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New Robust Estimators for Handling Multicollinearity and Outliers in the Poisson Model: Methods, Simulation and Applications

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Abstract: The Poisson maximum likelihood (PML) is used to estimate the coefficients of the Poisson regression model (PRM). Since the resulting estimators are sensitive to outliers, different studies have provided robust Poisson regression estimators to alleviate this problem. Additionally, the PML estimator is sensitive to multicollinearity. Therefore, several biased Poisson estimators have been provided to cope with this problem, such as the Poisson ridge estimator, Poisson Liu estimator, Poisson Kibria–Lukman estimator, and Poisson modified Kibria–Lukman estimator. Despite different Poisson biased regression estimators being proposed, there has been no analysis of the robust version of these estimators to deal with the two above-mentioned problems simultaneously, except for the robust Poisson ridge regression estimator, which we have extended by proposing three new robust Poisson one-parameter regression estimators, namely, the robust Poisson Liu (RPL), the robust Poisson Kibria–Lukman (RPKL), and the robust Poisson modified Kibria–Lukman (RPMKL). Theoretical comparisons and Monte Carlo simulations were conducted to show the proposed performance compared with the other estimators. The simulation results indicated that the proposed RPL, RPKL, and RPMKL estimators outperformed the other estimators in different scenarios, in cases where both problems existed. Finally, we analyzed two real datasets to confirm the results.

Keywords: robust Poisson ridge regression estimator; robust Poisson Liu estimator; robust Poisson Kibria–Lukman estimator; robust Poisson modified Kibria–Lukman estimator; mean squared error; multicollinearity; Monte Carlo simulations; outliers

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1. Introduction

In general, outliers are often data points that differ significantly from other data points. In other words, they represent uncommon dataset values. For many statistical models, outliers are an issue because they can either cause tests to miss important findings, or bias real results. There are some statistical methods used to detect outliers, such as index plot of Cook's distance and potential residual plot, see Hadi [1]. Several robust regression estimators have been provided to address the problem of outliers in different regression models, see Lawrence [2]. On the other hand, when the explanatory variables are highly correlated, this means that the model has a multicollinearity problem. This problem causes the maximum likelihood estimates to be unstable and inefficient. Therefore, many biased estimators have been provided to address this problem in different regression models, see e.g., Wu et al. [3], Arashi et al. [4], Algamal et al. [5], Rasheed et al. [6], Majid et al. [7], Asar and Algamal [8], and Jabur et al. [9].

The Poisson regression model (PRM) is popular for modeling and analyzing count data. The Poisson maximum likelihood (PML) method is used to obtain the estimators for the PRMs based on the algorithm of the iterative weighted least square (IWLS) technique. However, the presence of the multicollinearity problem in the PRM causes the PML estimators to be unreliable and have unexpected signs. To mitigate this, biased estimators have been suggested for the PRM, such as the Poisson ridge regression (PRR) estimator [10], Poisson Liu (PL) estimator [11], Poisson Kibria–Lukman (PKL) estimator [12], and Poisson modified Kibria–Lukman (PMKL) estimator [13]. These estimators' properties have been investigated via the mean squared error (MSE) criterion. Additionally, the authors have modified and adopted some approaches for estimating the biasing parameters, which have been described in many studies. For more details see: Månsson and Shukur [10], Månsson et al. [11], Amin et al. [14], Yehia [15], Lukman et al. [12], and Jadhav [16].

The PRM is a popular model for analyzing count data when the dependent variable (y_i) distribution is $Poisson(\mu_i)$ and $\mu_i(\beta) = \exp(x_i\beta)$, where x_i is the ith row in X matrix (that is an $n \times (p+1)$ with p explanatory variables), and β is an unknown $(p+1) \times 1$ coefficients vector, and the model is assumed to have an intercept. Then, the PML estimator is provided via the IWLS algorithm as follows:

$$\hat{\beta}_{PML} = \left(X'\hat{W}X\right)^{-1}X'\hat{W}\hat{z}, \tag{1}$$

where $\hat{W} = diag(\hat{\mu}_i)$, and $\hat{z}$ is a vector where the ith element equals $\hat{z}_i = \log\left[\hat{\mu}_i\left(\hat{\beta}_{PML}\right)\right] + \left[\frac{y_i - \hat{\mu}_i\left(\hat{\beta}_{PML}\right)}{\hat{\mu}_i\left(\hat{\beta}_{PML}\right)}\right]$. The matrix MSE (MSEM) of the PML estimator is given by:

