $\sigma(z)$ in the Borel's sense and the question on the Julia's rays for the functions $\vartheta_{j+1}(z)(j = \overline{0,3})$ similarly to how it was done in [10] for the function $\sigma(z)$. The obtained asymptotic formulas can be applied for an investigation of properties for the solutions of differential equations and their systems, in which the functions $\vartheta_{j+1}(z)$, $\vartheta'_{j+1}(z)/\vartheta_{j+1}(z)(j = \overline{0,3})$ play a role, similar to the main facts of the Nevanlinna theory used in the papers [13–16].

## 2. Preliminaries

We will use the main notions, facts and standard notations from the Nevanlinna theory of the meromorphic functions values distribution known from the paper [3]. Let us recall some of them. For the given function $g : D \to \mathbb{R}, D \subset \mathbb{R}$, we denote by $g^+, g^-$ such functions that $g^+(x) = g^-(x) = 0$, when $x \in \mathbb{R} \setminus D$, and $g^+(x) = (|g(x)| + g(x))/2, g^-(x) = (|g(x)| - g(x))/2$, if $x \in D$. Herewith, the equality $g(x) = g^+(x) - g^-(x), x \in D$ is true. Namely, $\ln^+ x = \ln x$, if $x \geq 1$, and $\ln^+ x = 0$, if $0 < x < 1$. The Nevanlinna characteristics of the meromorphic function $f$ are introduced by the equalities

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+\left|f(re^{i\varphi})\right| d\varphi,$$

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\ln r,$$

$$T(r,f) = m(r,f) + N(r,f).$$

Here the quantity $n(r,f)$ (also denoted by $n(r,\infty,f)$) is the number of the function poles $f$ (taking into account their multiplicities), in the disc $\{z \in \mathbb{C} : |z| \leqslant r\}, r \geqslant 0$. If $a \in \mathbb{C}$, then the notations $n(r,a,f)$, $N(r,a,f)$, $m(r,a,f)$ are used instead of $n(r, \frac{1}{f-a})$, $N(r, \frac{1}{f-a})$, $m(r, \frac{1}{f-a})$, respectively. The Nevanlinna defect of the meromorphic function $f$ at the point $a, a \in \bar{\mathbb{C}}$, is defined as follows:

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}.$$

If $\delta(a,f) > 0$, then $a$ is called an exceptional (defective) value in Nevanlinna's sense for the function $f$.

## 3. Main Results

Without loss of generality, we will assume that the conditions (1) and (2) hold. We denote $\eta_1 = \zeta(\frac{1}{2}) = |\eta_1|e^{i\psi}, \eta_1 = \eta_1(\lambda,\alpha)$. It is obvious that $\eta_1 = \eta_1(\lambda,\alpha)$. We will recall some statements and facts that are used below.

As it was noted in the proof of Theorem 1 in [9], the following relation holds under the conditions (1) and (2)

$$\ln|\sigma(z)| = V(z) + O(1),\ z \notin \Phi(d), \tag{11}$$

where

$$V(z) = \frac{\pi|z|^2}{2\lambda\sin\alpha} + \mathrm{Re}\Big[\Big(\eta_1 - \frac{\pi}{2\lambda\sin\alpha}\Big)z^2\Big], \tag{12}$$

where $\Phi(d)$ is given by the Formula (4). The Formula (11) follows from the relations (12.5) and (12.6) of the work [6]. From (11), it follows that, uniformly in $\varphi, \varphi \in [0, 2\pi]$, the following equality holds ($r \to \infty$)

$$\ln\left|\sigma(re^{i\varphi})\right| = H(\varphi)r^2 + o(r^2),\ z = re^{i\varphi} \in \mathbb{C} \setminus \Phi(d), \tag{13}$$

where the function $H(\varphi)$, being the indicator of the entire function $\sigma(z)$, is defined by the equality

$$H(\varphi) = |\eta_1| \cos(2\varphi + \psi) + \frac{\pi}{\lambda \sin \alpha} \sin^2 \varphi, \tag{14}$$

whereas $V(re^{i\varphi}) = H(\varphi)r^2$. Thus, Formula (11) refines the relation (1) from Theorem 1 of the work [5] taking into account the remark for this theorem.

