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Article

Proving, Refuting, Improving—Looking for a Theorem

Branislav Boričić

Faculty of Economics, University of Belgrade, Belgrade 11000, Serbia; boricic@ekof.bg.ac.rs

Abstract: Exploring the proofs and refutations of an abstract statement, conjecture with the aim to give a formal syntactic treatment of its proving–refuting process, we introduce the notion of extrapolation of a possibly unprovable statement having the form *if A, then B*, and propose a procedure that should result in the new statement *if A', then B'*, which is similar to the starting one, but provable. We think that this procedure, based on the extrapolation method, can be considered a basic methodological tool applicable to prove–refute–improve any conjecture. This new notion, extrapolation, presents a dual counterpart of the well-known interpolation introduced in traditional logic sixty-five years ago.

Keywords: extrapolation; interpolation; proving; refuting; improving

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1. Introduction

Lakatos' monumental play 'Proofs and Refutations' (see [1]) can be considered a demonstration of applying the proof–refutation (or conjecture–refutation) method as a practical realization of the falsificationism concept advocated and supported at that time, among other authors, by [2]. At the same time, the concept of proving–refuting–improving, demonstrated in the same play, can be used as an effective interactive class model.

Refutation, as an isolated process, plays an extremely important role in the development of a pupil's critical thinking and has a crucial place in each study program syllabus. We deem that examples of finding and treating incorrectness in some reasoning and argumentation are at least of equal didactic importance as those with correct derivations and proofs. Such examples present and help incite critical thinking.

First, let us explain in brief what we mean under the term 'extrapolation'. As we know, interpolation deals with finding statements C and D, which are in between A and B, when $A \vdash B$; i.e., 'A implies B', is provable, meaning that all three sequents $A \vdash C$, $C \vdash D$ and $D \vdash B$ are provable. In this case, the sequent $C \vdash D$ presents an interpolant for $A \vdash B$. On the other side, if $A \vdash B$ is refutable, i.e., $A \nvdash B$, then we are looking for two statements C and D, such that $C \vdash A$, $B \vdash D$ and $C \vdash D$ are all provable; in this case, the sequent $C \vdash D$ will be an extrapolant for $A \nvdash B$.

In this paper, we extend the proving–refuting method by its immediate result—improving—and place it in a wider logical context relating it with the well-known concept of interpolation, with a new concept, extrapolation, as its dual. Both these notions, extrapolation and interpolation, are closely connected with many aspects of abductive reasoning [3]. The improving process, based deeply on the extrapolation method, is presented through several examples. Let me repeat here that once, a long time ago, my teacher Aleksandar Kron told me: 'Oh, how many times I fell asleep with a proof, and woke up with a counterexample'. This was the essence of the proving–refuting–improving process, during the daily journey of any scientist from a conjecture to the truth (see [4]). This process, consisting of proving and refuting attempts producing an improvement of the starting conjecture, is presented formally as an methodological procedure for discovering better statements. In fact, this can be considered a kind of Hegelian–Marxist dialectic scheme: thesis–antithesis–synthesis. However, the crucial cognition is that the essential step of this procedure is based on extrapolation, which is a dual to the well-known logical feature of reasoning—the interpolation property. We introduce the notion of extrapolation as a counterpart of interpolation. We do this in general form, independently of the basic logic. Namely, our definition depends neither on language—we do not use connectives—nor on logic—we suppose that our deduction relation is not necessarily linked to classical logic. Pure propositional logics open the problem of existence of a minimal extrapolant, which seems particularly interesting in case of infinitely valued systems.

2. Interpolation and Extrapolation—A General Idea

A typical form of a scientific statement is that 'B follows from A', denoted by $A \vdash B$, expressing a causal relationship between A and B. Refutation of such a statement consists of argumentation presenting at least one example (interpretation) where A is satisfied, but B is not.

The turnstyle symbol will be used in an informal way, not connected to any particular logical system, but assuming its rudimentary structural properties such as identity ($A \vdash A$), weakening ($A \vdash B$ implies $A, C \vdash B$), permutation ($A, B \vdash C$ implies $B, A \vdash C$), contraction ($A, A \vdash B$ implies $A \vdash B$) and transitivity ($A \vdash B$ and $B \vdash C$ imply $A \vdash C$).

Establishing a statement $A \vdash B$ as a conjecture means that we believe that $A \vdash B$ holds, but also that this is partly under question; does $A \vdash B$? In order to obtain a final conclusion regarding the truthfulness of our conjecture, we try to prove and to refute it. This process implies finding examples supporting $A \vdash B$ and counterexamples refuting $A \vdash B$, as well as looking for similar statements $A' \vdash B'$ that are, by their nature, weaker than $A \vdash B$ in cases when $A \vdash B$ is refutable, and stronger than $A \vdash B$ in cases when $A \vdash B$ is provable.

Let us consider the two apparently simplest cases of causal connection: (i) $A \vdash B$ is not proven and (ii) $A \vdash B$ is proven, where A and B are two arbitrary sentences. In the second case (ii), we can assert that there are two propositions C and D such that the following statements are provable: $A \vdash C$, $C \vdash D$ and $D \vdash B$. If C and D are logically equivalent, then we recognize here a form of the well-known Craig interpolation theorem (see [5]), pointing out that the form presented here can be considered as its slight generalization. In a similar way, we will deal with the first case (i) and suppose that there are two propositions C and D such that the following statements are provable: $C \vdash A$, $B \vdash D$ and $C \vdash D$, obtaining a form that is somehow dual to interpolation (ii) and which could be treated as a kind of extrapolation.

