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# Fixed Point Theory for Multi-Valued Feng–Liu–Subrahmanyan Contractions

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**Abstract:** In this paper, we consider several problems related to the so-called multi-valued Feng–Liu–Subrahmanyan contractions in complete metric spaces. Existence of the fixed points and of the strict fixed points, as well as data dependence and stability properties for the fixed point problem, are discussed. Some results are presented, under appropriate conditions, and some open questions are pointed out. Our results extend recent results given for multi-valued graph contractions and multi-valued Subrahmanyan contractions.

**Keywords:** complete metric space; fixed point; data dependence; Ulam–Hyers stability; Ostrowski stability

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## 1. Introduction and Preliminaries

Let $(M, d)$ be a metric space. We denote by $P(M)$ the set of all nonempty subsets of $M$, by $P_{cl}(M)$ the set of all nonempty closed subsets of $M$, and by $P_{cl,b}(M)$ the set of all nonempty closed and bounded subsets of $M$.

The following notations are used throughout this paper:

(1) The distance between a point $m \in M$ and a set $A \in P(M)$:

$$D(m, A) := \inf\{d(m, a) \mid a \in A\}.$$

(2) The excess of $A$ over $B$, where $A, B \in P(M)$:

$$e(A, B) := \sup\{D(a, B) \mid a \in A\}.$$

(3) The Hausdorff–Pompeiu distance between the sets $A, B \in P(M)$:

$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

Notice that $H$ is a generalized metric (in the sense that it takes values in $\mathbb{R} \cup \{+\infty\}$) on $P_{cl}(M)$ and it is a metric on $P_{cl,b}(M)$.

Let $(M, d)$ be a metric space and $S : M \to P(M)$ be a multi-valued operator with nonempty values. A fixed point of $S$ is an element $m^* \in M$ such that $m^* \in S(m^*)$. A strict fixed point of $S$ is an element $m^* \in M$ such that $S(m^*) = \{m^*\}$. We denote by $Fix(S)$ the fixed point set of $S$ and by $SFix(S)$ the set of all strict fixed points of $S$. By $Graph(S) := \{(u, v) \mid v \in S(u)\}$, we denote the graph of $S$.

A multi-valued operator $S : M \to P(M)$ is said to be a multi-valued $K$-contraction if $K \in [0,1[$ and the following relation holds:

$$H(S(u), S(v)) \leq Kd(u,v), \text{ for all } (u,v) \in M \times M.$$

The main fixed point result for multi-valued contractions was given by Nadler in 1969; see [1]. The result was slightly improved in 1970 by Covitz and Nadler (see [2]), and it is known as the multi-valued contraction principle. It says that any multi-valued contraction on a complete metric space has at least one fixed point.

In the same context, $S$ is called a multi-valued graph contraction with constant $K$ if

$$H(S(u), S(v)) \leq Kd(u,v), \text{ for all } (u,v) \in Graph(S).$$

For the main fixed point result concerning multi-valued graph contractions, see [3].

Fixed point theorems for multi-valued (graph) contractions are important tools in various applications, from integral and differential inclusions to optimization and fractal theory. Moreover, strict fixed point theorems are important in the theory of generalized games (abstract economies), as well as in the study of the convergence, to the fixed point, of various iterative schemes (see [4–7]).

For fixed point results for multi-valued contractions and multi-valued graph contractions, see [3,8–10] and the references therein.

The following concept was introduced by Feng and Liu in [11].

**Definition 1.** *Let $(M,d)$ be a metric space, $S : M \to P(M)$ be a multi-valued operator, $b \in ]0,1[$, and $u \in M$. Consider the set*

$$I_b^u := \{v \in S(u) : bd(u,v) \leq D(u, S(u))\}.$$

*Then, $S$ is called a multi-valued $K$-contraction of Feng–Liu type if $K \in ]0,1[$ such that for each $u \in M$ there is $v \in I_b^u$ with the property:*

$$D(v, S(v)) \leq Kd(u,v).$$

It is easy to see that any multi-valued $K$-contraction is a multi-valued $K$-contraction of Feng–Liu type, but not reversely (for examples, see Remark 1 in the paper [11]). For fixed-point-results-related Feng–Liu operators, see [11–17].

The following definition was introduced in [18]. Some fixed point results for this class of multi-valued operators are given in the same paper. For the single-valued case, see [19,20].