Using the method of finding the asymptotics for $n(r, \wp), n(r, a, \wp)$ $(a \in \mathbb{C})$, introduced in [1] (pp. 420–422), we can show that the following relation holds for the arbitrary elliptic function $f$ $(a \in \bar{\mathbb{C}})$

$$n(r, a, f) = \frac{\pi s r^2}{D} + O(r),\ r \to \infty, \tag{15}$$

where $D$ is the area of fundamental parallelogram of its periods, and $s$ is the number of the poles of the function $f$ (taking into account multiplicities) located in this parallelogram. Herewith, the quantity $O(r)/r$ is uniformly bounded with respect to $a,\ a \in \bar{\mathbb{C}}$. Namely, if the conditions (1) and (2) hold, we obtain $(a \in \bar{\mathbb{C}})$

$$n(r, a, \wp) = \frac{2\pi r^2}{\lambda \sin \alpha} + O(r),\ r \to \infty. \tag{16}$$

Therefore, as $r \to \infty$, we get

$$n(r, 0, \sigma) = \frac{1}{2} n(r, \infty, \wp) = \frac{\pi r^2}{\lambda \sin \alpha} + O(r),\ N(r, 0, \sigma) = \frac{\pi r^2}{2\lambda \sin \alpha} + O(r). \tag{17}$$

The above formulas refine the corresponding relations, which were formulated in Theorem 3 in [5].

The following relations have been indicated in the work [5]

$$T(r, \sigma) = \frac{r^2}{2\pi} \int_0^{2\pi} H^+(\varphi) d\varphi + o(r^2), \tag{18}$$

$$m(r, 0, \sigma) = \frac{r^2}{2\pi} \int_0^{2\pi} H^-(\varphi) d\varphi + o(r^2), \tag{19}$$

using which, we get

$$\delta(0, \sigma) = \varliminf_{r \to \infty} \frac{m(r, 0, \sigma)}{T(r, \sigma)} = \int_0^{2\pi} H^-(\varphi) d\varphi \Big/ \int_0^{2\pi} H^+(\varphi) d\varphi.$$

Using Formula (14), we have

$$\int_0^{2\pi} H^-(\varphi) d\varphi = \int_0^{2\pi} H^+(\varphi) d\varphi - \int_0^{2\pi} H(\varphi) d\varphi = \int_0^{2\pi} H^+(\varphi) d\varphi - \frac{\pi^2}{\lambda \sin \alpha}.$$

Then, we get

$$\delta(0,\sigma) = 1 - \frac{\pi^2}{\lambda \sin\alpha}\left(\int_0^{2\pi} H^+(\varphi)d\varphi\right)^{-1}. \tag{20}$$

Thus, we have obtained the formula for finding the Nevanlinna defect $\delta(0,\sigma)$ of the Weierstrass sigma function $\sigma(z)$.

We note that the final calculation of $\delta(0,\sigma)$ done in the work [4] has some technical difficulties in the given case. Here, the number $\eta_1$ depends on two parameters $\lambda$ and $\alpha$, in terms of which, the set of $\varphi$ such that $H(\varphi) \geq 0$ should be described. For such calculations, the following relation can be useful

$$\eta_1 = \pi^2\left(\frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{1}{\sin^2 n\pi\tau}\right) = \frac{\pi^2}{3}\left(1 - 24e^{-2\pi\tau} + O\left(e^{-4\pi \mathrm{Im}\tau}\right)\right) \tag{21}$$

being true when $\mathrm{Im}\tau \to \infty, \tau = \omega_3/\omega_1$, which has been indicated in [10] (p. 7).

To prove the next theorem, we represent the indicator $H(\varphi)$ of the function $\sigma(z)$ in the form

$$H(\varphi) = \sqrt{A^2+B^2}\cos(2\varphi - \theta) + \frac{\pi}{2\lambda\sin\alpha}, \tag{22}$$

where $A = |\eta_1|\cos\psi - \frac{\pi}{2\lambda\sin\alpha}$, $B = -|\eta_1|\sin\psi$ and $A/\sqrt{A^2+B^2} = \cos\theta$, $B/\sqrt{A^2+B^2} = \sin\theta$, following from Formula (14).

Furthermore, we formulate and prove the statements related to the estimation of the type of the function $\sigma(z)$ , the refined asymptotic, the Nevanlinna characteristics, and the function values distribution.