$$MSEM\left(\hat{\beta}_{PML}\right) = \left(X'\hat{W}X\right)^{-1}. \tag{2}$$

Suppose that γ and λ_j are eigenvectors and eigenvalues of $X'\hat{W}X$ matrix, where $X'\hat{W}X = \gamma'\Lambda\gamma$, with $\Lambda = diag(\lambda_j)$. Hence, the scalar MSE of PML estimator is given by:

$$MSE\left(\hat{\beta}_{PML}\right) = trac\left(X'\hat{W}X\right)^{-1} = \sum_{j=1}^{p+1}\frac{1}{\lambda_j}. \tag{3}$$

The PRM assumes that the mean and variance of the outcome (y) are equal. However, data sets in many real-world applications contain outliers causing the variance of the outcome to be larger than the mean of the outcome, which is known as the problem of overdispersion. It is proposed here that outliers in count data can be more adequately addressed using a robust Poisson estimator, in comparison with the usual PML estimator. A few robust Poisson estimators have been provided, such as by Cantoni and Ronchetti [17], Hall and Shen [18], Chen et al. [19], Hosseinian and Morgenthaler [20], Chen et al. [21], Abonazel and Saber [22], Marazzi [23], and Abonazel et al. [24]. However, the robust Poisson estimator that was proposed by Hosseinian and Morgenthaler [20] is the most common estimator given via the weighted maximum likelihood method (WMLM) for handling outliers in y-direction [22]. The weighted function (W^*) of the WMLM depends on μ_i and two constants (c_1 and c_2):

$$W^* = \begin{cases} 1 & if\, \frac{m}{c_1} < \mu_i < c_1 m; \\ \frac{c_1\mu_i}{m} & if\, \mu_i < \frac{m}{c_1}; \\ \frac{c_2 m - \mu_i}{m} & if\, c_1 m < \mu_i < c_2 m; \\ 0 & otherwise \end{cases}$$

where m is known as the median of μ_i, c_1 = 2, and c_2 = 3. Then, the WMLM estimation equation is defined as:

$$\sum_{i=1}^{n}\frac{\mu_i'}{\mu_i}W^*(y_i - \mu_i)x_i = 0, \tag{4}$$

By solving the equations system in Equation (4) based on the Newton–Raphson method, we obtain a robust PML estimator (RPML), see Hosseinian [25] and Hosseinian and Morgenthaler [20] for more details. Then, the RPML estimator [26] is:

$$\hat{\beta}_{RPML} = \left(X'\hat{W}^*X\right)^{-1}X'\hat{W}^*\hat{z}^*, \tag{5}$$

where $\hat{W}^* = diag\left[\hat{\mu}_i\left(\hat{\beta}_{RPML}\right)\right]$, and $\hat{z}^*$ is a vector where the ith element equals $\hat{z}_i^* = \log\left[\hat{\mu}_i\left(\hat{\beta}_{RPML}\right)\right] + \left[\frac{y_i - \hat{\mu}_i\left(\hat{\beta}_{RPML}\right)}{\hat{\mu}_i\left(\hat{\beta}_{RPML}\right)}\right]$.

The MSEM and MSE of the RPML estimator are given by:

$$MSEM\left(\hat{\beta}_{RPML}\right) = \left(X'\hat{W}^*X\right)^{-1}, \tag{6}$$

$$MSE\left(\hat{\beta}_{RPML}\right) = \sum_{j=1}^{p+1}\frac{1}{\lambda_j^*}, \tag{7}$$

where γ^* and λ_j^* are the eigenvectors and eigenvalues of $X'\hat{W}^*X$, where $X'\hat{W}^*X = \gamma^{*\prime}\Lambda^*\gamma^*$ with $\Lambda^* = diag\left(\lambda_j^*\right)$.

