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On Holomorphic Contractibility of Teichmüller Spaces

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Abstract: The problem of the holomorphic contractibility of Teichmüller spaces $\mathbf{T}(0,n)$ of the punctured spheres ($n > 4$) arose in the 1970s in connection with solving algebraic equations in Banach algebras. Recently it was solved by the author. In the present paper, we give a refined proof of the holomorphic contractibility for all spaces $\mathbf{T}(0,n)$, $n > 4$ and provide two independent proofs of holomorphic contractibility for low-dimensional Teichmüller spaces, which has intrinsic interest.

Keywords: Teichmüller spaces; Fuchsian group; quasiconformal deformations; holomorphic contractibility; univalent function; Schwarzian derivative; holomorphic sections

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1. Preamble

1.1. Holomorphic Contractibility

A complex Banach manifold X is contractible to its point x_0 if there exists a continuous map $F : X \times [0,1] \to X$ with $F(x,0) = x$ and $F(x,1) = x_0$ for all $x \in X$. If map F can be chosen so that for every $t \in [0,1]$ the map $F_t : x \mapsto F(x,t)$ of X is holomorphic to itself and $F_t(x_0) = x_0$, then X is called **holomorphically contractible** to x_0.

The problem of holomorphic contractibility of Teichmüller spaces $\mathbf{T}(0,n)$ of the punctures spheres ($n > 4$) arose in the 1970s in connection with solving algebraic equations in Banach algebras. It was caused by the fact that in the space $\mathbb{C}^m$, $m > 1$, there are domains (even bounded) that are only topologically but not holomorphically contractible (see [1–5]).

The simplest example of holomorphically contractible domains in complex Banach spaces is given by starlike domains. However, all Teichmüller spaces of sufficiently great dimensions are not starlike (see [6,7]).

Earle [8] established the holomorphic contractibility of two modified Teichmüller spaces related to asymptotically conformal maps.

Recently, this problem was solved positively in [9]. It was established that all spaces $\mathbf{T}(0,n)$, $n > 4$, are holomorphically contractible.

Theorem 1. *Any space $\mathbf{T}(0,n)$ with $n > 4$ is holomorphically contractible.*

The proof of Lemma 3 in that paper contains a wrong assertion (which is not used here) that the map s_m, including the space $T(\Gamma_0)$ into $T(\Gamma_0^m)$, is a section of the forgetful map $\chi_m : T(X_{a^0}^m) \to T(X_{a^0})$. Such sections do not exist if $n > 6$.

In fact, s_m as an open holomorphic map (of a domain onto a manifold) was only used in the proof of Lemma 3 (and of Theorem 1), and the openness is preserved for the limit map $s = \lim_{m\to\infty} s_m$, which determines an $(n-3)$-dimensional complex submanifold $s(T(X_{a^0}))$ in the universal Teichmüller space $\mathbf{T}$.

In the present paper, we give a refined proof of holomorphic contractibility for all spaces $\mathbf{T}(0,n)$, $n > 4$ and provide two independent proofs of holomorphic contractibility for low-dimensional Teichmüller spaces (of dimensions two and three). The second proof

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has its own interest in view of the importance of the problem. Its underlying idea is different; the arguments do not extend to higher dimensions.

1.2. Teichmüller Spaces of Low Dimenshions

There are two Teichmüller spaces of dimension two: the space $\mathbf{T}(0,5)$ of the spheres with five punctures and the space $\mathbf{T}(1,2)$ of twice-punctured tori; these spaces are biholomorphically equivalent. Such spheres and tori are uniformized by the corresponding Fuchsian groups Γ and Γ' so that Γ is a subgroup of index two in Γ'; letting $\mathbf{T}(0,5) = \mathbf{T}(\Gamma)$, $\mathbf{T}(1,2) = \mathbf{T}(\Gamma')$, we have $\mathbf{T}(\Gamma') = \mathbf{T}(\Gamma)$.

In a similar way, the Teichmüller spaces $\mathbf{T}(0,6)$ of spheres with six punctures and $\mathbf{T}(2,0)$ of closed Riemann surfaces of genus 2 also are biholomorphically equivalent, and in terms of the corresponding Fuchsian groups Γ and Γ' we have the same relationship $\mathbf{T}(\Gamma') = \mathbf{T}(\Gamma)$. We state:

Theorem 2. *The spaces* $\mathbf{T}(0,5)$, $\mathbf{T}(1,2)$, $\mathbf{T}(0,6)$, $\mathbf{T}(2,0)$ *are holomorphically contractible.*

The Teichmüller space $\mathbf{T}(1,3)$ of tori with three punctures also has three dimensions; it will not be involved here.

2. Underlying Results

2.1. Teichmüller Spaces of Punctured Spheres

Consider the ordered n-tuples of points

$$\mathbf{a} = (0, 1, a_1, \dots, a_{n-3}, \infty), \quad n > 4, \tag{1}$$

with distinct $a_j \in \widehat{\mathbb{C}} \setminus \{1, -1, i\}$ and the corresponding punctured spheres

$$X_{\mathbf{a}} = \widehat{\mathbb{C}} \setminus \{0, 1, a_1 \dots, a_{n-3}, \infty\}, \quad \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

regarded as Riemann surfaces of genus zero. Fix a collection $\mathbf{a}^0 = (0, 1, a_1^0, \dots, a_{n-3}^0, 1, \infty)$ defining the base point $X_{\mathbf{a}^0}$ of Teichmüller space $\mathbf{T}(0,n) = \mathbf{T}(X_{\mathbf{a}^0})$ of n-punctured spheres. Its points are the equivalence classes $[\mu]$ of Beltrami coefficients from the ball

$$\mathrm{Belt}(\mathbb{C})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \|\mu\|_\infty < 1\},$$

under the relationship that $\mu_1 \sim \mu_2$ if the corresponding quasiconformal homeomorphisms $w^{\mu_1}, w^{\mu_2} : \ X_{\mathbf{a}^0} \to X_{\mathbf{a}}$ (the solutions of the Beltrami equation $\bar{\partial} w = \mu \partial w$ with $\mu = \mu_1, \mu_2$) are homotopic on $X_{\mathbf{a}^0}$ (and hence coincide at the points $0, 1, a_1^0, \dots, a_{n-3}^0, \infty$). This models $\mathbf{T}(0,n)$ as the quotient space

$$\mathbf{T}(0,n) = \mathrm{Belt}(\mathbb{C})_1 / \sim$$

with a complex Banach structure of dimension $n-3$ inherited from the ball $\mathrm{Belt}(\mathbb{C})_1$.