**Theorem 1.** *For the quantity of the form $\Delta_T[\sigma]$ of the function $T(r,\sigma)$ under conditions (1) and (2), the following relation holds:*

$$\frac{1}{2\pi}\int_0^{2\pi} H^+(\varphi)d\varphi = \Delta_T[\sigma] \geq \frac{1}{2\pi}\sqrt{A^2+B^2} + \frac{1}{4\lambda\sin\alpha}, \tag{23}$$

*where $A$ and $B$ are related to the equality (22).*

**Proof.** Since the entire function $\sigma(z)$ has the order $\rho = \rho[\sigma] = 2$, then we denote

$$\Delta_T[\sigma] = \overline{\lim_{r\to\infty}}\frac{T(r,\sigma)}{r^2},\ \Delta_M[\sigma] = \overline{\lim_{r\to\infty}}\frac{\ln M(r,\sigma)}{r^2},$$

where $M(r,\sigma) = \max\{|\sigma(z)| : |z| \leq r\}, r \geq 0$. Using (18), we obtain $\Delta_T[\sigma] = \frac{1}{2\pi}\int_0^{2\pi} H^+(\varphi)d\varphi$, hence the left-hand side in (23) is valid. According to the properties of the entire function indicator and according to the equality (22), we find

$$\Delta_M[\sigma] = \max\{H(\varphi) : 0 \leq \varphi \leq 2\pi\} = \sqrt{A^2+B^2} + \frac{\pi}{2\lambda\sin\alpha}.$$

From Theorem 1 [3] (p. 554), we get

$$\Delta_T[\sigma] \geq \frac{1}{2\pi}\Delta_M[\sigma] = \frac{1}{2\pi}\sqrt{A^2+B^2} + \frac{1}{4\lambda\sin\alpha},$$

so the right-hand side of the relation (23) is valid. Theorem 1 is proved. □

**Theorem 2.** *Under the conditions (1) and (2), the following formulas hold:*

$$\ln|\vartheta_1(z)| = U(z) + O(1),\ z \notin \Phi(d), \tag{24}$$

$$\ln\left|\vartheta_{j+1}(z)\right| = U(z) + O(|z|),\ z \notin \Phi_j(d)\ (j = \overline{1,3}), \tag{25}$$

$$\vartheta'_{j+1}(z)/\vartheta_{j+1}(z) = -\frac{2\pi i \mathrm{Im} z}{\lambda \sin\alpha} + O(1),\ z \notin \Phi_j(d)\ (j = \overline{0,3}), \tag{26}$$

*where*

$$U(z) = \frac{\pi\left(|z|^2 - \mathrm{Re} z^2\right)}{2\lambda \sin\alpha}, \tag{27}$$

$$\Phi_j(d) = \bigcup_{m,n\in\mathbb{Z}} \{z \in \mathbb{C} : \left|z - \Omega_{mn} + \omega_j\right| < d\},\ j = \overline{0,3}, \tag{28}$$

*and $d$ is the arbitrary constant, $\Phi_0(d) = \Phi(d)$, and $\Phi(d)$ is defined by the equality (4).*

**Proof.** Under conditions (1) and (2), the following equalities hold:

$$\vartheta_1(z) = \vartheta_1(0)\sigma(z)\exp(-\eta_1 z^2), \tag{29}$$

$$\vartheta_{j+1}(z) = \vartheta_{j+1}(0)\exp(-\eta_1 z^2 - \eta_j z)\frac{\sigma(z+\omega_j)}{\sigma(\omega_j)}\ (j = \overline{1,3}), \tag{30}$$

as the consequence of the formulas (6.8:3), (6.11:4) and (6.11:8) in the work [1]. Hence,

$$\ln|\vartheta_1(z)| = \ln|\sigma(z)| - \mathrm{Re}(\eta_1 z^2) + O(1), \tag{31}$$

$$\ln\left|\vartheta_{j+1}(z)\right| = \ln\left|\sigma(z+\omega_j)\right| - \mathrm{Re}(\eta_1 z^2) + O(|z|),\ j = \overline{1,3}. \tag{32}$$