Although the RPML estimator is robust for outliers, it is not suitable in the case of the multicollinearity problem in the PRM. On the other hand, the PL, PKL and PMKL estimators are robust for the multicollinearity problem, but not suitable in the case of outliers. Therefore, in this paper, we will combine the RPML estimator that was introduced by Hosseinian and Morgenthaler [20], into the PL, PKL and PMKL estimators, to obtain new robust estimators. The suggested estimators can efficiently handle both problems simultaneously in the PRM.

This paper is organized as follows. Section 2 presents the three proposed robust one-parameter estimators, theoretical comparisons, and biasing parameters selection techniques. The Monte Carlo simulation and empirical applications are described in Sections 3 and 4, respectively. Section 5 provides some concluding remarks.

2. Methodology

2.1. Poisson One-Parameter Regression Estimators

As mentioned above, multicollinearity is a frequent problem that occurs among the explanatory variables in many datasets. This causes the PML estimator to be unstable and inefficient. Therefore, different authors have developed several biased Poisson regression estimators as a remedy to multicollinearity, e.g., Månsson and Shukur [10] proposed the PRR estimator, Månsson et al. [11] proposed the PL estimator, Lukman et al. [12] proposed the PKL estimator, and Aladeitan et al. [13] proposed the PMKL estimator, where these estimators are:

$$\hat{\beta}_{PRR} = \left(X'\hat{W}X + kI\right)^{-1}X'\hat{W}X\hat{\beta}_{PML} \tag{8}$$

$$\hat{\beta}_{PL} = \left(X'\hat{W}X + I\right)^{-1}\left(X'\hat{W}X + dI\right)\hat{\beta}_{PML} \tag{9}$$

$$\hat{\beta}_{PKL} = \left(X'\hat{W}X + kI\right)^{-1}\left(X'\hat{W}X - kI\right)\hat{\beta}_{PML} \tag{10}$$

$$\hat{\beta}_{PMKL} = \left(X'\hat{W}X + kI\right)^{-1}\left(X'\hat{W}X - kI\right)\left(X'\hat{W}X + kI\right)^{-1}X'\hat{W}X\hat{\beta}_{PML} \tag{11}$$

The MSE of the PRR, PL, PKL, and PMKL estimators are given by:

$$MSE\left(\hat{\beta}_{PRR}\right) = \sum_{j=1}^{p+1}\frac{\lambda_j}{\left(\lambda_j + k\right)^2} + \sum_{j=1}^{p+1}\frac{k^2\alpha_j^2}{\left(\lambda_j + k\right)^2}, \tag{12}$$

$$MSE\left(\hat{\beta}_{PL}\right) = \sum_{j=1}^{p+1}\frac{\left(\lambda_j + d\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} + (1-d)^2\sum_{j=1}^{p+1}\frac{\alpha_j^2}{\left(\lambda_j + 1\right)^2}, \tag{13}$$

$$MSE(\hat{\beta}_{PKL}) = \sum_{j=1}^{p+1} \frac{(\lambda_j - k)^2}{\lambda_j(\lambda_j + k)^2} + 4k^2 \sum_{j=1}^{p+1} \frac{\alpha_j^2}{(\lambda_j + k)^2}, \quad (14)$$

$$MSE(\hat{\beta}_{PMKL}) = \sum_{j=1}^{p+1} \frac{\lambda_j(\lambda_j - k)^2}{(\lambda_j + k)^4} + k^2 \sum_{j=1}^{p+1} \frac{(k + 3\lambda_j)^2\alpha_j^2}{(\lambda_j + k)^4}. \quad (15)$$

where α_j is the jth item of $\gamma'\hat{\beta}_{PML}$.

2.2. Robust Poisson Ridge Regression Estimator

Recently, Abonazel and Dawoud [26] proposed the robust Poisson ridge regression (RPRR) estimator to eliminate the effects of the outliers and multicollinearity problems simultaneously in the PRM, where the RPRR is given by:

$$\hat{\beta}_{RPRR} = (X'\hat{W}^*X + kI)^{-1}X'\hat{W}^*X\hat{\beta}_{RPML}, \quad (16)$$

and the MSE of the RPRR estimator is:

$$MSE(\hat{\beta}_{RPRR}) = \sum_{j=1}^{p+1} \frac{\lambda_j^*}{(\lambda_j^* + k)^2} + \sum_{j=1}^{p+1} \frac{k^2\alpha_j^{*2}}{(\lambda_j^* + k)^2}, \quad (17)$$

where α_j^* is the jth item of $\gamma^{*'}\hat{\beta}_{RPML}$.