**Definition 2.** *Let $(M,d)$ be a metric space and $S : M \to P(M)$ be a multi-valued operator with nonempty values. Then, $S$ is said to be a multi-valued Subrahmanyan contraction if there exists a function $\psi : M \to [0,1[$ such that*

*(i) $H(S(u), S(v)) \leq \psi(u)d(u,v)$, for all $(u,v) \in Graph(S)$;*

*(ii) $\psi(v) \leq \psi(u)$, for every $(u,v) \in Graph(S)$.*

In this paper, we introduce a new class of multi-valued contraction type operators by combining the above two conditions: the multi-valued contraction condition of Feng–Liu type and the multi-valued Subrahmanyan contraction. As a consequence, we present existence and stability results for the fixed point inclusion $m \in S(m), m \in M$, where $(M,d)$ is a complete metric space and $S : M \to P(M)$ is a multi-valued Feng–Liu–Subrahmanyan contraction. The strict fixed point problem is also considered and some open questions are pointed out. Our results extend recent results given for multi-valued graph contractions and multi-valued Subrahmanyan contractions.

## 2. Main Results

Let $(M, d)$ be a metric space and $S : M \to P(M)$ be a multi-valued operator. For each $(m_0, m_1) \in Graph(S)$, the sequence $\{m_n\}_{n\in\mathbb{N}}$ with the property $m_{n+1} \in S(m_n),\ n \in \mathbb{N}$ is called the sequence of successive approximations for $S$ starting from $(m_0, m_1)$. We recall now the notion of multi-valued weakly Picard operator.

**Definition 3** ([21]). *Let $(M, d)$ be a metric space. Then, $S : M \to P(M)$ is called a multivalued weakly Picard operator if for each $u \in M$ and each $v \in S(u)$ there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ in $M$ such that*

*(i)* $m_0 = u,\ m_1 = v$;
*(ii)* $m_{n+1} \in S(m_n)$, *for all* $n \in \mathbb{N}$;
*(iii)* $\{m_n\}_{n\in\mathbb{N}}$ *is convergent in* $(M, d)$ *and its limit* $m^*(u, v)$ *is a fixed point of* $S$.

Let us recall the following important notions.

**Definition 4.** *Let $(M, d)$ be a metric space and $S : M \to P(M)$ be a multi-valued weakly Picard operator. Let us consider the multi-valued operator $S^\infty : Graph(S) \to P(Fix(S))$ defined by $S^\infty(u, v) := \{m^* \in Fix(S) \mid$ there is a sequence of successive approximations of $S$ starting from $(u, v)$ convergent to $m^*\}$. Then, $S$ satisfies the local retraction–displacement condition if there exists a selection $s^\infty$ of $S^\infty$ such that*

$$d(u, s^\infty(u, v)) \le C(u, v)d(u, v),\ \text{for all } (u, v) \in Graph(S),$$

*for some $C(u, v) > 0$.*

*When $C$ is independent of $u$ and $v$, then we say that $S$ satisfies the retraction–displacement condition.*

A similar concept is given now in our next definition.

**Definition 5.** *Let $(M, d)$ be a metric space and let $S : M \to P(M)$ be a multi-valued operator such that $Fix(S) \neq \emptyset$. Then, we say that $S$ satisfies the local strong retraction–displacement condition if there exists a set retraction $r : M \to Fix(S)$ such that*

$$d(m, r(m)) \le C(m)D(m, S(m)),\ \text{for all } m \in M, \tag{1}$$

*for some $C(m) > 0$.*

For related notions, examples, and results, see [1,18,21–24].

We now define the central concept of this paper, i.e., a multi-valued Feng–Liu–Subrahmanyan contraction on a metric space.

**Definition 6.** *Let $(M, d)$ be a metric space, $S : M \to P(M)$ be a multi-valued operator, $b \in ]0, 1[$, and $m \in M$. Consider the set*

$$I_b^u := \{v \in S(u) \mid bd(u, v) \le D(u, S(u))\}.$$

*Then, by definition, $S$ is a multi-valued Feng–Liu–Subrahmanyan contraction if there exists $\psi : M \to [0, b[$ such that for each $u \in M$ there is $v \in I_b^u$ with*

*(i)* $D(v, S(v)) \le \psi(u)d(u, v)$, *for all* $(u, v) \in Graph(S)$;
*(ii)* $\psi(v) \le \psi(u)$, *for every* $(u, v) \in Graph(S)$.

It is obvious that any multi-valued Subrahmanyan contraction is a multi-valued Feng–Liu–Subrahmanyan contraction, but the reverse implication, in general, does not hold.