Using (11) and (12), and also notations (28), we get ($j = \overline{1,3}$)

$$\ln\left|\sigma(z+\omega_j)\right| = V(z+\omega_j) + O(1) = V(z) + O(|z|),\ z \notin \Phi_j(d), \tag{33}$$

whereas $V(z+\omega_j) = V(z) + O(|z|)$. Hence, from Formulas (12) and (31)–(33), we have ($j = \overline{1,3}$)

$$\ln|\vartheta_1(z)| = V(z) - \mathrm{Re}(\eta_1 z^2) + O(1) = \frac{\pi\left(|z|^2 - \mathrm{Re} z^2\right)}{2\lambda\sin\alpha} + O(1),\ z \notin \Phi(d),$$

$$\ln\left|\vartheta_{j+1}(z)\right| = V(z) - \mathrm{Re}(\eta_1 z^2) + O(|z|) = \frac{\pi\left(|z|^2 - \mathrm{Re} z^2\right)}{2\lambda\sin\alpha} + O(|z|),\ z \notin \Phi_j(d).$$

Using notation (27), we obtain the relations (24) and (25).

The equalities (29) and (30) imply that

$$\vartheta'_1(z)/\vartheta_1(z) = \sigma'(z)/\sigma(z) - 2\eta_1 z = \zeta(z) - 2\eta_1 z, \tag{34}$$

$$\begin{aligned}\vartheta'_{j+1}(z)/\vartheta_{j+1}(z) &= \sigma'(z+\omega_j)/\sigma(z+\omega_j) - 2\eta_1 z - 2\eta_j = \\ &= \zeta(z+\omega_j) - 2\eta_1 z - 2\eta_j,\ j = \overline{1,3}.\end{aligned} \tag{35}$$

Let us rewrite the Formula (12.8) from [6] (p. 27) in the form

$$\zeta(z) = 2\eta_1 z - \frac{2\pi i}{\lambda\sin\alpha}\mathrm{Im} z + O(1),\ z \notin \Phi(d),$$

($\eta$ is defined by (12.8) and is such as $\eta = 2\eta_1$). Using the last formula, the relations (34) and (35), we have

$$\vartheta'_{j+1}(z)/\vartheta_{j+1}(z) = -\frac{2\pi i \mathrm{Im} z}{\lambda\sin\alpha} + O(1),\ z \notin \Phi_j(d),\ (j = \overline{0,3})$$

hence the relation (26) is valid. Theorem 2 is proved. □

**Remark 1.** *It follows from the Formulas (24)–(26) that, as $z = re^{i\varphi} \in \mathbb{C} \setminus \Phi_j(d), j = \overline{0,3}$, uniformly in $\varphi, \varphi \in [0, 2\pi]$, the following relations hold ($r \to \infty$):*

$$\ln\left|\vartheta_{j+1}(re^{i\varphi})\right| = S(\varphi)r^2 + o(r^2), \tag{36}$$

$$\vartheta'_{j+1}(re^{i\varphi})/\vartheta_{j+1}(re^{i\varphi}) = -\frac{2\pi i \sin\varphi}{\lambda \sin\alpha} r + o(r), \tag{37}$$

*where*

$$S(\varphi) = \frac{\pi \sin^2 \varphi}{\lambda \sin\alpha}, \tag{38}$$

*whereas $U(re^{i\varphi}) = S(\varphi)r^2$. Hence, the relations (24)–(26) refine the statement C of Theorem 2 from [7].*

**Theorem 3.** *Under conditions (1), (2) and for $j = \overline{0,3}$, the following relations take place ($r \to \infty$)*

$$n(r, 0, \vartheta_{j+1}) = \frac{\pi r^2}{\lambda \sin\alpha} + O(r),\ N(r, 0, \vartheta_{j+1}) = \frac{\pi r^2}{2\lambda \sin\alpha} + O(r), \tag{39}$$

$$T(r, \vartheta_{j+1}) = \frac{\pi r^2}{2\lambda \sin\alpha} + o(r^2), \tag{40}$$

$$m(r, a, \vartheta_{j+1}) = o(r^2)\ (a \in \mathbb{C}). \tag{41}$$

*There is no number, $a, a \in \mathbb{C}$, being the exceptional value of the function $\vartheta_{j+1}(z), j = \overline{0,3}$, in Nevanlinna's sense.*