Another canonical model of $\mathbf{T}(0,n) = \mathbf{T}(X_{\mathbf{a}^0})$ is obtained using the uniformization of Riemann surfaces and the holomorphic Bers embedding of Teichmüller spaces. Consider the upper and lower half-planes

$$U = \{z = x + iy \in \mathbb{C} : \ y > 0\}, \quad U^* = \{z = x + iy \in \mathbb{C} : \ y < 0\}$$

The holomorphic universal covering map $h : \ U \to X_{\mathbf{a}^0}$ provides a torsion-free Fuchsian group Γ_0 of the first kind acting discontinuously on $U \cup U^*$, and the surface $X_{\mathbf{a}^0}$ is represented (up to conformal equivalence) as the quotient space U/Γ_0. The functions $\mu \in L_\infty(X_{\mathbf{a}^0}) = L_\infty(\mathbb{C})$ are lifted to U as the Beltrami $(-1,1)$-measurable forms $\widetilde{\mu} d\bar{z}/dz$ on U with respect to Γ_0, satisfying $(\widetilde{\mu} \circ \gamma)\overline{\gamma'}/\gamma' = \widetilde{\mu}$, $\gamma \in \Gamma_0$ and forming the corresponding Banach space $L_\infty(U, \Gamma_0)$. We extend these $\widetilde{\mu}$ by zero to U^* and consider the unit ball

$\mathrm{Belt}(U, \Gamma_0)_1$ of this space $L_\infty(U, \Gamma_0)$. The corresponding quasiconformal maps $w^{\widehat{\mu}}$ are conformal on the half-plane U^*, and their *Schwarzian derivatives*.

$$S_w(z) = \left(\frac{w'''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2, \quad w = w^{\widehat{\mu}},$$

fill a bounded domain in the complex $(n-3)$-dimensional space $\mathbf{B}(\Gamma_0) = \mathbf{B}(U^*, \Gamma_0)$ of hyperbolically bounded holomorphic Γ_0-automorphic forms of degree -4 on U^* (i.e., satisfy $(\varphi \circ \gamma)(\gamma')^2 = \varphi,\ \gamma \in \Gamma_0$), with norm

$$\|\varphi\| = \sup_{U^*} 4y^2 |\varphi(z)|.$$

This domain models the *Teichmüller space* $\mathbf{T}(\Gamma_0)$ *of the group* Γ_0. It is canonically isomorphic to the space $\mathbf{T}(X_{\mathbf{a}^0})$ (and to a bounded domain in the complex Euclidean space $\mathbb{C}^{n-3}$).

The indicated map $\widehat{\mu} \to S_{w^{\widehat{\mu}}}$ determines a holomorphic map $\phi_{\mathbf{T}} : \mathrm{Belt}(U, \Gamma_0)_1 \to \mathbf{B}(\Gamma_0)$; it has only local holomorphic sections.

Note also that $\mathbf{T}(\Gamma_0) = \mathbf{T} \cap \mathbf{B}(\Gamma_0)$, where $\mathbf{T}$ is the universal Teichmüller space (modeled as a domain of the Schwarzian derivatives of all univalent functions on U^* admitting quasiconformal extension to U).

2.2. Connection with Interpolation of Univalent Functions

The collections (1) fills a domain $\mathcal{D}_n$ in $\mathbb{C}^{n-3}$ obtained by deleting from this space the hyperplanes $\{z = (z_1, \ldots, z_{n-3}) : \ z_j = z_l,\ j \neq l\}$, and with $z_1 = 0, z_2 = 1$. This domain represents the Torelli space of the spheres $X_{\mathbf{a}}$ and is covered by $\mathbf{T}(0, n)$, which is given by the following lemma (cf., e.g., [10]; [11], Section 2.8).

Lemma 1. *The holomorphic universal covering space of $\mathcal{D}_n$ is the Teichmüller space $\mathbf{T}(0, n)$. This means that for each punctured sphere $X_{\mathbf{a}}$, there is a holomorphic universal covering*

$$\pi_{\mathbf{a}} : \mathbf{T}(0, n) = \mathbf{T}(X_{\mathbf{a}}) \to \mathcal{D}_n.$$

The covering map $\pi_{\mathbf{a}}$ is well defined by

$$\pi_{\mathbf{a}} \circ \phi_{\mathbf{a}}(\mu) = (0, 1, w^\mu(a_1), \ldots, w^\mu(a_{n-3}), \infty),$$

where $\phi_{\mathbf{a}}$ denotes the canonical projection of the ball $\mathrm{Belt}(\mathbb{C})_1$ *onto the space* $\mathbf{T}(X_{\mathbf{a}})$.

Lemma 1 also yields that the truncated collections $\mathbf{a}_* = (a_1, \ldots, a_{n-3})$ provide the local complex coordinates on the space $\mathbf{T}(0, n)$ and define its complex structure.

These coordinates are simply connected with the Bers local complex coordinates on $\mathbf{T}(0, n)$ (related to basis of the tangent spaces to $\mathbf{T}(0, n)$ at its points, see [12]) via standard variation of quasiconformal maps of $X_{\mathbf{a}} = U/\Gamma_{\mathbf{a}}$

$$\begin{aligned} w^\mu(z) &= z - \frac{z(z-1)}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} d\xi d\eta + O(\|\mu\|_\infty^2) \\ &= z - \frac{z(z-1)}{\pi} \sum_{\gamma \in \Gamma_{\mathbf{a}}} \iint\limits_{U/\Gamma_{\mathbf{a}}} \frac{\mu(\gamma\zeta)|\gamma'(\zeta)|^2}{\gamma\zeta(\gamma\zeta-1)(\gamma\zeta-z)} d\xi d\eta + O(\|\mu\|_\infty^2). \end{aligned}$$

with a uniform estimate of the ratio $O(\|\mu\|_\infty^2)/\|\mu\|_\infty^2$ on compacts in $\mathbb{C}$ (see, e.g., [13]).

It turns out that one can obtain the whole space $\mathbf{T}(X_{\mathbf{a}^0})$ using only the similar equivalence classes $[\mu]$ of the Beltrami coefficients from the ball $\mu \in \mathrm{Belt}(U)_1$ (vanishing on U^*), requiring that the corresponding quasiconformal homeomorphisms w^μ are homotopic on the punctured sphere $X_{\mathbf{a}^0}$. Surjectivity of this holomorphic quotient map

$$\chi : \mathrm{Belt}(U)_1 \to \mathbf{T}(0, n),$$

is a consequence of the following interpolation result from [14].