To reduce the bias of the RPRR estimator, Abonazel and Dawoud [26] proposed the robust jackknife ridge estimator for the PRM. Furthermore, the robust modified jackknife ridge estimator for the PRM was provided by Arum et al. [27].

2.3. Proposed Robust Poisson One-Parameter Regression Estimators

Since the PL, PKL, and PMKL estimators are more efficient than the PRR estimator in the PRM, we propose a robust version of these estimators for the PRM as extensions of the one-parameter estimators to the area of the PRM in the presence of both multicollinearity and outliers. We can define the proposed estimators (RPL, RPKL, and RPMKL) as follows:

$$\hat{\beta}_{RPL} = (X'\hat{W}^*X + I)^{-1}(X'\hat{W}^*X + dI)\hat{\beta}_{RPML} \quad (18)$$

$$\hat{\beta}_{RPKL} = (X'\hat{W}^*X + kI)^{-1}(X'\hat{W}^*X - kI)\hat{\beta}_{RPML} \quad (19)$$

$$\hat{\beta}_{RPMKL} = (X'\hat{W}^*X + kI)^{-1}(X'\hat{W}^*X - kI)(X'\hat{W}^*X + kI)^{-1}X'\hat{W}^*X\hat{\beta}_{RPML} \quad (20)$$

The MSE of the RPL, RPKL, and RPMKL estimators are given by:

$$MSE(\hat{\beta}_{RPL}) = \sum_{j=1}^{p+1} \frac{(\lambda_j^* + d)^2}{\lambda_j^*(\lambda_j^* + 1)^2} + (1 - d)^2 \sum_{j=1}^{p+1} \frac{\alpha_j^{*2}}{(\lambda_j^* + 1)^2}, \quad (21)$$

$$MSE(\hat{\beta}_{RPKL}) = \sum_{j=1}^{p+1} \frac{(\lambda_j^* - k)^2}{\lambda_j^*(\lambda_j^* + k)^2} + 4k^2 \sum_{j=1}^{p+1} \frac{\alpha_j^{*2}}{(\lambda_j^* + k)^2}, \quad (22)$$

$$MSE(\hat{\beta}_{RPMKL}) = \sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^* - k)^2}{(\lambda_j^* + k)^4} + k^2 \sum_{j=1}^{p+1} \frac{(k + 3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^* + k)^4} \quad (23)$$

2.4. Superiority of Proposed Estimators

The theorems below specify the necessary and sufficient conditions of the proposed RPL, RPKL, and RPMKL estimators' superiority.

Theorem 1. *If* $\sum_{j=1}^{p+1} \frac{\left((1-d)\alpha_j^{*2}-2\right)}{(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1} \frac{(1+d)}{\lambda_j^*(\lambda_j^*+1)^2}$*, then* $MSE(\hat{\beta}_{RPL}) < MSE(\hat{\beta}_{RPML})$.

Proof. The MSE difference between the RPML and the RPL estimators is given by:

$$\begin{aligned}\Delta_1 &= MSE(\hat{\beta}_{RPL}) - MSE(\hat{\beta}_{RPML}) \\ &= \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2-(\lambda_j^*+1)^2+\lambda_j^*(1-d)^2\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}\end{aligned} \tag{24}$$

After using some algebraic calculations, Δ_1 in Equation (24) will be negative if $\sum_{j=1}^{p+1} \frac{\left((1-d)\alpha_j^{*2}-2\right)}{(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1} \frac{(1+d)}{\lambda_j^*(\lambda_j^*+1)^2}$, then $MSE(\hat{\beta}_{RPL}) < MSE(\hat{\beta}_{RPML})$. That means, the RPL estimator is better than the RPML estimator if $\sum_{j=1}^{p+1} \frac{\left((1-d)\alpha_j^{*2}-2\right)}{(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1} \frac{(1+d)}{\lambda_j^*(\lambda_j^*+1)^2}$. □

Theorem 2. *If* $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1} \frac{1}{(\lambda_j^*+k)^2}$*, then* $MSE(\hat{\beta}_{RPKL}) < MSE(\hat{\beta}_{RPML})$.

