axioms

MDPI

Article

Multiplicity of Positive Solutions to Nonlocal Boundary Value Problems with Strong Singularity

Chan-Gyun Kim

Department of Mathematics Education, Chinju National University of Education, Jinju 52673, Korea; cgkim75@cue.ac.kr

Abstract: In this paper, we consider generalized Laplacian problems with nonlocal boundary conditions and a singular weight, which may not be integrable. The existence of two positive solutions to the given problem for parameter λ belonging to some open interval is shown. Our approach is based on the fixed point index theory.

Keywords: generalized Laplacian problems; multiplicity of positive solutions; singular weight function; nonlocal boundary conditions

check for updates

Citation: Kim, C.-G. Multiplicity of Positive Solutions to Nonlocal Boundary Value Problems with Strong Singularity. *Axioms* **2022**, *11*, 7. https://doi.org/10.3390/axioms11010007

Academic Editor: Chiming Chen

Received: 18 November 2021
Accepted: 21 December 2021
Published: 23 December 2021

1. Introduction

Consider the following singular φ-Laplacian problem:

$$(q(t)\varphi(u'(t)))' + \lambda h(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1}$$

$$u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r), \tag{2}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $q \in C([0,1],(0,\infty))$, $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a parameter, $f \in C(\mathbb{R}_+,\mathbb{R}_+)$, $h \in C((0,1),\mathbb{R}_+)$, and the integrator functions α_i $(i = 1,2)$ are nondecreasing on $[0,1]$.

All integrals in (2) are meant in the sense of Riemann–Stieltjes. Throughout this paper, we assume the following hypotheses:

(F_1) There exist increasing homeomorphisms $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that:

$$\varphi(x)\psi_1(y) \le \varphi(yx) \le \varphi(x)\psi_2(y) \text{ for all } x,y \in [0,\infty). \tag{3}$$

(F_2) For $i = 1,2$, $\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0,1)$.

Let $\xi : [0,\infty) \to [0,\infty)$ be an increasing homeomorphism. Then, we denote by $\mathcal{H}_\xi$ the set:

$$\left\{ g \in C((0,1),\mathbb{R}_+) : \int_0^1 \xi^{-1}\left(\left|\int_s^{\frac{1}{2}} g(\tau)d\tau\right|\right)ds < \infty \right\}.$$

It is well known that if (F_1) is assumed, then:

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x,y \in \mathbb{R}_+ \tag{4}$$

and

$$L^1(0,1) \cap C(0,1) \subsetneq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$$

(see, e.g., ([1], Remark 1)).

It is not hard to see that any function of the form

$$\varphi(s) = \sum_{k=1}^{n} |s|^{p_k-2}s$$

satisfies the assumption (F_1) with $\psi_1(s) = \min\{s^{p_n-1}, s^{p_1-1}\}$ and $\psi_2(s) = \max\{s^{p_n-1}, s^{p_1-1}\}$ for $s \in \mathbb{R}_+$ (see, e.g., [1,2]). Here, $n \in \mathbb{N}$, $p_k \in (1,\infty)$ for $1 \le k \le n$ and $p_i \le p_j$ for $1 \le i \le j \le n$. If $n = 1$, it follows that $\varphi(s) = |s|^{p-2}s$ for some $p \in (1,\infty)$, that is, Equation (1) becomes the classical p-Laplacian one.

The study of problems with nonlocal boundary conditions is motivated by a variety of applications such as beam deflection [3], chemical reactor theory [4], and thermostatics [5]. For this reason, the existence of positive solutions for nonlocal boundary value problems has been extensively studied. For example, Liu [6] studied the multi-point boundary value problem, which is a special case of problem (1)–(2) with $\lambda = 1$. Under various assumptions of the nonlinearity f, the existence of positive solutions was shown. Bachouche, Djebali and Moussaoui [7] proved, under suitable assumptions of the nonlinearity $f = f(t,u,u')$ satisfying the L^1-Carathéodory condition, several existence results for positive solutions to φ-Laplacian boundary value problems involving linear bounded operators in the boundary conditions. Yang [8], by using the Avery– Peterson fixed point theorem, obtained the existence of at least three positive solutions to the p-Laplacian equation with integral boundary conditions. Goodrich [9] studied perturbed Volterra integral operator equations and, as an application, established the existence of at least one positive solution to the p-Laplacian differential equation with nonlocal boundary conditions. Jeong and Kim [10] obtained sufficient conditions on the nonlinearity f for the existence of multiple positive solutions to problem (1)–(2) with $\lambda = 1$. For the nonlinearity $f = f(t,s)$ satisfying $f(t,0) \not\equiv 0$, Kim [11] showed the existence, nonexistence and multiplicity of positive solutions to problem (1)–(2) by investigating the shape of the unbounded solution continuum. For the historical development of the theory of the problems with nonlocal boundary conditions, we refer the reader to the survey papers [12–15].