**Proof.** As it follows from (29) and (30), the points $\Omega^{(j)}_{mn} = \Omega_{mn} - \omega_j$, $j = \overline{0,3}$, where $m \in \mathbb{Z}, n \in \mathbb{Z}$, and $\Omega^{(0)}_{mn} = \Omega_{mn}$, are simple zeros of the function $\vartheta_{j+1}(z)$ because they are simple zeros of the function $\sigma(z + \omega_j), \omega_0 = 0$. Then, it follows from the equality $n(r, 0, \vartheta_1) = n(r, 0, \sigma)$ and relations (17) that two relations are valid as $j = 0$ in (39). It is obvious that, for $j = \overline{1,3}$, two formulas are also valid in (39) because zeros of the function $\vartheta_{j+1}(z)$ are generated by "shifting" of zeros of the function $\vartheta_1(z)$ on the vector $-\omega_j$ in this case.

Note that zeros of the functions $\vartheta_{j+1}(z), j = \overline{1,3}$, are located on the countable set of rays starting at $z = 0$. Thus, using relation (36), we can say that uniformly in $\varphi, \varphi \in [0, 2\pi] \setminus P_j(\delta)$, the following equality holds ($j = \overline{0,3}$):

$$\ln\left|\vartheta_{j+1}(re^{i\varphi})\right| = S(\varphi)r^2 + o(r^2),\ r \to \infty, \tag{42}$$

where $P_j(\delta)$ is the set for which $\mathrm{mes} P_j(\delta) = \delta$ for the arbitrary $\delta > 0$ and $S(\varphi)$ is defined by the equality (38). From (42), it follows that, as $r \to \infty$ and $a \in \mathbb{C}$ uniformly in $\varphi, \varphi \in [0, 2\pi] \setminus P_j(\delta)$, the following relations hold ($j = \overline{0,3}$) :

$$\ln^+\left|\vartheta_{j+1}(re^{i\varphi})\right| = \{S(\varphi)r^2 + o(r^2)\}^+ = S^+(\varphi)r^2 + o(r^2) = \frac{r^2 \pi \sin^2\varphi}{\lambda \sin\alpha} + o(r^2), \tag{43}$$

$$\ln^+ \frac{1}{\left|\vartheta_{j+1}(re^{i\varphi}) - a\right|} = \left\{-\ln\left|\vartheta_{j+1}(re^{i\varphi}) - a\right|\right\}^+ =$$

$$= \left\{-\ln\left|\vartheta_{j+1}(re^{i\varphi})\right| - \ln\left|1 - \frac{a}{\vartheta_{j+1}(re^{i\varphi})}\right|\right\}^+ = \left\{-\ln\left|\vartheta_{j+1}(re^{i\varphi})\right| + o(1)\right\}^+ =$$

$$= \left\{-S(\varphi)r^2 + o(r^2)\right\}^+ = \{-S(\varphi)\}^+ r^2 + o(r^2) = o(r^2). \tag{44}$$

Using (43), (44) and relation (7.14) from Theorem 7.4 [3] (p. 59), we obtain $(j=\overline{0,3})$

$$T(r,\vartheta_{j+1})=m(r,\vartheta_{j+1})=\frac{1}{2\pi}\int_0^{2\pi}\ln^+\left|\vartheta_{j+1}(re^{i\varphi})\right|d\varphi=$$

$$=\frac{r^2}{2\lambda\sin\alpha}\int_0^{2\pi}\sin^2\varphi d\varphi+o(r^2)=\frac{\pi r^2}{2\lambda\sin\alpha}+o(r^2),\ r\to\infty,$$

$$m(r,a,\vartheta_{j+1})=\frac{1}{2\pi}\int_0^{2\pi}\ln^+\frac{1}{\left|\vartheta_{j+1}(re^{i\varphi})-a\right|}d\varphi=o(r^2),\ r\to\infty\ (a\in\mathbb{C}).$$

Hence, the equalities (40) and (41) are valid. These equalities imply that, for $j=\overline{0,3}$, the following relation holds:

$$\delta(a,\vartheta_{j+1})=\varliminf_{r\to\infty}\frac{m(r,a,\vartheta_{j+1})}{T(r,\vartheta_{j+1})}=0\ (a\in\mathbb{C}),$$

so the second statement of Theorem 3 is true. Theorem 3 is proved. □

**Remark 2.** *From the relations (40), (41) and Theorem 4.1 [3] (p. 27), we have*

$$N(r,a,\vartheta_{j+1})=T(r,\vartheta_{j+1})-m(r,a,\vartheta_{j+1})+O(1)=\frac{\pi r^2}{2\lambda\sin\alpha}+o(r^2),\ r\to\infty,$$

*for every* $a\in\mathbb{C}$ *and* $j=\overline{0,3}$.