Our first main result is a fixed point theorem for a multi-valued Feng–Liu–Subrahmanyan contractions with closed graph.

**Theorem 1.** *Let $(M, d)$ be a complete metric space, and consider a multi-valued Feng–Liu–Subrahmanyan contraction $S : M \to P(M)$ with closed graph. Then, the following conclusions hold:*

*(a) $Fix(S) \neq \emptyset$;*

*(b) For every $u \in M$ there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ of successive approximations for $S$ starting at $m_0 = u$ which converges to a fixed point $m^*(u)$ of $S$, and the following apriori estimation holds:*

$$d(m_n, m^*(u)) \leq \left(\frac{\psi(u)}{b}\right)^n \frac{1}{b - \psi(u)} D(u, S(u)), \text{ for every } n \in \mathbb{N}.$$

*(c) The following local strong retraction–displacement type condition holds:*

$$d(u, m^*(u)) \leq \frac{1}{b - \psi(u)} D(u, S(u)), \text{ for all } u \in M.$$

**Proof.** Let $m_0 = u \in M$ be arbitrary and $b \in ]0, 1[$. Then, since $S(u) \in P_{cl}(M)$, the set $I_b^u$ is nonempty, for each $u \in M$. By Definition 6, there exist $\psi : M \to [0, b[$ and $m_1 \in I_b^u$ (i.e., $bd(u, m_1) \leq D(u, S(u))$) such that

(i) $D(m_1, S(m_1)) \leq \psi(m_0)d(m_0, m_1)$;

(ii) $\psi(m_1) \leq \psi(m_0)$.

In a similar way, there exists $m_2 \in I_b^{m_1}$ (i.e., $bd(m_1, m_2) \leq D(m_1, S(m_1))$) such that

(i) $D(m_2, S(m_2)) \leq \psi(m_1)d(m_1, m_2)$;

(ii) $\psi(m_2) \leq \psi(m_1)$.

Hence, we have

$$d(m_1, m_2) \leq \frac{1}{b} D(m_1, S(m_1)) \leq \frac{\psi(m_0)}{b} d(m_0, m_1)$$

and

$$D(m_2, S(m_2)) \leq \psi(m_1)d(m_1, m_2) \leq \frac{\psi(m_1)}{b} D(m_1, S(m_1)) \leq$$

$$\frac{\psi(m_1)}{b}\psi(m_0)d(m_0, m_1) \leq \frac{\psi(m_1)\psi(m_0)}{b^2} D(u, S(u)) \leq \left(\frac{\psi(u)}{b}\right)^2 D(u, S(u)).$$

In the next step, there exists $m_3 \in I_b^{m_2}$ (i.e., $bd(m_2, m_3) \leq D(m_2, S(m_2))$) such that

(i) $D(m_3, S(m_3)) \leq \psi(m_2)d(m_2, m_3)$;

(ii) $\psi(m_3) \leq \psi(m_2)$.

Hence, in this case, we have

$$d(m_2, m_3) \leq \frac{1}{b} D(m_2, S(m_2)) \leq \frac{\psi(m_1)}{b} d(m_1, m_2) \leq$$

$$\frac{\psi(m_1)}{b}\frac{\psi(m_0)}{b} d(m_0, m_1) \leq \left(\frac{\psi(m_0)}{b}\right)^2 d(m_0, m_1)$$

and

$$D(m_3, S(m_3)) \leq \psi(m_2)d(m_2, m_3) \leq \frac{\psi(m_2)}{b} D(m_2, S(m_2)) \leq$$

$$\frac{\psi(m_2)}{b}\frac{\psi(m_1)}{b} D(m_1, S(m_1)) \leq \left(\frac{\psi(m_1)}{b}\right)^2 D(m_1, S(m_1)) \leq \left(\frac{\psi(m_0)}{b}\right)^3 D(u, S(u)).$$

Inductively, there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ such that

(i) $D(m_{n+1}, S(m_{n+1})) \leq \psi(m_n)d(m_n, m_{n+1})$;

(ii) $\psi(m_{n+1}) \leq \psi(m_n)$;

(iii) $m_{n+1} \in I_b^{m_n}$, for $n \in \mathbb{N}$, i.e., $bd(m_n, m_{n+1}) \leq D(m_n, S(m_n))$.