We point out that the term 'duality' is used here in a quite different meaning than in classical two-valued logic. For each statement of the form $A \vdash B$, provable or unprovable, we consider a provable statement $C \vdash D$. If $A \vdash C$ and $D \vdash B$ are provable, then $C \vdash D$ is called an interpolant, while when $C \vdash A$ and $B \vdash D$ are provable, then $C \vdash D$ is called an extrapolant. Consequently, C and D as parts of an interpolant are in consequent of A and antecedent of B, respectively, but as parts of an extrapolant, they have 'dually' just the opposite roles; C is in antecedent of A and D is in consequent of B.

More accurately, if we suppose that $A \vdash B$ is any statement, provable or not, then (i) $C \vdash D$ is its *extrapolant* if all statements $C \vdash A$, $B \vdash D$ and $C \vdash D$ are provable; (ii) $C \vdash D$ is its *interpolant* if all statements $A \vdash C$, $D \vdash B$ and $C \vdash D$ are provable. We omit here more formal details such as variable sharing and the context of a particular logical system for the deduction relation.

Note that in the case that an interpolant exists, the original statement $A \vdash B$ is provable. However, in the case that an extrapolant exists, we can conclude nothing regarding the provability of $A \vdash B$. The most interesting cases in the sequel of this paper will be exactly those (i) when $A \nvdash B$, i.e., $A \vdash B$ is unprovable. The challenges before us here are how to find some 'good' extrapolants for $A \nvdash B$ and (ii) when $A \vdash B$ is provable, how to find its 'good' interpolants. This is because in both these cases, the statement $C \vdash D$ should present an improvement of $A \vdash B$, which will be explained below.

The term 'interpolation' is justified by the simple fact that we insert a new statement $C \vdash D$ in between A and B, with an obvious possibility to infer $A \vdash B$ from $A \vdash C$,

$C \vdash D$ and $D \vdash B$. Similarly, the extrapolation process involves looking for a statement C 'before' A, because $C \vdash A$, and a statement D 'after' B, because $B \vdash D$. Both requirements, interpolation and extrapolation, have some trivial solutions. If $A \vdash B$ is proven, then both forms $A \vdash A$ and $B \vdash B$ present possible interpolants. Furthermore, for any $A \vdash B$, all statements $\perp \vdash \top$, $\perp \vdash A$ and $B \vdash \top$ present its extrapolants, where we use the symbols $\top$ and $\perp$, respectively, to denote truth and absurdity constants. Later, after sharpening both notions, extrapolation and interpolation, following the spirit of Craig's interpolation theorem and practical applications of extrapolation, we will see that trivial solutions have no importance (as usual).

Example 1 (Lakatos' Proofs and Refutations). *In his famous work, by giving a picturesque presentation of the proving–refuting process, Lakatos (see [1]) begins with an incorrect and refutable formulation of Euler's Polyhedral Theorem. The dialog between a teacher and his pupils starts with the teacher's provocation: "In our last lesson we arrived at a conjecture concerning polyhedra, namely, that for all polyhedra $V - E + F = 2$, where V is the number of vertices, E the number of edges and F the number of faces. We tested it by various methods. But we have not yet proven it. Has anybody found a proof?" After that, through a few iterations, the teacher, together with his pupils, by using a proving–refuting–improving method, obtains and proves the correct form of Euler's Polyhedral Theorem: for all convex polyhedra, $V - E + F = 2$.*

Example 2 (Elementary Geometry). *Let $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denote any right triangle with sides a, b, c, where c is its hypothenuse, and $\mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denotes any triangle with sides a, b, c. Some of the known elementary geometric facts can be formulated by means of a deduction relation as follows:*

Triangle Inequality: $\mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{a} + \mathbf{b} > \mathbf{c}$.

Pythagorean Theorem: $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{a^2} + \mathbf{b^2} = \mathbf{c^2}$ *and* $a^2 + b^2 = c^2 \vdash \mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

We also have two obvious facts: $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c})$*; i.e., each right triangle is a triangle and, in elementary algebra, $a^2 + b^2 = c^2 \vdash a + b > c$ for any positive reals a, b, c (see [6]). In order to illustrate the extrapolation phenomenon in this context, we consider the following negative statement:*

$$\mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \nvdash \mathbf{a^2} + \mathbf{b^2} = \mathbf{c^2}$$

By the extrapolation approach, bearing in mind that $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $a^2 + b^2 = c^2 \vdash a + b > c$, we can infer the following statements: $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{a^2} + \mathbf{b^2} = \mathbf{c^2}$, $\mathbf{Tr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{a} + \mathbf{b} > \mathbf{c}$ and $\mathbf{RTr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \vdash \mathbf{a} + \mathbf{b} > \mathbf{c}$, as possible extrapolants. Deeming the proving–refuting–improving process one of the most important methods of knowledge growth, the author of this text, with a group of his brilliant students (Aleksanra Djoković, Bojana Tujković, Ivana Čekrdžić, Aleksandar Elezović, Doroteja Djordjević and Milan Perić), during the spring semester 2014, set up a musical performance under the title 'Proofs and refutations: devoted to the glorious triangle', at the Faculty of Economics, University of Belgrade. That performance was deeply inspired by [1] but, for the sake of better understanding the basic message, instead of Euler's Polyhedral Theorem, considered in the original Lakatos' play, we dealt with proofs and refutations of the Pythagorean Theorem.