**Theorem 4.** *For the functions* $h_{j+1}(z)=\vartheta'_{j+1}(z)/\vartheta_{j+1}(z)$ $(j=\overline{0,3})$ *under the conditions (1) and (2), the following relations hold* $(r\to\infty)$ :

$$n(r,h_{j+1})=\frac{\pi r^2}{\lambda\sin\alpha}+O(r),\ N(r,h_{j+1})=\frac{\pi r^2}{2\lambda\sin\alpha}+O(r), \tag{45}$$

$$T(r,h_{j+1})=\frac{\pi r^2}{2\lambda\sin\alpha}+O(r), \tag{46}$$

$$m(r,a,h_{j+1})=O(r)\ (a\in\overline{\mathbb{C}}). \tag{47}$$

*There is no number* $a, a\in\overline{\mathbb{C}}$, *being the exceptional value of the functions* $h_{j+1}(z)$ $(j=\overline{0,3})$ *in Nevanlinna's sense.*

**Proof.** Mentioned in the proof of Theorem 3 points $\Omega^{(j)}_{mn}=\Omega_{mn}-\omega_j$ $(m,n\in\mathbb{Z}),\Omega^{(0)}_{mn}=\Omega_{mn}$ are the simple poles of the functions $h_{j+1}(z)$ because they are the simple zeros of the functions $\vartheta_{j+1}(z)(j=\overline{0,3})$. This is why, from (39), we have $(j=\overline{0,3})$

$$n(r,h_{j+1})=n(r,0,\vartheta_{j+1})=\frac{\pi r^2}{\lambda\sin\alpha}+O(r),\ r\to\infty,$$

$$N(r,h_{j+1})=\frac{\pi r^2}{2\lambda\sin\alpha}+O(r),\ r\to\infty,$$

so both relations of (45) are valid, whereas $m(r,h_{j+1})=m(r,\vartheta'_{j+1}/\vartheta_{j+1})=O(\ln r), r\to\infty$; then, according to Theorem 1.3 [3] (p. 122), we obtain

$$T(r,h_{j+1})=m(r,h_{j+1})+N(r,h_{j+1})=\frac{\pi r^2}{2\lambda\sin\alpha}+O(r),\ r\to\infty\ (j=\overline{0,3}),$$

hence the relation (46) is true.

The equalities (34) and (35) imply that

$$h'_{j+1}(z) = \zeta'(z+\omega_j) - 2\eta_1 = -\wp(z+\omega_j) - 2\eta_1, \text{ where } j = \overline{0,3}, \omega_0 = 0.$$

Let us denote $F_j(z) = \wp(z+\omega_j)$. Then, using relations (2.5) from [3] (p. 128), we get $(a \in \mathbb{C})$

$$m(r,a,h_{j+1}) \le m(r, \frac{1}{h'_{j+1}}) + O(\ln r) = m(r, -2\eta_1, F_j) + O(\ln r). \tag{48}$$

The function $F_j(z)$ $(j = \overline{0,3})$ is the elliptic function with fundamental periods $2\omega_1, 2\omega_3$, and it has all the same values as the function $\wp(z)$ in its fundamental period parallelogram. Hence, from (16) under conditions (1) and (2), it follows that $(r \to \infty)$

$$n(r,a,F_j) = \frac{2\pi r^2}{\lambda \sin\alpha} + O(r),$$

$$N(r,a,F_j) = \frac{\pi r^2}{\lambda \sin\alpha} + O(r), \tag{49}$$

where the quantity $O(r)/r$ is uniformly bounded in $a, a \in \overline{\mathbb{C}}$. Putting $a = e^{i\theta}$ in (49) and using Cartan's identity (4.13) [3] (p. 33), we find $(j = \overline{0,3})$

$$T(r,F_j) = \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta}, F_j) d\theta + O(1) = \frac{\pi r^2}{\lambda \sin\alpha} + O(r),\ r \to \infty. \tag{50}$$