Lemma 2. *Given two cyclically ordered collections of points $(z_1, \dots, z_m)$ and $(\zeta_1, \dots, \zeta_m)$ on the unit circle $S^1 = \{|z| = 1\}$, there exists a holomorphic univalent function f in the closure of the unit disk $\mathbb{D} = \{|z| < 1\}$ such that $|f(z)| < 1$ for $z \in \overline{\mathbb{D}}$ distinct from $z_1, \dots, z_m$, and $f(z_k) = \zeta_k$ for all $k = 1, \dots, m$. Moreover, there exist univalent polynomials f with such an interpolation property.*

It follows that the function f given by Lemma 2 is actually holomorphic and univalent (hence, maps conformally) in a broader disk $\mathbb{D}_r$, $r > 1$.

First of all, $f'(z) \neq 0$ on the unit circle S^1. Indeed, if $f'(z_0) = 0$ at some point $z_0 \in S^1$, then in its neighborhood $f(z) = c_s(z - z_0)^s + O((z - z_0)^{s+1}) = c_s\zeta^s$, where $c_s \neq 0$ for some $s > 1$, which contradicts the injectivity of $f(z)$ on S^1. Therefore, f is univalent in some disk $\mathbb{D}_r = \{|z| < r\}$, $r > 1$.

Assuming, on the contrary, that f is not globally univalent in any admissible disk $\mathbb{D}_r$ with $r > 1$, one obtains the distinct sequences $\{z'_n\}$, $\{z''_n\} \subset \mathbb{D}_r$ with $f(z'_n) = f(z''_n)$ for any n, whose limit points z'_0, z''_0 lie on S^1. Then, in the limit, we have $f(z'_0) = f(z''_0)$, which in the case $z'_0 \neq z''_0$, contradicts the univalence of f on S^1 given by Lemma 2, and in the case $z'_0 = z''_0 = z_0$, contradicts the local univalence of f in a neighborhood of z_0.

The interpolating function f given by Lemma 2 is extended quasiconformally to the whole sphere $\widehat{\mathbb{C}}$ across any circle $\{|z| = r\}$ with $r > 1$ indicated above. Hence, given a cyclically ordered collection $(z_1, \dots, z_m)$ of points on S^1, then for any ordered collection $(\zeta_1, \dots, \zeta_m)$ in $\widehat{\mathbb{C}}$, there exists a quasiconformal homeomorphism $\widehat{f}$ of the sphere $\widehat{\mathbb{C}}$ carrying the points z_j to ζ_j, $j = 1, \dots, m$, and such that its restriction to the closed disk $\overline{\mathbb{D}}$ is biholomorphic on $\overline{\mathbb{D}}$.

Taking the quasicircles L passing through the points $\zeta_1, \dots, \zeta_m$ and such that each ζ_j belongs to an analytic subarc of L, one obtains quasiconformal extensions of the interpolating function f, which are biholomorphic on the union of $\overline{\mathbb{D}}$ and some neighborhoods of the initial points $z_1, \dots, z_m \in S^1$. Now Lemma 1 provides quasiconformal extensions of f lying in prescribed homotopy classes of homeomorphisms $X_{\mathbf{z}} \to X_{\mathbf{w}}$.

2.3. The Bers fiber space

Pick a space $\mathbf{T}(0, n) = \mathbf{T}(X_{\mathbf{a}^0})$ with $n \geq 5$ and let

$$X'_{\mathbf{a}^0} = X_{\mathbf{a}^0} \setminus \{a^0_{n-3}\} = U/\Gamma'_0.$$

Due to the Bers isomorphism theorem, *the space $T(X'_{\mathbf{a}^0})$ is biholomorphically isomorphic to the Bers fiber space*

$$\mathcal{F}(0, n) := \mathcal{F}(T(X_{\mathbf{a}^0}) = \{(\phi_{\mathbf{T}}(\mu), z) \in \mathbf{T}(X_{\mathbf{a}^0}) \times \mathbb{C} : \ \mu \in \mathrm{Belt}(U, \Gamma'_0)_1, \ z \in w^\mu(\mathbb{D})\}$$

over $\mathbf{T}(X_{\mathbf{a}^0})$ with holomorphic projection $\pi(\varphi, z) = \varphi$ ($\varphi \in T(X_{\mathbf{a}^0})$ (see [15]).

This fiber space is a bounded hyperbolic domain in $\mathbf{B}(\Gamma_0) \times \mathbb{C}$ and represents the collection of domains $D_\mu = w^\mu(U)$ (the universal covers of the surfaces $X_{\mathbf{a}^0}$) as a holomorphic family over the space $\mathbf{T}(0, n - 1) = T(X_{\mathbf{a}^0})$.

The indicated isomorphism between $\mathbf{T}(0, n + 1)$ and $\mathcal{F}(0, n)$ is induced by the inclusion map $j : \ \mathbb{D}_* \hookrightarrow \mathbb{D}$ forgetting the puncture at a^0_n, via

$$\mu \mapsto (S_{w^{\mu_1}}, w^{\mu_1}(\widehat{a}^0_{n-3})) \quad \text{with} \ \ \mu_1 = j_*\mu := (\mu \circ \widehat{j}_0)\overline{\widehat{j}'}/\widehat{j}', \tag{2}$$

where $\widehat{j}$ is the lift of j to U and $\widehat{a}^0_{n-3}$ is one of the points from the fiber over a^0_n under the quotient map $U \to U/\Gamma_0$.

Note also that the holomorphic universal covering maps $h : U^* \to U^*/\Gamma_0$ and $h' : U^* \to U^*/\Gamma'_0$ (and similarly, the corresponding covering maps in U) are related by

$$j \circ h' = h \circ \widehat{j},$$

where $\widehat{j}$ is the lift of j. This induces a surjective homomorphism of the covering groups $\theta : \Gamma_0 \to \Gamma'_0$ by

$$\widehat{j} \circ \gamma = \theta(\gamma) \circ \gamma, \quad \gamma \in \Gamma'_0$$

and the norm preserving isomorphism $\widehat{j}_* : \mathbf{B}(\Gamma_0) \to \mathbf{B}(\Gamma'_0)$ by

$$\widehat{j}_* \varphi = (\varphi \circ \widehat{j})(\widehat{j}')^2, \tag{3}$$

which projects to the surfaces $X_{\mathbf{a}^0}$ and $X'_{\mathbf{a}^0}$ as the inclusion of the space $Q(X_{\mathbf{a}^0})$ of holomorphic quadratic differentials corresponding to $\mathbf{B}(\Gamma_0)$ in the space $Q(X'_{\mathbf{a}^0})$ (cf. [16]).