Example 3 (Propositional Calculus). *Here, we present some more subtle examples of interpolants and extrapolants. Let $\wedge$ and $\vee$ denote the conjunction and disjunction connectives, respectively. (i) The form $p \vee q \vdash p \wedge q$ is unprovable, i.e., $p \vee q \nvdash p \wedge q$, and it can be improved by the following forms: $p \vdash p$, $q \vdash q$, $p \vdash p \vee q$, $q \vdash p \vee q$ and $p \wedge q \vdash p \vee q$; this is not a complete list of its extrapolants. (ii) The form $p \wedge q \vdash p \vee q$ is provable and it can be improved by the following interpolants: $p \wedge q \vdash p$, $p \vdash p \vee q$, $p \vdash p$ and $q \vdash q$; this is not a complete list of its interpolants. Let us note that the examples of extrapolants and interpolants considered here are compatible not only with classical, but also with many non-classical propositional logics.*

Example 4 (Set-Theoretic Interpretation). *Due to the immediate link between the set-theoretic inclusion relation and the classical implication connective, the interpolation and extrapolation have a rough illustrative and a quite simple set-theoretic interpretation. Namely, if for two sets A and B*

we have $A \subseteq B$, then the sets C and D, such that $A \subseteq C \subseteq D \subseteq B$, can be considered as the basic constituents of an interpolant $C \subseteq D$ for $A \subseteq B$. On the other side, if $A \nsubseteq B$, then the sets C and D, such that $C \subseteq A$, $B \subseteq D$ and $C \subseteq D$, define an extrapolant $C \subseteq D$ for $A \nsubseteq B$.

Example 5 (Impossibility Tradition)**.** *In spite of their great methodological and logical importance (see [7,8]), impossibility theorems raise a natural question: how they can be transformed into the corresponding relevant possibility results? Each such transformation is based on some proving–refuting–improving process that starts with the improving, i.e., with an extrapolation step. Let us discuss two simple cases of impossibility theorems.*

Incommensurability of the diagonal and the side of a square: if a is the side of a square and d is its diagonal, then $a = 1 \vdash d \notin \mathbf{N}$, i.e., $a = 1 \nvdash d \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. By replacing (weakening) $d \in \mathbf{N}$ with $d \in \mathbf{Q}$, bearing in mind that $d \in \mathbf{N} \vdash \mathbf{d} \in \mathbf{Q}$ where $\mathbf{Q}$ is the set of rational numbers, we also obtain by extrapolation an invalid statement $a = 1 \vdash d \in \mathbf{Q}$. The next iteration is finding an appropriate extrapolant for the statement $a = 1 \nvdash d \in \mathbf{Q}$. Obviously, by replacing $d \in \mathbf{Q}$ with $d \in \mathbf{R}$, bearing in mind that $d \in \mathbf{Q} \vdash \mathbf{d} \in \mathbf{R}$ where $\mathbf{R}$ is the set of reals, we obtain a valid positive statement $a = 1 \vdash d \in \mathbf{R}$, i.e., a possibility result.

Unsolvability of the equation $x^2 + a = 0$, $a \in \mathbf{R}$, in the field of reals: $a \in \mathbf{R} \wedge \mathbf{x^2} + \mathbf{a} = \mathbf{0} \nvdash \mathbf{x} \in \mathbf{R}$ leads to two simple positive possibilities. By antecedent weakening, from $a \leq 0 \vdash a \in \mathbf{R}$, we obtain $a \leq 0 \wedge x^2 + a = 0 \vdash x \in \mathbf{R}$, or, by consequent weakening, from $x \in \mathbf{R} \vdash \mathbf{x} \in \mathbf{C}$ where $\mathbf{C}$ is the set of complex numbers, we have $a \in \mathbf{R} \wedge \mathbf{x^2} + \mathbf{a} = \mathbf{0} \vdash \mathbf{x} \in \mathbf{C}$, i.e., solvability of that equation in the field of complex numbers.

In a similar way, but with more complex argumentation and context, Arrow's Impossibility Theorem (see [7–9]), the most popular and important result in Social Choice Theory during the last century, has generated a number of possibility results. Variations of the corresponding possibility theorems (see [10]), obtained by weakening the antecedent or consequent of Arrow's original theorem, can be considered as effective examples of applying the proving–refuting–improving process as well.

Here, we will explain why we do believe that both interpolants and extrapolants present improvements of our initial statement $A \vdash B$. (i) If $A \vdash B$ is not provable, then its extrapolant $C \vdash D$, which is provable, obtained from an unprovable statement, presents its improvement, bearing in mind that from the initial statement $A \vdash B$ of low quality (unprovable), we obtain its extrapolant $C \vdash D$, a statement of higher quality (provable). (ii) If $A \vdash B$ is provable, then its interpolant $C \vdash D$, which is provable together with $A \vdash C$ and $C \vdash B$, can be used as a sufficient condition to infer the initial statement $A \vdash B$, and from this point of view it can be considered as its essence—its improvement—enabling us to prove and better understand the meaning of the initial statement $A \vdash B$.