In this paper, we show the existence of two positive solutions to nonlocal boundary value problems (1)–(2) for λ belonging to some open interval in the case when either $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$. Here,

$$f_0 := \lim_{s\to 0} \frac{f(s)}{\varphi(s)} \text{ and } f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi(s)}.$$

For problems with zero Dirichlet boundary conditions, that is, $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$, there have been several works for problems with such assumptions on the nonlinearity f. For example, when $\varphi(s) = |s|^{p-2}s$ for some $p \in (1,\infty)$, $q \equiv 1$ and $h \in \mathcal{H}_\varphi$, Agarwal, Lü and O'Regan [16] investigated the existence of two positive solutions to problem (1)–(2). After that, Wang [17] obtained the same multiplicity results in [16] for generalized φ-Laplacian problems with the assumptions that φ satisfies (F_1) and $h \in C[0,1]$. Recently, Lee and Xu [18] extended the result of [17] to the singularly weighed φ-Laplacian problem under the assumptions that $q \equiv 1$ and $h \in \mathcal{H}_{\psi_1}$, that is, h may be singular at $t = 0$ and/or $t = 1$.

The aim of this paper is to generalize the results for the previous papers [16–18]. The main result is stated as follows:

Theorem 1. *Assume that (F_1), (F_2) and $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ hold.*

(1) *If $f_0 = f_\infty = \infty$, then there exist $\lambda^* \in (0,\infty)$ and $m^* \in (0,\infty)$ such that problem (1) has two positive solutions $u_1(\lambda)$ and $u_2(\lambda)$ for any $\lambda \in (0,\lambda^*)$. Moreover, $u_1(\lambda)$ and $u_2(\lambda)$ can be chosen with the property that:*

$$0 < \|u_1(\lambda)\|_\infty < m^* < \|u_2(\lambda)\|_\infty, \lim_{\lambda\to 0} \|u_1(\lambda)\|_\infty = 0 \text{ and } \lim_{\lambda\to 0} \|u_2(\lambda)\|_\infty = \infty.$$

(2) *If $f_0 = f_\infty = 0$, then there exist $\lambda_* \in (0,\infty)$ and $m_* \in (0,\infty)$ such that (1) has two positive solutions $u_1(\lambda)$ and $u_2(\lambda)$ for any $\lambda \in (\lambda_*,\infty)$. Moreover, $u_1(\lambda)$ and $u_2(\lambda)$ can be chosen with the property that:*

$$0 < \|u_1(\lambda)\|_\infty < m_* < \|u_2(\lambda)\|_\infty, \lim_{\lambda\to\infty} \|u_1(\lambda)\|_\infty = 0 \text{ and } \lim_{\lambda\to\infty} \|u_2(\lambda)\|_\infty = \infty.$$

The rest of this paper is organized as follows. In Section 2, preliminary results which are essential for proving Theorem 1 are provided. In Section 3, the proof of Theorem 1 is given. Finally, the summary of this paper is provided in Section 4.

2. Preliminaries

Throughout this section, we assume that $(F_1), (F_2)$ and $h \in \mathcal{H}_\varphi \setminus \{0\}$ hold. For convenience, we use some notations which were used by Jeong and Kim ([10]).