The Bers theorem is valid for the Teichmüller space $\mathbf{T}(X_0 \setminus \{x_0\})$ of any punctured hyperbolic Riemann surface $X_0 \setminus \{x_0\}$ and implies that $\mathbf{T}(X_0 \setminus \{x_0\})$ is biholomorphically isomorphic to the Bers fiber space $\mathcal{F}(\mathbf{T}(X_0))$ over $\mathbf{T}(X_0)$.

2.4. Holomorphic Curves and Holomorphic Sections

The group Γ_0 uniformizing the surface $X_{\mathbf{a}^0}$ acts discontinuously on the fiber space $\mathcal{F}(\Gamma_0)$ as a group of biholomorphic maps by

$$\gamma(\phi_{\mathbf{T}}(\mu), z) = (\phi_{\mathbf{T}}(\mu), \gamma^\mu z), \tag{4}$$

where $\mu \in \mathrm{Belt}(U, \Gamma_0)$, $z \in w^\mu(U))$, $\gamma \in \Gamma_0$, and

$$\gamma^\mu \circ w^\mu = w^\mu \circ \gamma$$

(see [15]). The quotient space

$$\mathcal{V}(0,n) := \mathcal{V}(\Gamma_0) = \mathbf{T}(0, n+1)/\Gamma_0$$

is called the *n-punctured Teichmüller curve* and depends only on the analytic type of the Γ_0 group. The projection $\pi : \mathcal{F}(0,n) \to \mathbf{T}(0,n)$ induces a holomorphic projection

$$\pi_1 : \mathcal{V}(0,n) \to \mathbf{T}(0,n). \tag{5}$$

This curve is also a complex manifold with fibers $\pi^{-1}(x) = X_{\mathbf{a}}$.

Due to the deep Hubbard–Earle–Kra theorem [16,17], *the projections* $\mathcal{V}(0,n) \to \mathbf{T}(0,n)$ *and (4) have no holomorphic sections for any* $n \geq 7$ (more generally, for all spaces $\mathbf{T}(\Gamma)$ corresponding to groups Γ without elliptic elements, excluding the spaces $\mathbf{T}(1,2) \simeq \mathbf{T}(0,5)$ and $\mathbf{T}(2,0) \simeq \mathbf{T}(0,6)$). Such sections exist for Γ groups containing elliptic elements.

In the exceptional cases of $\mathbf{T}(1,2)$ and $\mathbf{T}(2,0)$, there is a group Γ' that contains Γ as a subgroup of index two. Then, $\mathbf{T}(\Gamma') = \mathbf{T}(\Gamma)$, $\mathcal{F}(\Gamma') = \mathcal{F}(\Gamma)$, and the elliptic elements $\gamma \in \Gamma'$ produce the indicated holomorphic sections s as the maps

$$\phi_{\mathbf{T}}(\mu) \mapsto (\phi_{\mathbf{T}}(\mu), w^\mu(z_0)), \tag{6}$$

where z_0 is a fixed point of γ in the half-plane U. These sections are called the *Weierstrass sections* (in view of their connection with the Weierstrass points of the hyperelliptic surface U/Γ). We describe these sections following [16].

We also consider the *punctured fiber space* $\mathcal{F}_0(\Gamma)$ to be the largest open dense subset of $\mathcal{F}(\Gamma)$ on which Γ acts freely and let

$$\mathcal{V}'(\Gamma) = \mathcal{F}_0(\Gamma)/\Gamma.$$

For Γ with no elliptic elements, the universal covering space for $\mathcal{V}'(g,n) = \mathcal{V}'(\Gamma)$ is $\mathbf{T}(g, n+1)$.

If Γ contains elliptic elements γ, then any holomorphic section $\mathbf{T}(\Gamma) \to \mathcal{F}(\Gamma)$ is determined by map (6) so that $w^\mu(z_0)$ is exactly one fixed point of corresponding map (4) in the fiber $w^\mu(U)$. These holomorphic sections take their values in the set $\mathcal{V}(\Gamma) \setminus \mathcal{V}'(\Gamma)$ and do not provide, in general, sections of projection (5).

In the case of spaces $\mathbf{T}(1,2)$ and $\mathbf{T}(2,0)$, either of the corresponding curves $\mathcal{V}(1,2)$ or $\mathcal{V}(2,0)$ has a unique biholomorphic self-map γ of order two, which maps each fiber into itself. The fixed-point locus of γ is a finite set of connected closed complex submanifolds of $\mathcal{V}'(g,n)$, and the restriction of map (5) to one of these submanifolds is a holomorphic map onto $\mathbf{T}(0,n)$; its inverse is a Weierstrass section. The restriction of γ to each fiber is a conformal involution of the corresponding hyperelliptic Riemann surface interchanging its sheets, and the fixed points of γ are the Weierstrass points on this surface.

In dimension one, there are three biholomorphically isomorphic Teichmüller spaces $\mathbf{T}(1,0)$, $\mathbf{T}(1,1)$ and $\mathbf{T}(0,4)$ (see, e.g., [15,18]). We shall use the last two spaces. Their fiber space $\mathcal{F}(0,4)$ is isomorphic to $\mathbf{T}(0,5)$.

As a special case of the Hubbard–Earle–Kra theorem [16,17], we have:

Lemma 3. *(a) The curve $\mathcal{V}(0,4)$ has, for any of its points x, a unique holomorphic section s with $s(\pi_1(x)) = x$.*

(b) If $\dim \mathcal{V}(g,n)' > 1$, only curves $\mathcal{V}(1,2)'$ and $\mathcal{V}(2,0)'$ have holomorphic sections, which are their Weierstrass sections.

In particular, curve $\mathcal{V}(2,0)$ has six disjoint holomorphic sections corresponding to the Weierstrass points of hyperelliptic surfaces of genus two.

3. Holomorphic Maps of $\mathbf{T}(0,n)$ into Universal Teichmüller Space and Holomorphic Contractibility

3.1. Equivalence Relations

The universal Teichmüller space $\mathbf{T} = \mathrm{Teich}(U)$ is the space of quasisymmetric homeomorphisms of the unit circle factorized by Möbius maps; all Teichmüller spaces have their isometric copies in $\mathbf{T}$.

The canonical complex Banach structure on $\mathbf{T}$ is defined by the factorization of the ball of the Beltrami coefficients

$$\mathrm{Belt}(U)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|U^* = 0, \ \|\mu\|_\infty < 1\}$$

(i.e., supported in the upper-half plane), letting $\mu_1, \mu_2 \in \mathrm{Belt}(U)_1$ be equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} coincide on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial U^*$ (hence, on $\overline{U^*}$). Such μ and the corresponding maps w^μ are called $\mathbf{T}$-*equivalent*. The equivalence classes $[w^\mu]_{\mathbf{T}}$ are in one-to-one correspondence with the Schwarzian derivatives S_w in U^*, which fill a bounded domain in the space $\mathbf{B} = \mathbf{B}(U^*)$ (see Section 2.1).