3. Extrapolation—More Formally

Let us discuss a more subtle aspect of extrapolation including some views of relevance logic. A deduction of B from hypothesis A is acceptable relevance logic if this deduction employs every element of A. Another syntactic relevance principle, known as *variable sharing condition*, is that if A entails B, then $\mathrm{At}A \cap \mathrm{At}B \neq \emptyset$, where $\mathrm{At}A$ denotes the set of all atomic formulae, i.e., propositional letters, occurring in A (see [11]). Variable sharing is not sufficient, but it is a necessary condition for relevance.

Now, we can formulate more ambitious expectations, including some kind of variable sharing principle.

Interpolation property: If $\mathrm{At}A \cap \mathrm{At}B \neq \emptyset$ and $A \vdash B$, then there exist C and D such that $\mathrm{At}C \cup \mathrm{At}D \subseteq \mathrm{At}A \cap \mathrm{At}B$, $A \vdash C$, $D \vdash B$ and $C \vdash D$.

Extrapolation property: If $\mathrm{At}A \cap \mathrm{At}B \neq \emptyset$ and $A \nvdash B$, then there exist C and D such that $\mathrm{At}C \cup \mathrm{At}D \subseteq \mathrm{At}A \cap \mathrm{At}B$, $C \vdash A$, $B \vdash D$ and $C \vdash D$.

The interpolation property is defined in accordance with Craig's well-known approach (see [5]). The extrapolation property tends to find relevant, non-trivial and, in some sense, minimal statements C and D establishing an extrapolant.

Let us note here that Craig's original definition deals with only one formula C, such that $A \vdash C$ and $C \vdash B$, as an interpolant for $A \vdash B$. In this spirit, it would be possible to redefine our notion of extrapolant C for $A \nvdash B$ so that $C \vdash A$ and $B \vdash C$. It is not difficult to see that this approach with one formula playing the role of interpolant (or extrapolant) is logically equivalent to our definition employing two formulae in both cases.

The logical, methodological, philosophical and, even algebraic aspects of interpolation have been analyzed, discussed and explained in detail as a necessary part of most textbooks in logic (see [12,13]). Here, we will attempt to elucidate the logical sense of extrapolation. Bearing in mind the following derivation:

$$\dfrac{\dfrac{C \vdash A \quad \dfrac{A \vdash B}{C, A \vdash B, D}\text{weakening} \times 2 \qquad B \vdash D}{C, C \vdash D, D}\text{cut} \times 2}{C \vdash D}\text{contraction} \times 2$$

the extrapolation can be considered to be a weakening of the antecedent and the consequent of $A \vdash B$, respectively, by special statements C and D, such that $C \vdash A$ and $B \vdash D$ (Instead of $\{A, B\} \vdash \{C, D\}$, we will write simply $A, B \vdash C, D$, which, according to the traditional classical logic proof-theoretic interpretation, can be understood as $A \wedge B \vdash C \vee D$). The procedure will be satisfiable when, from an unprovable statement, we obtain a provable one, i.e., when, in fact, from $A \nvdash B$, we obtain $C \vdash D$, where C and D are in the corresponding causal connections with A and B, respectively. In practice, when we search for an adequate statement, instead of reasoning starting with the explicit application of weakening rules, as above, the pure derivation with the cut rules

$$\dfrac{C \vdash A \qquad A \vdash B \qquad B \vdash D}{C \vdash D}\text{cut} \times 2$$

hides the presence of weakening. On the other side, we have to emphasize that it would be wrong to understand the extrapolation just as a simple weakening, because it is a very restricted and specific weakening in order to find the relevant extrapolant.

Extrapolation is formally, in the context of classical logic, equivalent to the left and the right side weakening rules, bearing in mind the following derivations

$$\dfrac{C, A \vdash A \quad A \vdash B}{C, A \vdash B} \text{ and } \dfrac{A \vdash B \quad B \vdash B, D}{A \vdash B, D}$$

Nevertheless, the extrapolation, as defined, seems more restrictive and suggests some kind of 'relevant' weakening. Namely, the above two derivations are classically, and even intuitionistically, admissible, but not from the point of view of relevance logic. This is the reason why the extrapolation can be essentially considered as a process partly supported by relevant logic principles, bearing in mind that variable sharing conditions for C with A and B with D are satisfied, but not necessary for C with D.

In case of an unprovable statement $A \vdash B$, when we look for some of its improvements $C \vdash D$, in order to avoid trivial solutions and to find the best one, if possible, we define the notion of minimal sentences:

Minimal extrapolants: Suppose $A \vdash B$ is not proven and $C \vdash D$ is its extrapolant. C will be called *a minimal sentence* for A, B and D, in this order, if for each C', such that $C \vdash C'$ is provable and $C' \vdash C$ is unprovable, one of the statements $C' \vdash A$ or $C' \vdash D$ is unprovable. In a dual way, D will be called *a minimal sentence* for A, B and C, in this order, if for each D', such that $D \vdash D'$ is unprovable and $D' \vdash D$ is provable, one of the statements $B \vdash D'$ or $C \vdash D'$ is unprovable. In cases when both hold, C is a minimal sentence for A, B and D, and D is a minimal sentence for A, B and C; then, the statement $C \vdash D$ is called *a minimal extrapolant* for $A \vdash B$.

The central question now is the following one: does a minimal extrapolant exist (and when)? It depends on the logical context, clearly. For instance, in m-valued propositional logics, due to the existence of finitely many nonequivalent formulae over any finite set of atomic formulae (propositional letters), we always have the possibility to choose the minimal sentences. The next question is: does the minimal nontrivial extrapolant exist (and when)? Moreover, how could a nontrivial statement be characterized?