The usual maximum norm in a Banach space $C[0,1]$ is denoted by:

$$\|u\|_\infty := \max_{t\in[0,1]} |u(t)| \text{ for } u \in C[0,1],$$

and let

$$\alpha_h := \inf\{x \in (0,1) : h(x) > 0\}, \beta_h := \sup\{x \in (0,1) : h(x) > 0\},$$

$$\bar{\alpha}_h := \sup\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (\alpha_h, x)\},$$

$$\bar{\beta}_h := \inf\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (x, \beta_h)\},$$

$$\gamma_h^1 := \frac{1}{4}(3\alpha_h + \bar{\alpha}_h) \text{ and } \gamma_h^2 := \frac{1}{4}(\bar{\beta}_h + 3\beta_h).$$

Then, since $h \in C((0,1), \mathbb{R}_+) \setminus \{0\}$, we have two cases, either:

$$\text{(i) } 0 \le \alpha_h < \bar{\alpha}_h \le \bar{\beta}_h < \beta_h \le 1$$

or

$$\text{(ii) } 0 \le \alpha_h = \bar{\beta}_h < \beta_h \le 1 \text{ and } 0 \le \alpha_h < \bar{\alpha}_h = \beta_h \le 1.$$

Consequently,

$$h(t) > 0 \text{ for } t \in (\alpha_h, \bar{\alpha}_h) \cup (\bar{\beta}_h, \beta_h), \text{ and } 0 \le \alpha_h < \gamma_h^1 < \gamma_h^2 < \beta_h \le 1. \tag{5}$$

Let $\rho_h := \rho_1 \min\{\gamma_h^1, 1 - \gamma_h^2\} \in (0,1)$, where

$$q_0 := \min_{t\in[0,1]} q(t) > 0 \text{ and } \rho_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1].$$

Then

$$\mathcal{K} := \{u \in C([0,1], \mathbb{R}_+) : u(t) \ge \rho_h \|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2]\}$$

is a cone in $C[0,1]$. For $r > 0$, let:

$$\mathcal{K}_r := \{u \in \mathcal{K} \ : \ \|u\|_\infty < r\}, \partial\mathcal{K}_r := \{u \in \mathcal{K} \ : \ \|u\|_\infty = r\}$$

and $\overline{\mathcal{K}}_r := \mathcal{K}_r \cup \partial\mathcal{K}_r$. Let

$$C_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right) \min\left\{\int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds, \int_{\gamma_h}^{\gamma_h^2} \psi_2^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right\};$$

$$C_2 := \psi_1^{-1}\left(\frac{1}{q_0}\right) \max\left\{A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds, A_2 \int_{\gamma_h}^1 \psi_1^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right\}.$$

Here, $\gamma_h := \dfrac{\gamma_h^1 + \gamma_h^2}{2}$ and $A_i := (1 - \hat{\alpha}_i)^{-1} \ge 1$ for $i = 1, 2$. Clearly, by (5),

$$C_1 > 0 \text{ and } C_2 > 0.$$

Define continuous functions $f_*, f^* : \mathbb{R}_+ \to \mathbb{R}_+$ by, for $m \in \mathbb{R}_+$,

$$f_*(m) := \min\{f(y) : \rho_h m \le y \le m\} \text{ and } f^*(m) := \max\{f(y) : 0 \le y \le m\}.$$

Define $R_1, R_2 : (0,\infty) \to (0,\infty)$ by:

$$R_1(m) := \frac{1}{f_*(m)}\varphi\left(\frac{m}{C_1}\right) \text{ and } R_2(m) := \frac{1}{f^*(m)}\varphi\left(\frac{m}{C_2}\right) \text{ for } m \in (0,\infty).$$

By (4) and (F_2), $\psi_2^{-1}(y) \le \psi_1^{-1}(y)$ for all $y \in \mathbb{R}_+$ and $A_i = (1-\hat{\alpha}_i)^{-1} \ge 1$ for $i = 1,2$. Consequently, $0 < C_1 < C_2$ and

$$0 < R_2(m) < R_1(m) \text{ for all } m \in (0,\infty). \tag{6}$$

Remark 2. *(1) For any $L \in C(\mathbb{R}_+, \mathbb{R}_+)$, let $L_c := \lim_{m\to c} \frac{L(m)}{\varphi(m)}$ for $c \in \{0,\infty\}$. Then it is easy to prove that:*

$$(f_*)_c = (f^*)_c = 0 \text{ if } f_c = 0 \text{ , and } (f_*)_c = (f^*)_c = \infty \text{ if } f_c = \infty. \tag{7}$$