Theorem 4.1 [3] (p. 27), equalities (50) and relations (49) (where we put $a = -2\eta_1$) to imply $(j = \overline{0,3})$

$$m(r, -2\eta_1, F_j) = T(r,F_j) - N(r, -2\eta_1, F_j) + O(1) = O(r),\ r \to \infty.$$

Hence, taking into account (48), we have $m(r,a,h_{j+1}) = O(r),\ r \to \infty\ (a \in \mathbb{C})$. It is obvious that, for $a = \infty$, such a relation is also valid. Thus, the relation (47) is proved. From Formulas (46) and (47), we obtain

$$\delta(a, h_{j+1}) = \varliminf_{r\to\infty} \frac{m(r,a,h_{j+1})}{T(r,h_{j+1})} = 0,\ j = \overline{0,3}\ (a \in \overline{\mathbb{C}}),$$

so the second statement of the Theorem is true. Theorem 4 is proved. □

**Theorem 5.** *For the arbitrary elliptic function $f \ne const$, the following relations hold $(r \to \infty)$ :*

$$N(r,a,f) = \frac{\pi s r^2}{2D} + O(r), \tag{51}$$

$$T(r,f) = \frac{\pi s r^2}{2D} + O(r), \tag{52}$$

$$m(r,a,f) = O(r), \tag{53}$$

*where $a \in \overline{\mathbb{C}}$, $D$ and $s$ are the quantities related to the Formula (15).*

*There is no number $a, a \in \overline{\mathbb{C}}$, being the exceptional value of the function $f$ in Nevanlinna's sense.*

**Proof.** From Formula (15), we get $(a \in \overline{\mathbb{C}})$

$$N(r,a,f) = \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t} dt + n(0,a,f)\ln r = \frac{\pi s r^2}{2D} + O(r),$$

where $r \to \infty$, so the relation (51) is true. Using again Cartan's identity (4.13) [3] (p. 33) and (51), we obtain $(r \to \infty)$

$$T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{i\theta}, f) d\theta + O(1) = \frac{\pi s r^2}{2D} + O(r),$$

hence the relation (52) is valid. According to Theorem 4.1 [3] (p. 27) and the relations (51) and (52), we get $(a \in \overline{\mathbb{C}})$

$$m(r,a,f) = T(r,f) - N(r,a,f) + O(1) = O(r),\ r \to \infty.$$

It proves the equality (53). The Formulas (52) and (53) imply

$$\delta(a,f) = \varliminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 0\ (a \in \overline{\mathbb{C}}).$$

Hence, the second statement of Theorem is valid. Theorem 5 is proved. □

**Remark 3.** *Using (52), we obtain*

$$\rho[f] = \varlimsup_{r\to\infty} \frac{\ln T(r,f)}{\ln r} = 2,$$

$$\sigma[f] = \varlimsup_{r\to\infty} \frac{T(r,f)}{r^2} = \frac{\pi s}{2D},$$

*so the arbitrary elliptic function $f \neq const$ has the order $\rho[f] = 2$ and normal type with the value of the type $\sigma[f] = \frac{\pi s}{2D}$.*

## 4. Conclusions

In this paper, we have found the Formula (10) for obtaining the quantity of Nevanlinna defect $\delta(0,\sigma)$ of the Weierstrass sigma function $\sigma(z)$ for the value $a = 0$. We have obtained the asymptotic Formulas (24)–(26) for the Jacobi theta functions $\vartheta_{j+1}(z), j = \overline{0,3}$, and their logarithmic derivatives, where the reminders are estimated more accurately than in corresponding formulas of the paper [8]. On the basis of such formulas, we have indicated the asymptotics of Nevanlinna characteristics of these functions, and we have proved that there is no number $a, a \in \mathbb{C}$, being the exceptional value in Nevanlinna's sense for these functions. In addition, for the functions $\vartheta'_{j+1}(z)/\vartheta_{j+1}(z)$ $(j = \overline{0,3})$, the last conclusion is also true for $a = \infty$. We have found the asymptotics of Nevanlinna characteristics for the arbitrary elliptic function. It allows for concluding that there is no number $a, a \in \overline{\mathbb{C}}$, being the exceptional value in Nevanlinna's sense for it.