Example 6. *Obviously, for any two sentences A and B, such that $A \nvdash B$ and $p \in \mathrm{At}A \cap \mathrm{At}B$, the statement $p \wedge \neg p \vdash p \rightarrow p$ presents an extrapolant. This is a trivial example.*

Example 7. *Let us consider again some extrapolants $p \wedge q \vdash p \vee q$, $p \vdash p \vee q$, $p \wedge q \vdash p$ and $p \vdash p$ of the statement $p \vee q \vdash p \wedge q$. In the case of extrapolant $p \wedge q \vdash p \vee q$, the statement $p \wedge q$ is not minimal for $p \vee q$, $p \wedge q$ and $p \vee q$ because there is a statement, p such that $p \wedge q \vdash p$, and both $p \vdash p \vee q$ and $p \wedge q \vdash p$ are provable. On the other hand, the statement p is a minimal one for $p \vee q$, $p \wedge q$ and $p \vee q$, and this is a way to find a new and 'better' extrapolant $p \vdash p \vee q$. In the case of this extrapolant $p \vdash p \vee q$, although p is a minimal for $p \vee q$, $p \wedge q$ and $p \vee q$, the proposition $p \vee q$ is not minimal for $p \vee q$, $p \wedge q$ and p because, for the statement p, we have that $p \vdash p \vee q$ and both $p \vdash p \vee q$ and $p \vdash p$ are provable, while p is a minimal statement for $p \vee q$, $p \wedge q$ and p. Let us note also that the examples considered here have a general character and are compatible with both classical and intuitionistic propositional logics.*

Example 8. *In set-theoretic interpretation, when $A \not\subseteq B$, the parts of minimal extrapolants will be the sets in between $C = A \cap B$ and $D = A \cup B$ with respect to the inclusion relation. In general, $C = A \cap B \subseteq B = D$ will be a minimal extrapolant for $C = A \cap B (\subseteq A)$, $A(\not\subseteq B)$ and $B \subseteq D = B$, and $C = A(\subseteq D = A \cup B)$ will be a minimal extrapolant for $A = C(\subseteq A)$, $A(\not\subseteq B)$ and $B(\subseteq D = A \cup B)$.*

4. More Examples

The importance of propositional language is founded, inter alia, on its simplicity. Propositional context is usually suitable for explaining and understanding the differences between various philosophical concepts for the foundations of mathematics. For instance, the spirit of essential variations between Platonism, intuitionism and relevance is already visible on the level of classical, intuitionistic and relevant propositional logics. On the other side, the founding of any serious mathematical theory needs much more than a propositional language. Here, we will try to present the idea of extrapolation in the context of the first-order predicate language.

The general symbolic form of an Impossibility Theorem stating that 'there does not exist an object x such that A implies B', is

$$\neg\exists x(A \rightarrow B)$$

The first-order sentence $\neg\exists x(A(x) \rightarrow B(x))$ can be presented in a classically equivalent way as $\neg(\forall x A(x) \rightarrow \exists x B(x))$, or a bit more informally as "$\forall x A(x)$ does not imply $\exists x B(x)$", i.e., $\forall x A(x) \nvdash \exists x B(x)$. Here, we want to describe an application of extrapolation method on

$$\forall x A(x) \nvdash \exists x B(x)$$

Namely, we are looking for sentences C and D such that $C \vdash \forall x A(x)$, $\exists x B(x) \vdash D$ and $C \vdash D$, where the last statement presents an extrapolant and, simultaneously, a transformation of an 'impossibility' result into a 'possibility' one.

On the level of general first-order languages examples, we analyze an 'impossibility case'.

Example 9. *Let us consider the following statement: $\forall x(A \vee B) \nvdash \exists x(A \wedge B)$, having exactly the form of an impossibility theorem. If we try to weaken the antecedent $\forall x(A \vee B)$ by (1) $\forall x A \vee \forall x B$ or by (2) $\forall x A$, and the consequent $\exists x(A \wedge B)$ by (3) $\exists x A \wedge \exists x B$ or by (4) $\exists x A$, we do not obtain extrapolants by combining (1) with (3), (1) with (4) or (2) with (3); only the combination (2) with (4) gives an extrapolant, because $\forall x A \vdash \forall x(A \vee B)$, $\exists x(A \wedge B) \vdash \exists x A$, and $\forall x A \vdash \exists x A$.*

We also consider some relationships between binary relations properties.

Example 10. *The logic of preferences is usually based on axioms concerning some properties of a binary relation P, called a preference relation. For instance, the list of axioms contains irreflexivity (Ir), $\forall x \neg P(y,x)$, asymmetry (As), $\forall x \forall y (P(x,y) \to \neg P(y,x))$, transitivity (Tr), $\forall x \forall y \forall z (P(x,y) \wedge P(y,z) \to P(x,z))$ and connectivity (Cn), $\forall x \forall y \forall z (P(x,y) \to P(x,z) \vee P(z,y))$. It is an easy exercise to show that* $\mathrm{Cn} \nvdash \mathrm{Tr}$*, but, bearing in mind that* $\mathrm{As} \wedge \mathrm{Cn} \vdash \mathrm{Cn}$, $\mathrm{As} \wedge \mathrm{Cn} \vdash \mathrm{Tr}$ *and* $\mathrm{Tr} \vdash \mathrm{Tr}$*, we conclude that* $\mathrm{As} \wedge \mathrm{Cn} \vdash \mathrm{Tr}$ *presents an extrapolant and an improvement of the initial statement. In a similar way, we can find that the same statement,* $\mathrm{As} \wedge \mathrm{Cn} \vdash \mathrm{Tr}$ *is an extrapolant for both* $\mathrm{Ir} \wedge \mathrm{Cn} \nvdash \mathrm{Tr}$ *and* $\mathrm{As} \wedge \mathrm{Tr} \nvdash \mathrm{Cn}$*.*