For the reader's convenience, we give the proof for the case $(f_*)_\infty = (f^*)_\infty = 0$ if $f_\infty = 0$. The proofs for other cases are similar. Indeed, let $\epsilon > 0$ be given and let $f_\infty = 0$ be assumed. Then, there exists $M > 0$ such that:

$$\frac{f(s)}{\varphi(s)} < \epsilon \text{ for all } s \ge M, \tag{8}$$

and

$$f^*(s) \le f^*(M) + f(x_{M,s}) \text{ for } s \ge M.$$

Here $x_{M,s}$ is the point in $[M,s]$ satisfying

$$f(x_{M,s}) = \max\{f(x) : M \le x \le s\}.$$

By (8), for $s \ge M$,

$$0 \le \frac{f_*(s)}{\varphi(s)} \le \frac{f^*(s)}{\varphi(s)} \le \frac{f^*(M)}{\varphi(s)} + \frac{f(x_{M,s})}{\varphi(x_{M,s})} \le \frac{f^*(M)}{\varphi(s)} + \epsilon,$$

which implies

$$0 \le \limsup_{s\to\infty} \frac{f_*(s)}{\varphi(s)} \le \limsup_{s\to\infty} \frac{f^*(s)}{\varphi(s)} \le \epsilon. \tag{9}$$

Consequently, $(f_*)_\infty = (f^*)_\infty = 0$, since (9) is true for all $\epsilon > 0$.
(2) By (3) and (7), for $i \in \{1,2\}$,

$$\lim_{m\to 0^+} R_i(m) = 0 \text{ if } f_0 = \infty, \text{ and } \lim_{m\to\infty} R_i(m) = 0 \text{ if } f_\infty = \infty; \tag{10}$$

$$\lim_{m\to 0^+} R_i(m) = \infty \text{ if } f_0 = 0, \text{ and } \lim_{m\to\infty} R_i(m) = \infty \text{ if } f_\infty = 0. \tag{11}$$

For $g \in \mathcal{H}_\varphi$, consider the following problem:

$$\begin{cases} (q(t)\varphi(u'(t)))' + g(t) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r). \end{cases} \tag{12}$$

Define a function $T : \mathcal{H}_\varphi \to C[0,1]$ by $T(0) = 0$ and, for $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$T(g)(t) = \begin{cases} A_1 \int_0^1 \int_0^r I_g(s,\sigma)ds d\alpha_1(r) + \int_0^t I_g(s,\sigma)ds, & \text{if } 0 \le t \le \sigma, \\ -A_2 \int_0^1 \int_r^1 I_g(s,\sigma)ds d\alpha_2(r) - \int_t^1 I_g(s,\sigma)ds, & \text{if } \sigma \le t \le 1, \end{cases} \tag{13}$$

where

$$I_g(s,x) := \varphi^{-1}\left(\frac{1}{q(s)}\int_s^x g(\tau)d\tau\right) \text{ for } s,x \in (0,1)$$

and $\sigma = \sigma(g)$ is a constant satisfying:

$$A_1 \int_0^1 \int_0^r I_g(s,\sigma) ds d\alpha_1(r) + \int_0^\sigma I_g(s,\sigma) ds = -A_2 \int_0^1 \int_r^1 I_g(s,\sigma) ds d\alpha_2(r) - \int_\sigma^1 I_g(s,\sigma) ds. \tag{14}$$

For any $g \in \mathcal{H}_\varphi$ and any σ satisfying (14), $T(g)$ is monotone increasing on $[0,\sigma)$ and monotone decreasing on $(\sigma, 1]$. We notice that $\sigma = \sigma(g)$ is not necessarily unique, but $T(g)$ is independent of the choice of σ satisfying (14) (see [10], [Remark 2]).