Proof. The MSE difference between the RPML and the RPKL estimators is given by:

$$\Delta_2 = MSE(\hat{\beta}_{RPKL}) - MSE(\hat{\beta}_{RPML}) = \sum_{j=1}^{p+1} \frac{(\lambda_j^* - k)^2 - (\lambda_j^* + k)^2 + \lambda_j^* 4k^2\alpha_j^{*2}}{\lambda_j^*(\lambda_j^* + k)^2}. \tag{25}$$

After using some algebraic calculations, Δ_2 in Equation (25) will be negative if $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1} \frac{1}{(\lambda_j^*+k)^2}$, then $MSE(\hat{\beta}_{RPKL}) < MSE(\hat{\beta}_{RPML})$. That means, the RPKL estimator is better than the RPML estimator if $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1} \frac{1}{(\lambda_j^*+k)^2}$. □

Theorem 3. *If* $\sum_{j=1}^{p+1} \frac{k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+k)^4-\lambda_j^{*2}(\lambda_j^*-k)^2}{\lambda_j^*(\lambda_j^*+k)^4}$*, then* $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPML})$.

Proof. The MSE difference between the RPML and the RPMKL estimators is given by:

$$\begin{aligned}\Delta_3 &= MSE(\hat{\beta}_{RPMKL}) - MSE(\hat{\beta}_{RPML}) \\ &= \sum_{j=1}^{p+1} \frac{\lambda_j^{*2}(\lambda_j^*-k)^2-(\lambda_j^*+k)^4+k^2\lambda_j^*(k+3\lambda_j^*)^2\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^4}.\end{aligned} \tag{26}$$

After using some algebraic calculations, Δ_3 in Equation (26) will be negative if $\sum_{j=1}^{p+1} \frac{k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+k)^4-\lambda_j^{*2}(\lambda_j^*-k)^2}{\lambda_j^*(\lambda_j^*+k)^4}$, then $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPML})$. That means, the RPMKL estimator is better than the RPML estimator if $\sum_{j=1}^{p+1} \frac{k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+k)^4-\lambda_j^{*2}(\lambda_j^*-k)^2}{\lambda_j^*(\lambda_j^*+k)^4}$. □

Theorem 4. *If* $\sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1} \frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2}$*, then* $MSE(\hat{\beta}_{RPL}) < MSE(\hat{\beta}_{RPRR})$.

Proof. The MSE difference between the RPRR and the RPL estimators is given by:

$$\begin{aligned}\Delta_4 &= MSE\left(\hat{\beta}_{RPL}\right) - MSE\left(\hat{\beta}_{RPRR}\right) \\ &= \sum_{j=1}^{p+1}\left[\frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} - \frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2}\right].\end{aligned} \tag{27}$$

After using some algebraic calculations, Δ_4 in Equation (27) will be negative if $\sum_{j=1}^{p+1}\frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1}\frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2}$, then $MSE\left(\hat{\beta}_{RPL}\right) < MSE\left(\hat{\beta}_{RPRR}\right)$. That means, the RPL estimator is better than the PRRR estimator if $\sum_{j=1}^{p+1}\frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} < \sum_{j=1}^{p+1}\frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2}$. □

Theorem 5. *If* $\sum_{j=1}^{p+1}\frac{3k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1}\frac{2\lambda_j^*-k}{\lambda_j^*(\lambda_j^*+k)^2}$, *then* $MSE\left(\hat{\beta}_{RPKL}\right) < MSE\left(\hat{\beta}_{RPRR}\right)$.

Proof. The MSE difference between the RPRR and the RPKL estimators is given by:

$$\begin{aligned}\Delta_5 &= MSE\left(\hat{\beta}_{RPKL}\right) - MSE\left(\hat{\beta}_{RPRR}\right) \\ \Delta_5 &= MSE\left(\hat{\beta}_{RPKL}\right) - MSE\left(\hat{\beta}_{RPRR}\right).\end{aligned} \tag{28}$$