Hence, we have

$$d(m_n, m_{n+1}) \leq \frac{1}{b} D(m_n, S(m_n)) \leq \frac{\psi(m_{n-1})}{b} d(m_{n-1}, m_n) \leq \cdots \leq \left(\frac{\psi(m_0)}{b}\right)^n d(m_0, m_1)$$

and

$$D(m_{n+1}, S(m_{n+1})) \leq \psi(m_n)d(m_n, m_{n+1}) \leq \frac{\psi(m_n)}{b}D(m_n, S(m_n)) \leq \cdots \leq$$

$$\left(\frac{\psi(m_1)}{b}\right)^n D(m_1, S(m_1)) \leq \left(\frac{\psi(m_0)}{b}\right)^{n+1} D(u, S(u)).$$

In order to show that the sequence $\{m_n\}_{n\in\mathbb{N}}$ is Cauchy, we can estimate

$$d(m_n, m_{n+p}) \leq d(m_n, m_{n+1}) + \cdots + d(m_{n+p-1}, m_{n+p}) \leq$$

$$\left(\frac{\psi(m_0)}{b}\right)^n d(m_0, m_1) + \cdots + \left(\frac{\psi(m_0)}{b}\right)^{n+p-1} d(m_0, m_1) =$$

$$\left(\frac{\psi(m_0)}{b}\right)^n \left[1 + \frac{\psi(m_0)}{b} + \cdots + \left(\frac{\psi(m_0)}{b}\right)^{p-1}\right] d(m_0, m_1) \leq$$

$$\left(\frac{\psi(m_0)}{b}\right)^n \frac{b}{b - \psi(m_0)} d(m_0, m_1) \to 0 \text{ as } n, p \to \infty.$$

We also observe that

$$d(m_n, m_{n+p}) \leq \left(\frac{\psi(m_0)}{b}\right)^n \frac{b}{b - \psi(m_0)} d(m_0, m_1), \text{ for each } n, p \in \mathbb{N}^*. \tag{2}$$

It follows that $\{m_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $(M, d)$ and, thus, there exists $m^*(u) \in M$ such that $\{m_n\}_{n\in\mathbb{N}}$ is convergent to $m^*(u) \in M$. From the condition that $S$ has a closed graph, we deduce that $m^*(u)$ is a fixed point for $S$.

In addition, for $p \to +\infty$ in (2), we have

$$\begin{aligned} d(m_n, m^*(u)) \quad &\leq \quad \left(\frac{\psi(u)}{b}\right)^n \frac{b}{b - \psi(u)} d(u, m_1) \leq \\ &\left(\frac{\psi(u)}{b}\right)^n \frac{1}{b - \psi(u)} D(u, S(u)), \text{ for every } n \in \mathbb{N}. \end{aligned} \tag{3}$$

Taking $n = 0$ in (3), it follows that $d(u, m^*(u)) \leq \frac{1}{b-\psi(u)} D(u, S(u)),$ for all $u \in M$. □

**Example 1.** *Let $S : M := \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by*

$$S(u, v) = \begin{cases} \{(u, \frac{v+|u|+v|v-|u||}{2+|v-|u||})\}, \ (u, v) \in M, v = |u| \\ \{(0, 1), (0, -1)\}, (u, v) \in M, v \neq |u|. \end{cases} \tag{4}$$

*Then, $S$ is a multi-valued Feng–Liu–Subrahmanyan contraction with $\psi(u, v) := \frac{(v-|u|)^2+3|v-|u||+2}{(v-|u|)^2+3|v-|u||+4}$. Notice that $Fix(S) = \{(u, v) \in M : v = |u|\}$ and $S$ is not a multi-valued Feng–Liu operator since $\sup_{(u,v)\in M} \psi(u, v) = 1$.*

We recall now some stability concepts for the fixed point inclusion $m \in S(m)$.

**Definition 7.** *Let $(M, d)$ be a metric space and $S : M \to P(M)$ be a multi-valued operator. We say that the fixed point inclusion*

$$m \in S(m), \ m \in M \tag{5}$$

*is local Ulam–Hyers stable if for any $\varepsilon > 0$ and any $\varepsilon$-solution $u$ of the fixed point inclusion (5) (i.e., $u \in M$ with the property $D(u, S(u)) \leq \varepsilon$), there exist $C = C(u) > 0$ and $m^* = m^*(u) \in Fix(S)$ with*

$$d(u, m^*) \leq C\varepsilon.$$

*If C does not depend on u, then we say that the fixed point inclusion is Ulam–Hyers stable (see [25] for related results).*

A local data dependence property is given in our next definition.