Lemma 3. ([10], [Lemma 2]) *Assume that $(F_1),(F_2)$ and $g \in \mathcal{H}_\varphi$ hold. Then $T(g)$ is a unique solution to problem (12), satisfying the following properties:*
(i) $T(g)(t) \geq \min\{T(g)(0), T(g)(1)\} \geq 0$ *for* $t \in [0,1]$;
(ii) *for any* $g \not\equiv 0$, $\max\{T(g)(0), T(g)(1)\} < \|T(g)\|_\infty$;
(iii) σ *is a constant satisfying (14) if and only if* $T(g)(\sigma) = \|T(g)\|_\infty$;
(iv) $T(g)(t) \geq \rho_1 \min\{t, 1-t\} \|T(g)\|_\infty$ *for* $t \in [0,1]$ *and* $T(g) \in \mathcal{K}$.

Define a function $F : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by

$$F(\lambda, u)(t) := \lambda h(t) f(u(t)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{K} \text{ and } t \in (0,1).$$

Clearly, $F(\lambda, u) \in \mathcal{H}_\varphi$ for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$, since $h \in \mathcal{H}_\varphi$. Let us define an operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by

$$H(\lambda, u) := T(F(\lambda, u)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}.$$

By Lemma 3 (iv), $H(\mathbb{R}_+ \times \mathcal{K}) \subseteq \mathcal{K}$, and consequently H is well defined. Moreover, u is a solution to BVP (1)–(2) if and only if $H(\lambda, u) = u$ for some $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$.

Lemma 4. ([11], [Lemma 4]) *Assume that $(F_1),(F_2)$ and $h \in \mathcal{H}_\varphi \setminus \{0\}$ hold. Then, the operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous.*

Finally, we recall a well-known theorem of the fixed point index theory.

Theorem 5. ([19,20]) *Assume that, for some $m > 0$, $\mathcal{H} : \overline{\mathcal{K}}_m \to \mathcal{K}$ is completely continuous. Then the following assertions are true:*

(i) $i(\mathcal{H}, \mathcal{K}_m, \mathcal{K}) = 1$ *if* $\|\mathcal{H}(u)\|_\infty < \|u\|_\infty$ *for* $u \in \partial\mathcal{K}_m$;

(ii) $i(\mathcal{H}, \mathcal{K}_m, \mathcal{K}) = 0$ *if* $\|\mathcal{H}(u)\|_\infty > \|u\|_\infty$ *for* $u \in \partial\mathcal{K}_m$.

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1.

Proof of Theorem 1. (1) Since $f_0 = f_\infty = \infty$, from (10), it follows that, for $i = 1, 2$,

$$\lim_{m \to 0} R_i(m) = \lim_{m \to \infty} R_i(m) = 0. \tag{15}$$

We can choose $\lambda^* > 0$ and $m^* > 0$ satisfying:

$$\lambda^* = \max\{R_2(m) : m \in \mathbb{R}_+\} \text{ and } R_2(m^*) = \lambda^*.$$

Let $\lambda \in (0, \lambda^*)$ be fixed. By (6), there exist $m_1 = m_1(\lambda), m_2 = m_2(\lambda)$, $M_1 = M_1(\lambda), M_2 = M_2(\lambda)$ such that:

$$m_1 < m_2 < m^* < M_2 < M_1$$

and

$$\max\{R_1(m_1), R_1(M_1)\} < \lambda < \min\{R_2(m_2), R_2(M_2)\}.$$

Since $\lambda < R_2(m_2)$,

$$0 \leq \lambda f(v(t)) \leq \lambda f^*(m_2) = \frac{\lambda}{R_2(m_2)}\varphi\left(\frac{m_2}{C_2}\right) < \varphi\left(\frac{m_2}{C_2}\right) \text{ for } t \in [0,1]. \tag{16}$$

Let $u \in \partial\mathcal{K}_{m_2}$ be given and let σ be a number satisfying $H(\lambda,u)(\sigma) = \|H(\lambda,u)\|_\infty$. We have two cases: either (i) $\sigma \in (0,\gamma_h)$ or (ii) $\sigma \in [\gamma_h,1)$. We only give the proof for the case (i), since the case (ii) can be proved in a similar manner. First, we show that:

$$\|H(\lambda,u)\|_\infty \leq A_1 \int_0^\sigma I_{F(\lambda,u)}(s,\sigma) \text{ for } s \in [0,\sigma]. \tag{17}$$