5. A Proving–Refuting–Improving Procedure

Each theorem, or more generally, each scientific statement, can be expressed in the following form: *if* Γ, *then* Δ. Γ presents a set of hypotheses (given context or a theory) and Δ is a consequence (conclusion). This is the reason why the basic form we use in this part of the paper is $\Gamma \vdash \Delta$, an informal deduction relation (entailment) $\vdash$ between two finite sets of sentences Γ (antecedent) and Δ (consequent), with the intended meaning that it is possible to infer a conclusion Δ, interpreted as a disjunction of all elements of Δ, from the hypotheses set Γ, interpreted as a conjunction of all elements of Γ. The Greek capitals $\Gamma, \Delta, \ldots$, with or without subscripts or superscripts, will be used as metavariables over finite sets of sentences denoted by Latin capitals $A, B, C, D, \ldots$ We also use $\Gamma \models \Delta$ with the usual model theoretic, meaning that if all elements of Γ are true, then at least one element of Δ is true. This will be the context enabling us to express that $\Gamma \vdash \Delta$, or $A \vdash B$, is provable or unprovable, and that $\Gamma \models \Delta$, or $A \models B$, is refutable or irrefutable.

The idea of a proving–refuting–improving procedure has been hinted at by [4]. Here, we will develop it further. In both cases when $\Gamma \vdash \Delta$ is provable or unprovable, i.e., when $\Gamma \models \Delta$ is valid or refutable, we define the following four sets: Γ-antecedent, Γ-consequent, Δ-antecedent and Δ-consequent, respectively, as $\Gamma_{\mathrm{ant}} = \{A_1^a, \ldots, A_m^a\}$, $\Gamma_{\mathrm{con}} = \{A_1^c, \ldots, A_m^c\}$, $\Delta_{\mathrm{ant}} = \{B_1^a, \ldots, B_n^a\}$ and $\Delta_{\mathrm{con}} = \{B_1^c, \ldots, B_n^c\}$, corresponding to the sets $\Gamma = \{A_1, \ldots, A_m\}$ and $\Delta = \{B_1, \ldots, B_n\}$, such that, for each i $(1 \le i \le m)$, $A_i^a \vdash A_i$ and $A_i \vdash A_i^c$ are provable, and for each j $(1 \le j \le n)$, $B_j^a \vdash B_j$ and $B_j \vdash B_j^c$ are provable.

The main problem here is to define concrete content of sets Γ_{ant}, Γ_{con}, Δ_{ant} and Δ_{con} in this general case, because the condition that $A_i^a \vdash A_i$ is provable has infinitely many solutions for A_i^a. On the other hand, each particular problem in some specific part of mathematics gives the researcher a freedom to use his intuition during the process of 'looking for a better theorem'.

The two elementary steps in our proving–refuting–improving procedure as follows:

Step (i): if $\Gamma \vdash \Delta$ is not proven or $\Gamma \models \Delta$ is refuted, we are looking for some $A_i^a \in \Gamma_{\mathrm{ant}}$ or some $B_j^c \in \Delta_{\mathrm{con}}$ for which the provability of $\Gamma' \vdash \Delta'$ can be reconsidered, where $\Gamma' \cup \Delta'$ is obtained from $\Gamma \cup \Delta$ by substituting at least one occurrence of A_i by A_i^a in Γ or at least one occurrence of B_j by B_j^c in Δ;

Step (ii): if $\Gamma \vdash \Delta$ is proven, or $\Gamma \models \Delta$ is not refuted, we are looking for some $A_i^c \in \Gamma_{\mathrm{con}}$ or $B_j^a \in \Delta_{\mathrm{ant}}$ for which the provability of $\Gamma' \vdash \Delta'$ can be reconsidered, where $\Gamma' \cup \Delta'$ is obtained from $\Gamma \cup \Delta$ by substituting at least one occurrence of A_i by A_i^c in Γ or at least one occurrence of B_j by B_j^a in Δ.

In both cases (i) and (ii), the result will be a statement $\Gamma' \vdash \Delta'$. If $\Gamma' \vdash \Delta'$ is provable, then the procedure can be stopped and $\Gamma' \vdash \Delta'$ will present a generalized extrapolant or interpolant for $\Gamma \vdash \Delta$ in cases (i) and (ii), respectively. Otherwise, if we cannot decide if $\Gamma' \vdash \Delta'$ is provable or if $\Gamma' \vdash \Delta'$ is refutable, then we proceed with step (i) on $\Gamma' \vdash \Delta'$.