**Definition 8.** *Let $(M, d)$ be a metric space and $S : M \to P(M)$ be a multi-valued operator. By definition, the fixed point inclusion*

$$m \in S(m), m \in M$$

*has the local data dependence property if, for any multi-valued operator, $T : M \to P(M)$, satisfying:*

*(i) $Fix(T) \neq \emptyset$;*

*(ii) There exists $\eta > 0$ such that $H(S(m), T(m)) \leq \eta$, for all $m \in M$,the following implication holds: for each $u \in Fix(T)$ there exist $C = C(u) > 0$ and $m^* = m^*(u) \in Fix(S)$ such that $d(u, m^*) \leq C(u)\eta$.*

The well-posedness of the fixed point inclusion $m \in S(m)$ is defined as follows (see [26,27]):

**Definition 9.** *Let $(M, d)$ be a metric space and let $S : M \to P(M)$ be a multi-valued operator such that $Fix(S) \neq \emptyset$. Suppose there exists $r : M \to Fix(S)$, a set retraction. Then, the fixed point inclusion $m \in S(m)$ is called well-posed in the sense of Reich and Zaslavski if for each $v \in Fix(S)$ and for any sequence $\{v_n\}_{n \in \mathbb{N}} \subset r^{-1}(v)$ such that*

$$D(v_n, S(u_n)) \to 0 \text{ as } n \to \infty,$$

*we have that*

$$v_n \to v \text{ as } n \to \infty.$$

Finally, we recall the notion of Ostrowski stability property for a fixed point inclusion (see [23]).

**Definition 10.** *Let $(M, d)$ be a metric space and let $S : M \to P(M)$ be a multi-valued operator such that $Fix(S) \neq \emptyset$. Suppose there exists $r : M \to Fix(S)$, a set retraction. Then, the fixed point inclusion $m \in S(m)$ is said to have the Ostrowski stability property if for each $m^* \in Fix(S)$ and for any sequence $\{w_n\}_{n \in \mathbb{N}} \subset r^{-1}(m^*)$ such that:*

$$D(w_{n+1}, S(w_n)) \to 0 \text{ as } n \to \infty,$$

*we have that*

$$w_n \to m^* \text{ as } n \to \infty.$$

Two abstract results concerning some stability properties of a multi-valued operator are given in our next results.

**Theorem 2.** *Let $(M, d)$ be a metric space and let $S : M \to P(M)$ be a multi-valued operator satisfying the local strong retraction–displacement condition such that $Fix(S) \neq \emptyset$. Then, the fixed point inclusion $m \in S(m)$ has the local Ulam–Hyers stability property and satisfies the local data dependence property.*

**Proof.** Suppose there exists a set retraction $r : M \to Fix(S)$ such that

$$d(m, r(m)) \leq C(m)D(m, S(m)), \text{ for all } m \in M, \tag{6}$$

for some $C(m) > 0$.

Let $\varepsilon > 0$ and $u \in M$ with the property $D(u, S(u)) \leq \varepsilon$. Then, by (6), there exists $C = C(u) > 0$ such that

$$d(u, r(u)) \leq C(u)D(u, S(u)) \leq C(u)\varepsilon.$$

Thus, $m^*(u) = r(u) \in Fix(S)$ and the local Ulam–Hyers stability property is established.

For the local data dependence property, let us consider any multi-valued operator $T : M \to P(M)$ such that $Fix(T) \neq \emptyset$ and for which there exists $\eta > 0$ such that $H(S(m), T(m)) \leq \eta$, for all $m \in M$. Take $t \in Fix(T)$. Then, by (6), there exists $C = C(t) > 0$ such that

$$d(t, r(t)) \leq C(t)D(t, S(t)) \leq C(t)\eta.$$

Since $r(t) \in Fix(S)$, the local data dependence property is proven. □

By the above abstract result, we immediately obtain the following stability properties for multi-valued Feng–Liu–Subrahmanyan contractions.

**Theorem 3.** *Let $(M, d)$ be a complete metric space and $S : M \to P(M)$ be a multi-valued Feng–Liu–Subrahmanyan contraction with closed graph. Then, the fixed point inclusion (5) is local Ulam–Hyers stable and satisfies the local data dependence property.*

**Proof.** By Theorem 1, we know that $Fix(S) \neq \emptyset$ (conclusion (a)) and $S$ satisfies the local strong retraction–displacement condition (see conclusion (c)). The result follows by Theorem 2. □

**Example 2.** *Let $S : M := \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by*

$$S(u, v) = \begin{cases} \{(u, \frac{v+|u|+v|v-|u||}{2+|v-|u||})\}, \ (u, v) \in M, v = |u| \\ \{(0, 1), (0, -1)\}, (u, v) \in M, v \neq |u|. \end{cases} \tag{7}$$