Since $I_{F(\lambda,u)}(s,x) \geq 0$ for $x \geq s$ and $I_{F(\lambda,u)}(s,x) \leq 0$ for $x \leq s$,

$$\begin{aligned}
&\int_0^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) \\
= \;& -\int_0^\sigma \int_r^\sigma I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_\sigma^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) \leq 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
H(\lambda,u)(\sigma) &= A_1 \int_0^1 \int_0^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds \\
&= A_1\left[\int_0^1 \int_0^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \left(1 - \int_0^1 d\alpha_1(r)\right)\int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds\right] \\
&= A_1\left[\int_0^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds\right] \\
&\leq A_1 \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds.
\end{aligned}$$

From (4), (16), (17) and the definition of C_2, it follows that:

$$\begin{aligned}
\|H(\lambda,u)\|_\infty &\leq A_1 \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&< A_1 \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau \frac{1}{q_0}\varphi\left(\frac{m_2}{C_2}\right)\right)ds \\
&\leq A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\varphi^{-1}\left(\frac{1}{q_0}\varphi\left(\frac{m_2}{C_2}\right)\right) \\
&\leq A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\psi_1^{-1}\left(\frac{1}{q_0}\right)\frac{m_2}{C_2} \leq m_2 = \|u\|_\infty.
\end{aligned}$$

By Theorem 5 (i),

$$i(H(\lambda,\cdot),\mathcal{K}_{m_2},\mathcal{K}) = 1. \tag{18}$$

Let $v \in \partial\mathcal{K}_{m_1}$ be given. Since $\lambda > R_1(m_1)$ and $\rho_h m_1 \leq v(t) \leq m_1$ for $t \in [\gamma_h^1,\gamma_h^2]$, and

$$\lambda f(v(t)) \geq \lambda f_*(m_1) = \frac{\lambda}{R_1(m_1)}\varphi\left(\frac{m_1}{C_1}\right) > \varphi\left(\frac{m_1}{C_1}\right) \text{ for } t \in [\gamma_h^1,\gamma_h^2]. \tag{19}$$

Let σ be a constant satisfying $H(\lambda,v)(\sigma) = \|H(\lambda,v)\|_\infty$. Then we have two cases: either (i) $\sigma \in [\gamma_h,1)$ or (ii) $\sigma \in (0,\gamma_h)$. We only give the proof for the case (i), since the case (ii) can be proved in a similar manner. By Lemma 3 (i), $H(\lambda,v)(0) \geq 0$, and it follows from (4), (19) and the definition of C_1 that:

$$
\begin{aligned}
\|H(\lambda,v)\|_\infty &= H(\lambda,v)(0)+\int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau)f(v(\tau))d\tau\right)ds \\
&> \int_{\gamma_h^1}^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\frac{1}{\|q\|_\infty}\varphi\left(\frac{m_1}{C_1}\right)\right)ds \\
&\geq \int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\varphi^{-1}\left(\frac{1}{\|q\|_\infty}\varphi\left(\frac{m_1}{C_1}\right)\right) \\
&\geq \int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\frac{m_1}{C_1} \geq m_1=\|v\|_\infty.
\end{aligned}
$$

By Theorem 5 (ii),

$$i(H(\lambda,\cdot),\mathcal{K}_{m_1},\mathcal{K})=0. \tag{20}$$

From (18), (20) and the additivity property,

$$i(H(\lambda,\cdot),\mathcal{K}_{m_2}\setminus\overline{\mathcal{K}}_{m_1},\mathcal{K})=-1.$$

Then there exists $u_\lambda^1\in\mathcal{K}_{m_2}\setminus\overline{\mathcal{K}}_{m_1}$ such that $H(\lambda,u_\lambda^1)=u_\lambda^1$ by the solution property. Consequently, problem (1)–(2) has a positive solution u_λ^1 satisfying $\|u_\lambda^1\|_\infty\in(m_1,m_2)$.