The map $\phi_{\mathbf{T}} : \ \mu \to S_{w^\mu}$ is holomorphic and descends to a biholomorphic map of the space $\mathbf{T}$ onto this domain, which we will identify with $\mathbf{T}$. As was mentioned above, it contains the Teichmüller spaces of all hyperbolic Riemann surfaces and of Fuchsian groups as complex submanifolds.

On this ball, we also define another equivalence relationship, letting $\mu, \ \nu \in \mathrm{Belt}(U)_1$ be equivalent if $w^\mu(a_j^0) = w^\nu(a_j^0)$ for all j and the homeomorphisms $w^\mu, \ w^\nu$ are homotopic on the punctured sphere $X_{\mathbf{a}^0}$. Let us call such μ and ν *strongly n-equivalent*.

Lemma 4. *If the coefficients $\mu, \nu \in \text{Belt}(U)_1$ are $\mathbf{T}$-equivalent, then they are also strongly n-equivalent.*

The proof of this lemma is given in [19].

In view of Lemmas 1 and 4, the above factorizations of the ball $\text{Belt}(U)_1$ generate (by descending to the equivalence classes) a holomorphic map χ of the underlying space $\mathbf{T}$ into $\mathbf{T}(0, n) = \mathbf{T}(X_{\mathbf{a^0}})$.

This map is a split immersion (has local holomorphic sections), which is a consequence, for example, of the following existence theorem from [13], which we present here as

Lemma 5. *Let D be a finitely connected domain on the Riemann sphere $\widehat{\mathbb{C}}$. Assume that there are a set E of positive, two-dimensional Lebesgue measures and a finite number of points $z_1, \dots, z_m$ distinguished in D. Let $\alpha_1, \dots, \alpha_m$ be non-negative integers assigned to $z_1, \dots, z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.*

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{sj}, s = 0, 1, \dots, \alpha_j, \ j = 1, 2, \dots, m$, which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \le \varepsilon, \ \ |w_{1j} - 1| \le \varepsilon, \ \ |w_{sj}| \le \varepsilon \ (s = 0, 1, \dots a_j, \ j = 1, \dots, m),$$

there exists a quasiconformal automorphism h of domain D, which is conformal on $D \setminus E$ and satisfies

$$h^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, \dots, \alpha_j, \ j = 1, \dots, m.$$

Moreover, the Beltrami coefficient μ_h of h on E satisfies $\|\mu_h\|_\infty \le M\varepsilon$. The constants ε_0 and M depends only upon the sets D, E and the vectors $(z_1, \dots, z_m)$ and $(\alpha_1, \dots, \alpha_m)$.

3.2. Surjectivity

In fact, we have more, given by the following theorem.

Theorem 3. *Map χ is surjective and generates an open holomorphic map s of the space $\mathbf{T}(0, n) = \mathbf{T}(X_{\mathbf{a^0}})$ into the universal Teichmüller space $\mathbf{T}$, embedding $\mathbf{T}(0, n)$ into $\mathbf{T}$ as a $(n - 3)$-dimensional submanifold.*

In particular, this theorem corrects the assertion of Lemma 3 in [9] (mentioned in the preamble).

Proof of Theorem 3. The surjectivity of χ is a consequence of Lemma 2. To construct s, we take a dense subset

$$e = \{x_1, \ x_2, \ \dots\} \subset X_{\mathbf{a}^0} \cap \mathbb{R}$$

accumulating to all points of $\mathbb{R}$ and considering the punctured spheres $X^m_{\mathbf{a}^0} = X_{\mathbf{a}^0} \setminus \{x_1, \dots, x_m\}$ with $m > 1$. The equivalence relations on $\text{Belt}(\mathbb{C})_1$ for $X^m_{\mathbf{a}^0}$ and $X_{\mathbf{a}^0}$ generate the corresponding holomorphic map $\chi_m : \mathbf{T}(X^m_{\mathbf{a}^0}) \to \mathbf{T}(X_{\mathbf{a}^0})$. $\square$

Uniformizing the surfaces $X_{\mathbf{a}^0}$ and $X^m_{\mathbf{a}^0}$ by the corresponding torsion-free Fuchsian groups Γ_0 and Γ^m_0 of the first kind acting discontinuously on $U \cup U^*$ and applying the construction from Section 2.3 to U^*/Γ_0 and U^*/Γ^m_0 (forgetting the additional punctures), one obtains, similar to (3), the norm-preserving isomorphism $\widehat{j}_{m,*} : \ \mathbf{B}(\Gamma_0) \to \mathbf{B}(\Gamma^m_0)$ by

$$\widehat{j}_{m,*}\varphi = (\varphi \circ \widehat{j})(\widehat{j}')^2,$$

which projects to the surfaces $X_{\mathbf{a}^0}$ and $X^m_{\mathbf{a}^0}$ as the inclusion of the space $Q(X_{\mathbf{a}^0})$ of quadratic differentials corresponding to $\mathbf{B}(\Gamma_0)$ into the space $Q(X^m_{\mathbf{a}^0})$, and (since projection η_m has local holomorphic sections) geometrically, this relation yields a holomorphic embedding of the space $\mathbf{T}(\Gamma_0)$ into $\mathbf{T}(\Gamma^m_0)$ as a $(n - 3)$-dimensional submanifold. We denote this embedding by s_m.

To investigate the limit function for $m \to \infty$, we compose the maps s_m with the canonical biholomorphic isomorphisms

$$\eta_m : \mathbf{T}(X^m_{\mathbf{a}^0}) \to \mathbf{T}(\Gamma^m_0) = \mathbf{T} \cap \mathbf{B}(\Gamma^m_0) \quad (m = 1, 2, \dots).$$

Then the elements of $\mathbf{T}(\Gamma^m_0)$ are given by

$$\widehat{s}_m(X_{\mathbf{a}}) = \eta_m \circ s_m(X_{\mathbf{a}}),$$

and this is a collection of the Schwarzians $S_{f^m}(z)$ corresponding to the points $X_{\mathbf{a}}$ of $\mathbf{T}(X_{\mathbf{a}^0})$. Therefore, for any surface $X_{\mathbf{a}}$, we have

$$\widehat{s}_m(X_{\mathbf{a}}) = S_{f^m}(z). \tag{7}$$

Each Γ^m_0 is the covering group of the universal cover $h_m : U^* \to X_{\mathbf{a}^m_0}$, which can be normalized (conjugating appropriately Γ^m_0) by $h_m(-i) = -i$, $h'_m(-i) > 0$. Take the fundamental polygon P_m obtained as the union of the circular m-gon in U^* centered at $z = -i$ with zero angles at the vertices and its reflection with respect to one of the boundary arcs. These polygons increasingly exhaust the half-plane U^*; hence, by the Carathéodory kernel theorem, the maps h_m converge to the identity map locally uniformly in U^*.