As further research, it is possible, using the Formula (10), to obtain the value of the defect $\delta(0,\sigma)$ in terms of parameters $\lambda, \alpha$ in finite form. Herewith, the Formula (21) from the paper [10] can be useful. Another important problem is the obtaining of the asymptotic values and asymptotic curves [3] (p. 223) of the functions $\sigma(z), \zeta(z), \vartheta_{j+1}(z), \vartheta'_{j+1}(z)/\vartheta_{j+1}(z) (j = \overline{0,3})$. It would be desirable to investigate the questions related to its Julia rays and Julia sets of points [1] (pp. 572–573) for the functions $\vartheta_{j+1}(z)$, in the same way as it had been done in the paper [10] for the function $\sigma(z)$. To investigate two previous problems, one can apply asymptotic Formulas (24)–(26), formulated in this paper. These formulas could also be useful for investigation of the differential equations solutions, where the above-mentioned functions are.

**Author Contributions:** Conceptualization, M.K. and Y.K.; methodology, M.K. and Y.K.; formal analysis, M.K. and Y.K.; writing-original draft preparation, M.K. and Y.K.; writing-review and editing, M.K. and Y.K. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported by the Grant of the Ministry of Education and Science of Ukraine, 0120U102630.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Markushevich, A.I. *Theory of Analytic Functions*; Nauka: Moscow, Russia, 1968; Volume 2. (In Russian)
2. Nevanlinna, R. *Eindeutige Analytische Funktionen*; Springer: Berlin/Heidelberg, Germany, 2013.
3. Gol'dberg, A.A.; Ostrovskii, I.V. *Distribution of Values of Meromorphic Functions*; Nauka: Moscow, Russia, 1970. (In Russian)
4. Gol'dberg, A.A. The distribution of values of the Weierstrass sigma-function. *Izv. Vyssh. Uchebn. Zaved.* **1966**, *1*, 43–46. (In Russian)
5. Korenkov, N.E. On the distribution of values of the Weierstrass sigma-function. *Mat. Sb.* **1976**, *1*, 240–242.
6. Lyubarskii, Y.I.; Sodin, M.L. *Analogs of Functions of the Sine Type for Convex Domains*; Preprint; Institute for Low Temperature Physics and Engineering, Ukrainian Academy of Sciences: Kharkov, Russia, 1986. (In Russian)
7. Korenkov, N.E. Asymptotic properties of the Weierstrass sigma-functions and the Jacobi theta-functions. *Ukr. Math. J.* **1989**, *41*, 612–614. [CrossRef]
8. Korenkov, M.E.; Kharkevych, Y.I. On the asymptotics of associated sigma-functions and Jacobi theta-functions. *Ukr. Math. J.* **2019**, *70*, 1326–1330. [CrossRef]
9. Zajac, J.; Korenkov, M.E.; Kharkevych, Y.I. On the asymptotics of some Weierstrass functions. *Ukr. Math. J.* **2015**, *67*, 154–158. [CrossRef]
10. Bergweiler, W. Lebesgue measure of Julia sets and escaping sets of certain entire functions. *Fundam. Math.* **2018**, *242*, 281–301. [CrossRef]
11. Zhuan, Y. On Nevanlinna's error terms. *Duke Math. J.* **1991**, *64*, 243–260.
12. Favorov, S.J. Sunyer-i-Balaguer's almost elliptic functions and Yosida's normal functions. *J. d'Anal. Math.* **2008**, *104*, 307–340. [CrossRef]
13. Laine, I. *Nevanlinna Theory and Complex Differential Equations*; Walter de Gruyter & Co.: Berlin, Germany, 1993.
14. Mokhon'ko, A.A. Malmquist theorem for solutions of differential equations in a neighborhood of a logarithmic singular point. *Ukr. Math. J.* **2004**, *56*, 577–585. [CrossRef]
15. Steinmetz, N. *Nevanlinna Theory, Normal Families, and Algebraic Differential Equations*; Springer International Publishing AG: Cham, Switzerland, 2017.
16. Mokhonko, A.A.; Mokhonko, A.Z. On meromorphic solutions of the systems of linear differential equations with meromorphic coefficients. *Ukr. Math. J.* **2020**, *71*, 1402–1418. [CrossRef]