Finally, in the sequel, we apply the same procedure on $\Gamma' \vdash \Delta'$; i.e., firstly, we try to prove $\Gamma' \vdash \Delta'$ or to falsify $\Gamma' \models \Delta'$. If $\Gamma' \vdash \Delta'$ is not proven or $\Gamma' \models \Delta'$ is falsifiable, then we apply the procedure (i) on $\Gamma' \vdash \Delta'$ in order to obtain a new statement $\Gamma'' \vdash \Delta''$. If $\Gamma' \vdash \Delta'$ is provable or $\Gamma' \models \Delta'$ is not refuted, then we apply the procedure (ii) on $\Gamma' \vdash \Delta'$

in order to obtain a new statement $\Gamma'' \vdash \Delta''$. This process is called the *proving–refuting–improving procedure*.

Let us point out that a similar form of a generalized interpolant appears in S. Maehara's approach to interpolation in the context of sequent calculi (see [13]).

Note that the sentence '$\Gamma \vdash \Delta$ is not proven' does not exclude the case that $\Gamma \vdash \Delta$ can be provable, and sentence '$\Gamma \models \Delta$ is not refuted' does not exclude the case that $\Gamma \models \Delta$ can be refutable. Namely, if some fact is not proven, maybe, in the future, it could be proven, and if some fact has not been refuted up to now, it could be refuted later.

The above procedure, part (i), proving–refuting–improving, was based on methodological ideas promoted by Popper–Lakatos' proof–refutation (also known as conjecture–refutation) falsificationism (see [1,2]). Furthermore, the transformation of $\Gamma \vdash \Delta$ into $\Gamma' \vdash \Delta'$, generally, can be considered a kind of Hegelian–Marxist dialectic scheme: thesis–antithesis–synthesis, which is obviously parallel with our scheme consisting of (i) and (ii), defining the process of proving–refuting–improving.

The statement $\Gamma' \vdash \Delta'$ presents an improvement of $\Gamma \vdash \Delta$ in case (i), in a sense that from an unprovable statement $\Gamma \vdash \Delta$, we obtain a statement $\Gamma' \vdash \Delta'$, which may be provable; but if $\Gamma \vdash \Delta$ is provable, then $\Gamma' \vdash \Delta'$ is provable as well. On the other hand, the statement $\Gamma' \vdash \Delta'$ presents an improvement of $\Gamma \vdash \Delta$ in case (ii), in the sense that from a provable statement $\Gamma \vdash \Delta$, we obtain a provable statement $\Gamma' \vdash \Delta'$ from which $\Gamma \vdash \Delta$ can be derived; i.e., $\Gamma' \vdash \Delta'$ is more general than $\Gamma \vdash \Delta$. These are the reasons to treat $\Gamma' \vdash \Delta'$ as an improvement of $\Gamma \vdash \Delta$ in both cases. This also means that any possible application of our procedure to a provable statement cannot produce an unprovable statement.

If reconsideration of $\Gamma \vdash \Delta$ provides a statement $\Gamma' \vdash \Delta'$, consisting of some new elements of $\Gamma_{\text{ant}} \cup \Gamma_{\text{con}} \cup \Delta_{\text{ant}} \cup \Delta_{\text{con}}$, then, obviously, $\Gamma' \vdash \Delta'$ presents an improvement of $\Gamma \vdash \Delta$. More accurately, we can justify our procedure by some kind of soundness statement:

Theorem 1.

(i) *If the statement $\Gamma' \vdash \Delta'$ is obtained from $\Gamma \vdash \Delta$ by applying step (i), then $\Gamma' \vdash \Delta'$ can be inferred from $\Gamma \vdash \Delta$;*

(ii) *If the statement $\Gamma' \vdash \Delta'$ is obtained from $\Gamma \vdash \Delta$ by applying step (ii), then $\Gamma \vdash \Delta$ can be inferred from $\Gamma' \vdash \Delta'$.*

Proof. By induction on $n + m$—the number of statements belonging to $\Gamma \cup \Delta$: in case (i), from both, $\Gamma \vdash \Delta$ and $A_i^a \vdash A_i$, and $\Gamma \vdash \Delta$ and $B_j \vdash B_j^c$, by the hypothetical syllogism rule, we can infer $\Gamma' \vdash \Delta'$. In case (ii), from both pairs, $\Gamma' \vdash \Delta'$ and $A_i \vdash A_i^c$, and from $\Gamma' \vdash \Delta'$ and $B_j^a \vdash B_j$, by the hypothetical syllogism rule, we can infer $\Gamma \vdash \Delta$. $\square$

Let us note that in the particular case when A_i^a is true or when B_j^c is a false statement, applying step (i) of our procedure produces the effects of enthymematic reasoning (see [14]).

A rare and unexpected case, which is not covered by (i) and (ii), is when the statement $\Gamma \vdash \Delta$ is undecidable, i.e., the case when it is possible to show that $\Gamma \vdash \Delta$ is neither provable nor refutable. Such examples are connected with highly formalized concepts and will not be our focus.

This procedure can be considered a sequence of consecutive attempts to falsify a statement and then to save it as a supplementary conjecture or to give it a new semantic interpretation. In this way, a progressive improvement of the initial claim is enabled.