Since the set of punctures e is dense on $\mathbb{R}$, it completely determines the equivalence classes $[w^\mu]$ and S_{w^μ} as the points of the universal space $\mathbf{T}$. The limit function $\widehat{s} = \lim_{m\to\infty} \widehat{s}_m$ maps $\mathbf{T}(X_{\mathbf{a}^0}) = \mathbf{T}(0, n)$ into the space $\mathbf{T}$ and also is canonically defined by the marked spheres $X_{\mathbf{a}}$.

Similar to (7), the function $\widehat{s}$ is represented as the Schwarzian of some univalent function f^n on U^* with a quasiconformal extension to $\widehat{\mathbb{C}}$ determined by $X_{\mathbf{a}}$. Then, by the well-known property of elements in the functional spaces with sup norms, $\widehat{s}$ is also holomorphic in the $\mathbf{B}$-norm on $\mathbf{T}$.

Lemma 5 yields that $\widehat{s}$ is a locally open map from $\mathbf{T}(X_{\mathbf{a}^0})$ to $\mathbf{T}$. Therefore, the image $\widehat{s}(\mathbf{T}(X_{\mathbf{a}^0}))$ is an $(n-3)$-dimensional complex submanifold in $\mathbf{T}$, biholomorphically equivalent to $\mathbf{T}(\Gamma_0)$. The proof of Theorem 2 is completed.

The holomorphy property indicated above is based on the following lemma of Earle [20].

Lemma 6. *Let E, T be open subsets of complex Banach spaces X, Y and $B(E)$ be the Banach space of holomorphic functions on E with sup norm. If $\varphi(x,t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \varphi(x,t)$ is holomorphic for each $x \in E$, then map φ is holomorphic.*

The holomorphy of $\varphi(x,t)$ in t for fixed x implies the existence of complex directional derivatives

$$\varphi'_t(x,t) = \lim_{\zeta\to 0} \frac{\varphi(x, t+\zeta v) - \varphi(x,t)}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\xi|=1} \frac{\varphi(x, t+\xi v)}{\xi^2} d\xi,$$

while the boundedness of φ in the sup norm provides the uniform estimate

$$\|\varphi(x, t+c\zeta v) - \varphi(x,t) - \varphi'_t(x,t)cv\|_{B(E)} \le M|c|^2,$$

for sufficiently small $|c|$ and $\|v\|_Y$.

3.3. Explicit Construction of Holomorphic Homotopy

Now we may construct the desired holomorphic homotopy of $\mathbf{T}(0,n) = \mathbf{T}(X_{\mathbf{a}^0})$ into its base point and establish the general result:

Pick a collection $\mathbf{a}^0 = (0, 1, a^0_1, \dots, a^0_{n-3}, \infty)$ and the marked surface $X_{\mathbf{a}^0}$ as indicated above, and consider its Teichmüller spaces $\mathbf{T}(X_{\mathbf{a}^0})$ and $\mathbf{T}(\Gamma_0)$.

Using the canonical embedding of $\mathbf{T}(0,n)$ in $\mathbf{T}$ via $\mathbf{T}(\Gamma_0)$, we define on the space $\mathbf{T}(\Gamma_0)$ a holomorphic homotopy applying the maps

$$W^\mu = \sigma^{-1} \circ w^\mu \circ \sigma, \quad \mu \in \text{Belt}(U)_1; \quad \sigma(\zeta) = i(1+\zeta)/(1-\zeta), \ \zeta \in \mathbb{D},$$

and $w_t^\mu(z) := w^\mu(z,t) = \sigma \circ W_t^\mu \circ \sigma^{-1}(z)$; then,

$$S_{w^\mu}(\cdot, t) = t^2 S_{w^\mu}(\cdot) = t^{-2}(S_{W^\mu} \circ \ \sigma^{-1})(\sigma')^{-2}. \tag{8}$$

This point-wise equality determines a bounded holomorphic map by Lemma 6 $\eta(\varphi,t) = S_{w_t^\mu} : \ \mathbf{T} \times \mathbb{D} \to \mathbf{T}$ with $\eta(\mathbf{0},t) = \mathbf{0}$, $\eta(\varphi,0) = \mathbf{0}$, $\eta(\varphi,1) = \varphi$; its boundedness follows from the estimate

$$S_{W_t^\mu}(\zeta) < 6|t|^2/(|\zeta|^2-1)^2, \quad \zeta \in U^*.$$

We apply homotopy (8) to $\varphi = S_{w^\mu} \in \mathbf{T}(\Gamma_0)$. Since it is not compatible with the group Γ_0, there are images $\varphi_t := \eta(\varphi,t) = S_{w_t^\mu}$ that are located in $\mathbf{T}$ outside of $\mathbf{T}(\Gamma_0)$. Map $\chi \circ \eta(\varphi,t)$ carries these images to the points of the space $\mathbf{T}(0,n) = \mathbf{T}(X_{\mathbf{a}^0})$. We compose this map with the holomorphic map s given by Lemma 3 and with a biholomorphism $\xi : \ s(\mathbf{T}(X_{\mathbf{a}^0})) \to \mathbf{T}(\Gamma_0)$, getting the function

$$\Theta(\varphi,t) = \xi \circ s \circ \chi \circ \eta(\varphi,t) \tag{9}$$

which maps holomorphically $\mathbf{T}(\Gamma_0) \times \mathbb{D}$ into $\mathbf{T}(\Gamma_0)$ with $\Theta(\varphi,0) = \mathbf{0}$.

A crucial step in constructing is to establish that function (9) extends holomorphically to the limit points $(\varphi,1)$ representing the initial Schwarzians S_{w^μ}. This property does not extend (in the $\mathbf{B}$-norm) to all points of $\mathbf{T}$.

To prove the limit holomorphy, fix a point $\varphi_0 \in \mathbf{T}(\Gamma_0)$ and consider, in its small neighborhood V_0, the local coordinates $\mathbf{a}_* = (a_1,\ldots,a_{n-3})$ introduced above.