By the similar argument above, one can show the existence of another positive solution u_λ^2 to problem (1) satisfying $\|u_\lambda^2\|_\infty\in(M_2,M_1)$. Moreover, by (15), we may choose $m_2(\lambda),M_2(\lambda)$ satisfying $m_2(\lambda)\to 0$ and $M_2(\lambda)\to\infty$ as $\lambda\to 0^+$, and thus (1) has two positive solutions u_λ^1,u_λ^2 for any $\lambda\in(0,\lambda^*)$ satisfying $\|u_\lambda^1\|_\infty\to 0$ and $\|u_\lambda^2\|_\infty\to\infty$ as $\lambda\to 0^+$.

(2) Since $f_0=f_\infty=0$, from (11), it follows that, for $i=1,2$,

$$\lim_{m\to 0}R_i(m)=\lim_{m\to\infty}R_i(m)=\infty. \tag{21}$$

We can choose $\lambda_*>0$ and $m_*>0$ satisfying

$$\lambda_*=\min\{R_1(m):m\in\mathbb{R}_+\}\text{ and }R_1(m_*)=\lambda_*.$$

Let $\lambda\in(\lambda_*,\infty)$ be fixed. By (6), there exist $m_1=m_1(\lambda),m_2=m_2(\lambda)$, $M_1=M_1(\lambda),M_2=M_2(\lambda)$ such that

$$m_2<m_1<m_*<M_1<M_2$$

and

$$\max\{R_1(m_1),R_1(M_1)\}<\lambda<\min\{R_2(m_2),R_2(M_2)\}.$$

By the argument similar to those in the proof of (1),

$$i(H(\lambda,\cdot),\mathcal{K}_{m_1}\setminus\overline{\mathcal{K}}_{m_2},\mathcal{K})=i(H(\lambda,\cdot),\mathcal{K}_{M_2}\setminus\overline{\mathcal{K}}_{M_1},\mathcal{K})=-1.$$

Thus, problem (1)–(2) has two positive solutions u_λ^1,u_λ^2 for any $\lambda\in(\lambda_*,\infty)$ satisfying $\|u_\lambda^1\|_\infty\in(m_2,m_1)$ and $\|u_\lambda^2\|_\infty\in(M_1,M_2)$. Moreover, by (21), we may choose $m_1(\lambda),M_1(\lambda)$ satisfying $m_1(\lambda)\to 0$ and $M_1(\lambda)\to\infty$ as $\lambda\to\infty$, and thus (1)–(2) has two positive solutions u_λ^1,u_λ^2 for any $\lambda\in(\lambda_*,\infty)$ satisfying $\|u_\lambda^1\|_\infty\to 0$ and $\|u_\lambda^2\|_\infty\to\infty$ as $\lambda\to\infty$.

4. Conclusions

In this paper, we establish the existence of two positive solutions to nonlocal boundary value problems (1)–(2) for λ belonging to some open interval in the case when either $f_0=f_\infty=\infty$ or $f_0=f_\infty=0$.

Let φ be an odd function satisfying $\varphi(x)=x+x^2$ for $x\in\mathbb{R}_+$. Then, φ satisfies (F_1) with $\psi_1(y)=\min\{y,y^2\}$ and $\psi_2(y)=\max\{y,y^2\}$. Define $h:(0,1)\to\mathbb{R}_+$ by:

$$h(t)=0\text{ for }t\in[0,\tfrac{1}{4}]\text{ and }h(t)=(t-\tfrac{1}{4})(1-t)^{-c}\text{ for }t\in(\tfrac{1}{4},1).$$

Then, since $\psi_1^{-1}(s) = s$ for all $s \geq 1$, $h \in \mathcal{H}_{\psi_1} \setminus L^1(0,1)$ for any $c \in [1,2)$. We give some examples for nonlinearity f to illustrate the main result (Theorem 1).