In order to visualize the transformation process of $\Gamma \vdash \Delta$ into $\Gamma' \vdash \Delta'$ with the help of Γ_{ant}, Γ_{con}, Δ_{ant} and Δ_{con}, we give a $2D$-presentation of relationships between elements of Γ and Δ, with or without subscripts or superscripts:

$$\begin{array}{ccccccc} A_1^a & & A_m^a & & B_1^a & & B_n^a \\ \top & & \top & & \top & & \top \\ A_1, & \ldots, & A_m & ?\vdash & B_1, & \ldots, & B_m \\ \top & & \top & & \top & & \top \\ A_1^c & & A_m^c & & B_1^c & & B_n^c \end{array}$$

where, for instance, the first column

$$\begin{array}{c} A_1^a \\ \top \\ A_1 \\ \top \\ A_1^c \end{array}$$

of this $2D$-presentation means that both $A_1^a \vdash A_1$ and $A_1 \vdash A_1^c$ are provable. Consequently, by some replacements of A_i with A_i^a or with A_i^c, $(1 \leq i \leq m)$, and some replacements of B_j with B_j^a or with B_j^c, $(1 \leq j \leq n)$, we obtain this new form $\Gamma' \vdash \Delta'$. The symbol '$? \vdash$', appearing above, stands for '$\nvdash$' or '$\vdash$'.

6. Concluding Remarks

An unproven statement of hypothetical character, a conjecture, is usually treated in one of the following two ways: we try to prove it, or we try to refute it. Then, for a proven statement, we try to find its interpolants, in order to simplify its proof and to better understand the nature of its proof, but for a refuted, i.e., unprovable, statement, we look for its extrapolants, trying to find a similar and relevant but provable statement.

Briefly, if we start with a statement of the form $A \vdash B$, then we have, syntactically, two possibilities to obtain from $A \vdash B$ a better statement: if $A \vdash B$ is unproven, we will look for its extrapolant presenting a provable statement relevant for $A \vdash B$, but if $A \vdash B$ is proven, then we will find its interpolant relevant for $A \vdash B$, better explaining the nature of $A \vdash B$. Namely, the basic principle respected in the process of transforming $A \vdash B$ into a 'better statement' $A' \vdash B'$ is that all side statements occurring in derivations, such as $C \vdash A$ and $B \vdash D$, are provable, except the principal statement $A \vdash B$, which can be, but does not have to be, provable, and that each step in the considered derivation is made strictly in accordance with the sound logical inference rules.

In working versions of this paper, we used the term 'algorithm' for the proving–refuting–improving process, but later we accepted the term 'procedure' as the appropriate one. Namely, it is not clear if the step transforming $\Gamma \nvdash \Delta$ into $\Gamma' \vdash \Delta'$ is well defined, in the sense that we do not know if the problem of provability of both $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ is decidable.

Finally, let us note that while the phenomenon of interpolation is usually treated as a property of an axiomatic theory or a logical system, because even some natural propositional logics do not possess it (see [15]), extrapolation, although observed as a dual to interpolation, presents essentially a method of transforming an unprovable statement $A \vdash B$ into a 'similar', but provable one: $A' \vdash B'$.

We also point out that if there is a grain of suspicion that a counterexample to our conjecture exists, it will be of great didactic importance in developing and stirring the critical reasoning of students and researchers. This has to find a central place in all study programs as a basic goal of education, together with stimulating creative thinking.

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References

1. Lakatos, I. Proofs and Refutations. *Br. J. Philos. Sci.* **1963**, *14*, 1–25. 120–139. 221–245. 296–342. [CrossRef]
2. Popper, K.R. *Conjectures and Refutations: The Growth of Scientific Knowledge*; Basic Books: New York, NY, USA, 1962.
3. Gabbay, D.M.; Woods, J. *The Reach of Abduction*; Elsevier: Amsterdam, The Netherlands, 2005.
4. Boričić, B. Model, proving and refuting. In *Quantitative Models in Economics*; Kočović, J., Eds.; Faculty of Economics, University of Belgrade: Belgrade, Serbia, 2018; pp. 3–19.
5. Craig, W. Three uses of the Herbrand–Gentzen theorem in relating model theory and proof theory. *J. Symb. Log.* **1957**, *22*, 269–285. [CrossRef]
6. Birkhoff, G.D.; Beatley, R. *Basic Geometry*; AMS Chelsea Publishing: Providence, RI, USA, 1959.
7. Boričić, B. Logical and historical determination of impossibility theorems by Arrow and Sen. *Econ. Ann.* **2007**, *52*, 7–20. [CrossRef]
8. Boričić, B. Dictatorship, liberalism and the Pareto rule: possible and impossible. *Econ. Ann.* **2009**, *54*, 45–54. [CrossRef]
9. Arrow, K. *Social Choice and Individual Values*; John Wiley: New York, NY, USA, 1963.
10. Sen, A. Quasi–Transitivity, Rational Choice and Collective Decisions. *Rev. Econ. Stud.* **1969**, *36*, 381–393. [CrossRef]
11. Anderson, A.R.; Belnap, N.D., Jr. *Entailment: The Logic of Relevance and Necessity*; Princeton University Press: Princeton, NJ, USA, 1975; Volume I.
12. Chang, C.C.; Keisler, H.J. *Model Theory*; North-Holland Publishing: Amsterdam, The Netherlands, 1973.
13. Takeuti, G. *Proof Theory*; North-Holland Publishing: Amsterdam, The Netherlands, 1975.
14. Anderson, A.R.; Belnap, N.D., Jr. Enthymemes. *J. Philos.* **1961**, *58*, 713–723. [CrossRef]
15. Maksimova, L.L. Interpolation properties of superintuitionistic logics. *Stud. Log.* **1979**, *38*, 419–428. [CrossRef]