Both maps η and Θ are holomorphic in the points (φ_0,t) of this neighborhood for all t with $|t| < 1$. On the other hand, coordinates $\mathbf{a}_*$ are determined by the corresponding quasiconformal maps w_t^μ and, together with these maps, are uniformly continuous in t in the closed disk $\{|t| \le 1\}$. This follows from the uniform boundedness of dilatations given by the estimate

$$k(w_t^\mu) = \|\mu_t\|_\infty \le |t|\|\mu\|_\infty < 1 \tag{10}$$

(which holds for generic holomorphic motions) and from non-increasing the Kobayashi metric $d_X(\cdot,\cdot)$ under holomorphic maps. Since this metric on Teichmüller spaces equals their intrinsic Teichmüller metric $\tau_{\mathbf{T}(\Gamma_0)}$, one gets from (10),

$$\tau_{\mathbf{T}(\Gamma_0)}(\mathbf{0},\Theta(\varphi,t)) = d_{\mathbf{T}(\Gamma_0)}(\mathbf{0},\Theta(\varphi,t)) \le \tanh^{-1}(|t|\|\mu\|_\infty).$$

Hence, function $\Theta(\varphi,t)$ determines a normal family on $V_0 \cap \mathbf{T}(\Gamma_0)$.

Applying the classical Weierstrass theorem about the locally uniform convergent sequences of holomorphic functions in finite-dimensional domains, one derives that the limit function

$$\Theta(\varphi,1) = \lim_{t\to 1} \Theta(\varphi,t)$$

is also holomorphic on $V_0 \cap \mathbf{T}(\Gamma_0)$ and then on $\mathbf{T}(\Gamma_0)$, which completes the construction of the desired holomorphic homotopy on $\mathbf{T}(0,n)$.

4. Second Proof of Holomorphic Contractibility for Low-Dimensional Teichmüller Spaces

The previous section implies the proof of holomorphic contractibility for all spaces $\mathbf{T}(0,n)$ with $n \ge 5$, which also yields, in particular, Theorem 2. In this section, we provide another proof of this important theorem; it relies on the intrinsic features of the two and three-dimensional Teichmüller spaces mentioned in Section 2.4.

(a) *Case $n = 5$ (dimension two)*. It is enough to establish holomorphic contractibility of the space $\mathbf{T}(0,5) \simeq \mathbf{F}(0,4)$ for the spheres

$$X_{\mathbf{a}} = \widehat{\mathbb{C}} \setminus \{0, 1, a_1, a_2, \infty\}.$$

The fibers of $\mathbf{T}(0,5)$ are the spheres with quadruples of punctures $\{0, 1, a_1, \infty\}$.

We start with the construction of the needed holomorphic homotopy of the space $\mathbf{T}(0,5)$ to its base point $X_{\mathbf{a}^0}$ and first apply the assertion (a) of Lemma 3 of holomorphic sections over $\mathbf{T}(0,4)$. It implies that for any point

$$x = (S_{w^{\mu_1}}, w^{\mu_1}(\widehat{a}^0_{n-3})) \in \mathbf{T}(0,5)$$

a unique holomorphic section $s : \mathbf{T}(0,5) \to \mathbf{T}(0,4)$ with $s(\pi_1(x)) = x$. This section has a common point with each fiber $\pi^{-1}(x)$ over $\mathbf{T}(0,4)$.

Since $\mathbf{T}(0,4)$ is (up to a biholomorphic equivalence) a simply connected bounded Jordan domain $D \subset \mathbb{C}$ containing the origin, there is a holomorphic isotopy $h(\zeta, t) : D \times [0,1] \to D$ with $h(\zeta, 0) = \zeta$, $h(z, 1) = 0$. Using this isotopy, we define a homotopy $h_1(\varphi, t)$ on $\mathbf{T}(0,5)$, which carries each point $x = (S_{w^\mu}, w^\mu(\widehat{a}^0_2)) \in \mathbf{T}(0,5)$ to its image on the section s passing from x; that is,

$$h_1(\varphi, w^\mu(\widehat{a}^0_2)) = (h(\varphi), \widetilde{a}_2), \quad \varphi = S_{w^\mu}, \ \ \mu \in \mathrm{Belt}(\mathbb{C})_1, \tag{11}$$

where $\widetilde{a}_2$ is the common point of the fiber $h(\varphi)$ and the selected section s. The holomorphy of this homotopy in variables $x = (S_{w^\mu}, w^\mu(\widehat{a}^0_2))$ for any fixed $t \in [0,1]$ follows from Lemmas 1, 2, and the Bers isomorphism theorem. The limit map

$$h_1^*(x) = \lim_{t \to 1} h_1(x, t),$$

carries each fiber $w^\mu(U)$ to the initial half-plane U.

There is a canonical holomorphic isotopy

$$h_2(\zeta, t) : \ U \times [0,1] \to U \tag{12}$$

of U into its point corresponding to the origin of $\mathbf{T}(0,5)$. Now make $\mathbf{h}(x,t)$ equal to $h_1(x, 2t)$ for $t \le 1/2$ and equal to $h_2(x, 2t-1)$ for $x = \zeta \in U$ and $1/2 \le t \le 1$.

This function is holomorphic at $x \in \mathbf{T}(0,5)$ for any fixed $t \in [0,1]$ and hence provides the desired holomorphic homotopy of the space $\mathbf{T}(0,5)$ into its base point.

(b) *Case $n = 6$ (dimension three)*. This case is more complicated.

We prescribe to each ordered sextuple $\mathbf{a} = \{0, 1, a_1, a_2, a_3, \infty\}$ of distinct points the corresponding punctured sphere

$$X_{\mathbf{a}} = \widehat{\mathbb{C}} \setminus \{0, 1, a_1, a_2, a_3, \infty\} \tag{13}$$

and the two-sheeted closed hyperelliptic surface $\widehat{X}_{\mathbf{a}}$ of genus two with the branch points $0, 1, a_1, a_2, a_3, \infty$. The corresponding Teichmüller spaces $\mathbf{T}(0,6)$ and $\mathbf{T}(2,0)$ coincide up to a natural biholomorphic isomorphism. Note also that the collection $\mathbf{a} = \{0, 1, a_1, a_2, a_3, \infty\}$ provides the local complex coordinates on spaces $\mathbf{T}(0,6)$ and $\mathbf{T}(2,0)$.

In view of the symmetry of hyperelliptic surfaces, it suffices to deal with the Beltrami differentials $\mu d\bar{z}/dz$ on $\widehat{X}_{\mathbf{a}}$, which are compatible with a conformal involution $J_{\mathbf{a}}$ of $\widehat{X}_{\mathbf{a}}$, hence, satisfying $\mu(J_{\mathbf{a}} z) = \mu(z) J'_{\mathbf{a}} / \overline{J'_{\mathbf{a}}}$. In other words, these μ are obtained by lifting to $\widehat{X}_{\mathbf{a}}$ of the Beltrami coefficients on $X_{\mathbf{a}}$. This extends Lemma 2 and its consequences on holomorphy in the neighborhoods of the boundary interpolation points to the corresponding two-sheeted disks on hyperelliptic surfaces.