Let

$$f_1(s) = \begin{cases} s^{\frac{1}{2}}, & \text{for } s \in [0,1]; \\ s^3, & \text{for } s \in (1,\infty) \end{cases} \text{ and } f_2(s) = s^{\frac{3}{2}} \text{ for } s \in \mathbb{R}_+.$$

Then,

$$(f_1)_0 = (f_1)_\infty = \infty \text{ and } (f_2)_0 = (f_2)_\infty = 0.$$

Consequently, by Theorem 1, problem (1)–(2) with $f = f_1$ has two positive solutions for all small $\lambda > 0$, and problem (1)–(2) with $f = f_2$ has two positive solutions for all large $\lambda > 0$.

As shown in the examples of nonlinearity $f = f(s)$ above, $f(0)$ may be 0. What this means is that nonnegative solutions may be trivial ones. The existence of an unbounded solution component to problem (1)–(2) can be obtained as in the paper [11], where the nonlinearity $f = f(t,s)$ satisfies $f(t,0) \not\equiv 0$, but we cannot get any information about positive solutions from the solution component. Thus, the fixed point index theory was used in order to show the existence of two positive solutions to problem (1)–(2).

Funding: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular Boundary Value Problems Involving φ-Laplacian. *Mathematics* **2019**, *7*, 654. [CrossRef]
2. Karakostas, G.L. Positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function. *Electron. J. Differ. Equ.* **2004**, *68*, 1–12.
3. Infante, G.; Pietramala, P. A cantilever equation with nonlinear boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *15*, 1–14. [CrossRef]
4. Infante, G.; Pietramala, P.; Tenuta, M. Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 2245–2251. [CrossRef]
5. Cabada, A.; Infante, G.; Tojo, F.A.F. Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications. *Topol. Methods Nonlinear Anal.* **2016**, *47*, 265–287. [CrossRef]
6. Liu, B. Positive solutions of a nonlinear four-point boundary value problems. *Appl. Math. Comput.* **2004**, *155*, 179–203. [CrossRef]
7. Bachouche, K.; Djebali, S.; Moussaoui, T. ϕ-Laplacian BVPS with linear bounded operator conditions. *Arch. Math. (Brno)* **2012**, *48*, 121–137. [CrossRef]
8. Yang, Y.Y.; Wang, Q.R. Multiple positive solutions for p-Laplacian equations with integral boundary conditions. *J. Math. Anal. Appl.* **2017**, *453*, 558–571. [CrossRef]
9. Goodrich, C.S. Perturbed integral operator equations of Volterra type with applications to p-Laplacian equations. *Mediterr. J. Math.* **2018**, *15*, *47*,[CrossRef]
10. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular φ-Laplacian Nonlocal Boundary Value Problems when φ is a Sup-multiplicative-like Function. *Mathematics* **2020**, *8*, 420. [CrossRef]
11. Kim, C.G. Existence and Multiplicity Results for Nonlocal Boundary Value Problems with Strong Singularity. *Mathematics* **2020**, *8*, 680. [CrossRef]
12. Whyburn, W.M. Differential equations with general boundary conditions. *Bull. Amer. Math. Soc.* **1942**, *48*, 692–704. [CrossRef]
13. Conti, R. Recent trends in the theory of boundary value problems for ordinary differential equations. *Boll. Un. Mat. Ital.* **1967**, *22*, 135–178.
14. Krall, A.M. The development of general differential and general differential-boundary systems. *Rocky Mt. J. Math.* **1975**, *5*, 493–542. [CrossRef]
15. Štikonas, A. A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions. *Nonlinear Anal. Model. Control* **2014**, *19*, 301–334. [CrossRef]
16. Agarwal, R.P.; Lü, H.; O'Regan, D. Eigenvalues and the one-dimensional p-Laplacian. *J. Math. Anal. Appl.* **2002**, *266*, 383–400. [CrossRef]
17. Wang, H. On the structure of positive radial solutions for quasilinear equations in annular domains. *Adv. Differ. Equ.* **2003**, *8*, 111–128.
18. Lee, Y.H.; Xu, X. Existence and multiplicity results for generalized Laplacian problems with a parameter. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 403–424. [CrossRef]

19. Deimling, K. *Nonlinear Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1985. [CrossRef]
20. Guo, D.J.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press, Inc.: Boston, MA, USA, 1988.