*Then, $S$ is a multi-valued Feng–Liu–Subrahmanyan contraction with closed graph and $Fix(S) = \{(u, v) \in M : v = |u|\}$. By Theorem 2 and Theorem 3, the fixed point inclusion $m \in S(m)$ is local Ulam–Hyers stable and satisfies the local data dependence property.*

**Remark 1.** *It is an open question to obtain the well-posedness property in the sense of Reich and Zaslavski and the Ostrowski stability property for the fixed point inclusion $m \in S(m), m \in M$ for a multi-valued Feng–Liu–Subrahmanyan contraction with closed graph defined on a complete metric space $(M, d)$. For example, if $\psi$ has the following property:*

$$(P) \text{ there exists } q > 0 \text{ such that } \psi(m) \leq b - q, \text{ for all } m \in M,$$

*then, under the assumption given in Theorem 1, the fixed point inclusion $m \in S(m)$ has the well-posedness property in the sense of Reich and Zaslavski. Indeed, by Theorem 1 (a,b) we know that $Fix(S) \neq \emptyset$ and there exists a retraction $r : M \to Fix(S)$ given by $r(u) := \{m^*(u) :$ and there exists a sequence of successive approximations starting from $u$ converging to $m^*(u)\}$.*

*If we take $v \in Fix(S)$ and any sequence $\{v_n\}_{n\in\mathbb{N}} \subset r^{-1}(v)$ such that*

$$D(v_n, S(u_n)) \to 0 \text{ as } n \to \infty,$$

*then, by Theorem 1 (c), we have that*

$$d(v_n, v) \leq \frac{1}{b - \psi(v_n)} D(v_n, S(v_n)) \leq \frac{1}{q} D(v_n, S(v_n)) \to 0, \text{ as } n \to \infty.$$

Another open question is to obtain strict fixed point theorems for multi-valued Feng–Liu–Subrahmanyan contractions with closed graph defined on a complete metric space $(M, d)$. For example, we have the following strict fixed point result for multi-valued Feng–Liu–Subrahmanyan contractions, which generalize some theorems in [18,28]. As a matter of fact, the conclusion of the next theorem is $Fix(S) = SFix(S) \neq \emptyset$, which is a quite a usual assumption in various iteration methods for multi-valued operators.

**Theorem 4.** *Let $(M, d)$ be a complete metric space and $S : M \to P(M)$ be a multi-valued Feng–Liu–Subrahmanyan contraction with closed graph. Suppose that*

*(a) $S(S(m)) \subset S(m)$, for each $m \in M$;*

*(b) If $A \in P_{cl}(M)$ with $S(A) = A$, then $A$ is a singleton.*

*Then, $Fix(S) = SFix(S) \neq \emptyset$.*

**Proof.** By Theorem 1, we have that $Fix(S) \neq \emptyset$. Let $m^* \in Fix(S)$. By the assumption (a) of this theorem, we obtain that $S(m^*) \subset S(S(m^*)) \subset S(m^*)$. Thus, $S(S(m^*)) = S(m^*)$, i.e., $S(m^*)$ is a fixed set for $S$. By the assumption (b), we obtain that $S(m^*)$ is a singleton. Hence, $S(m^*) = \{m^*\}$. We also observe that $Fix(S) \subset SFix(S)$. Thus, $Fix(S) = SFix(S) \neq \emptyset$. □

## 3. Conclusions

In this work, we introduced, in the context of a metric space $(M, d)$, the class of multi-valued Feng–Liu–Subrahmanyan contractions, and we presented a fixed point theory for these kind of multi-valued operators. More precisely, if $S : M \to P(M)$ is a multi-valued Feng–Liu–Subrahmanyan contraction, we proved the following:

- An existence and approximation result for the fixed point inclusion $m \in S(m), m \in M$;
- An existence result for the strict fixed point problem $S(m) = \{m\}, m \in M$;
- The Ulam–Hyers stability property for the fixed point inclusion $m \in S(m), m \in M$;
- The data dependence property for the solution of the fixed point inclusion $m \in S(m), m \in M$;
- A partial answer for the well-posedness property in the sense of Reich and Zaslavski for the fixed point inclusion $m \in S(m), m \in M$.

Two open questions concerning the well-posedness property and the existence of the strict fixed points for multi-valued Feng–Liu–Subrahmanyan contractions are highlighted.

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