We fix a base point of $\mathbf{T}(2,0)$, determining a Fuchsian group Γ for which $\mathbf{T}(\Gamma) = \mathbf{T}(2,0)$. The corresponding Teichmüller curve $\mathcal{V}(2,0)$ is a 4-dimensional, complex analytic manifold with projection $\pi_1 : \mathcal{V}(2,0) \to \mathbf{T}(2,0)$ onto $\mathbf{T}(2,0)$ such that for every $\varphi \in \mathbf{T}(2,0)$ the fiber $\pi_1^{-1}(\varphi)$ is a hyperelliptic surface, determined by φ (see Section 2.4).

Due to assertion (b) of Lemma 3, this curve has, for any point

$$\widehat{X}_{\mathbf{a}} = (S_{w^{\mu_1}}, w^{\mu_1}(\widehat{a}^0_{n-3})) \in \mathbf{T}(2,0))$$

six distinct holomorphic sections $\widehat{s}_1, \dots, \widehat{s}_6$, corresponding to the Weierstrass points of the surface $X_{\mathbf{a}}$, with $\widehat{s}_j(\pi_1(X_{\mathbf{a}})) = X_{\mathbf{a}}$, and either from these sections has one common point with every fiber over $\mathbf{T}(2,0)$. We lift these sections to the Bers fiber space $\mathcal{F}(\Gamma)$ covering $\mathcal{V}(2,0)$.

As mentioned in Section 2.4, these sections are generated by the space $\mathcal{F}(\Gamma') = \mathcal{F}(\Gamma)$ corresponding to the extension Γ' of Γ, for which Γ is a subgroup of index two. Every section $\widehat{s}_j$ acts on $\mathbf{T}(\Gamma')$ via (6), where z_0 is now the corresponding Weierstrass point of hyperelliptic surface $\widehat{X}_{\mathbf{a}}$, and $\widehat{s}_j$ is compatible with action (2) of the Bers isomorphism.

Thus each $\widehat{s}_j$ descends to a holomorphic map $s_j : \mathbf{T}(0,6) \to \mathcal{V}(0,6)$ of the underlying space $\mathbf{T}(0,6)$ for the punctured spheres (10). We choose one from these maps and denote it by s.

The features of sections $\widehat{s}_j$ provide that the descended map s also determines, for each point $z_0 \in X_{\mathbf{a}}$, its unique image on every fiber $w^{\mu}(X_{\mathbf{a}})$ with $\mu \in \mathrm{Belt}(X_{\mathbf{a}})_1$, and this image is the point $w^{\mu}(z_0)$.

The next preliminary construction consists of embedding space $\mathbf{T}(0,5)$ into $\mathbf{T}(0,6)$, using the forgetting map (3). Its image $j_*\mathbf{T}(0,5)$ is a connected submanifold in $\mathbf{T}(0,6)$, and the corresponding fibers of the curve $\mathcal{V}(0,6)$ over the points $j_*\varphi \in j_*\mathbf{T}(0,5)$ are the surfaces $w^{j_*\mu}(X_{\mathbf{a}})$ with $j_*\mu(z) = \mu(\widehat{j}(z))\widehat{j}'(z)/\overline{\widehat{j}'(z)}$. The covering domains $w^{j_*\mu}(U)$ over these surfaces fill a submanifold $\widetilde{\mathbf{T}}(0,7) \subset \mathbf{T}(0,7)$, which is biholomorphically equivalent to the space $\mathbf{T}(0,6)$.

Using the biholomorphic equivalence of space $\mathbf{T}(0,5)$ to its image $j_*\mathbf{T}(0,5)$ in $\mathbf{T}(0,6)$, we carry over to $j_*\mathbf{T}(0,5)$ the result of the previous step (a) on the holomorphic contractibility of $\mathbf{T}(0,5)$, which provides a holomorphic homotopy

$$h(j_*\varphi, t) : j_*\mathbf{T}(0,5) \times [0,1] \to j_*\mathbf{T}(0,5) \quad \text{with } h(j_*\varphi, 0) = j_*\varphi, \;\; h(j_*\varphi, 1) = \mathbf{0} \tag{14}$$

(here, $\mathbf{0}$ stands for the origin of $j_*\mathbf{T}(0,5)$).

Now we may construct the desired holomorphic homotopy of $\mathbf{T}(0,6)$, contracting this space to its origin.

First, regarding $\mathbf{T}(0,6)$ as the Bers fiber space $\mathcal{F}(0,5)$ over $\mathbf{T}(0,5)$ (whose fibers are the covers of surfaces $X_{\mathbf{a}'}$ with collections of five punctures $\mathbf{a}' = (0,1,a_1,a_2,\infty)$), we apply homotopy (11) and define, for any pair $x = (j_*\varphi, z)$ with $\varphi \in \mathbf{T}(0,5)$ and $z \in X_{\mathbf{a}}$, the map

$$\widetilde{h}_1((j_*\varphi, z), t) = (h(j_*\varphi), t), w_t^{j_*\mu}(z)), \quad \varphi \in \mathbf{T}(0,5), \tag{15}$$

noting that the image point $w_t^{j_*\mu}(z)$ is uniquely determined on surface $w^{h(j_*\mu)}(X_{\mathbf{a}})$ by map s, as indicated above.

The pairs $(j_*\varphi, z)$ are located in the space $\mathcal{F}(0,6)$ and fill a three-dimensional submanifold $\widetilde{\mathbf{T}}(0,6)$ biholomorphically equivalent to $\mathbf{T}(0,6)$.

Homotopy (15) is well defined on $\widetilde{\mathbf{T}}(0,6) \times [0,1]$ and contracts the set $\widetilde{\mathbf{T}}(0,6)$ into fiber $\widetilde{U}$ over the base point. It is holomorphic with respect to the space variable $x = (j_*\varphi, z)$ for any fixed $t \in [0,1]$ and continuous in both variables.

In view of the biholomorphic equivalence of $\widetilde{\mathbf{T}}(0,6)$ to $\mathbf{T}(0,6)$, (15) generates a holomorphic homotopy $h_1(x,t)$ of the space $\mathbf{T}(0,6)$ onto the initial fiber (half-plane) U over the origin of $\mathbf{T}(0,5)$.

It remains to combine this homotopy h_1 with the additional homotopy (12) of U into its point corresponding to the origin of $\mathbf{T}(0,6)$. This provides the desired homotopy h and completes the proof of Theorem 1.

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