After using some algebraic calculations, Δ_5 in Equation (28) will be negative if $\sum_{j=1}^{p+1}\frac{3k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1}\frac{2\lambda_j^*-k}{\lambda_j^*(\lambda_j^*+k)^2}$, then $MSE\left(\hat{\beta}_{RPKL}\right) < MSE\left(\hat{\beta}_{RPRR}\right)$. That means, the RPKL estimator is better than the PRRR estimator if $\sum_{j=1}^{p+1}\frac{3k\alpha_j^{*2}}{(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1}\frac{2\lambda_j^*-k}{\lambda_j^*(\lambda_j^*+k)^2}$. □

Theorem 6. *If* $\sum_{j=1}^{p+1}\frac{k\lambda_j^*\alpha_j^{*2}\left(2\lambda_j^*+k\right)}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1}\frac{\lambda_j^{*2}}{(\lambda_j^*+k)^4}$, *then* $MSE\left(\hat{\beta}_{RPMKL}\right) < MSE\left(\hat{\beta}_{RPRR}\right)$.

Proof. The MSE difference between the RPRR and the RPMKL estimators is given by:

$$\begin{aligned}\Delta_6 &= MSE\left(\hat{\beta}_{RPMKL}\right) - MSE\left(\hat{\beta}_{RPRR}\right) \\ &= \sum_{j=1}^{p+1}\left[\frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}-\lambda_j^*(\lambda_j^*+k)^2-k^2(\lambda_j^*+k)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4}\right].\end{aligned} \tag{29}$$

After using some algebraic calculations, Δ_6 in Equation (29) will be negative if $\sum_{j=1}^{p+1}\frac{k\lambda_j^*\alpha_j^{*2}\left(2\lambda_j^*+k\right)}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1}\frac{\lambda_j^{*2}}{(\lambda_j^*+k)^4}$, then $MSE\left(\hat{\beta}_{RPMKL}\right) < MSE\left(\hat{\beta}_{RPRR}\right)$. That means, the RPMKL estimator is better than the PRRR estimator if $\sum_{j=1}^{p+1}\frac{k\lambda_j^*\alpha_j^{*2}\left(2\lambda_j^*+k\right)}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1}\frac{\lambda_j^{*2}}{(\lambda_j^*+k)^4}$. □

Theorem 7. *If* $\sum_{j=1}^{p+1}\frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1}\frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$, *then* $MSE\left(\hat{\beta}_{RPKL}\right) < MSE\left(\hat{\beta}_{RPL}\right)$.

Proof. The MSE difference between the RPL and the RPKL estimators is given by:

$$\begin{aligned}\Delta_7 &= MSE\left(\hat{\beta}_{RPKL}\right) - MSE\left(\hat{\beta}_{RPL}\right) \\ &= \sum_{j=1}^{p+1}\left[\frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} - \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}\right].\end{aligned} \tag{30}$$

After using some algebraic calculations, Δ_7 in Equation (30) will be negative if $\sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$, then $MSE(\hat{\beta}_{RPKL}) < MSE(\hat{\beta}_{RPL})$. That means, the RPKL estimator is better than the RPL estimator if $\sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$. □

Theorem 8. *If* $\sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$, *then* $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPL})$.

Proof. The MSE difference between the RPL and the RPMKL estimators is given by:

$$\begin{aligned}\Delta_8 &= MSE(\hat{\beta}_{RPMKL}) - MSE(\hat{\beta}_{RPL}) \\ &= \sum_{j=1}^{p+1}\left[\frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} - \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}\right]. \end{aligned} \tag{31}$$

After using some algebraic calculations, Δ_8 in Equation (31) will be negative if $\sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$, then $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPL})$. That means, the RPMKL estimator is better than the RPL estimator if $\sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2}$. □

Theorem 9. *If* $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}\left(5\lambda_j^{*2}-2\lambda_j^*k-3k^2\right)}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2\left(2\lambda_j^*+k\right)}{\lambda_j^*(\lambda_j^*+k)^4}$, *then* $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPKL})$.

Proof. The MSE difference between the RPKL and the RPMKL estimators is given by:

$$\begin{aligned}\Delta_9 &= MSE(\hat{\beta}_{RPMKL}) - MSE(\hat{\beta}_{RPKL}) \\ &= \sum_{j=1}^{p+1}\left[\frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} - \frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2}\right]. \end{aligned} \tag{32}$$

After using some algebraic calculations, Δ_9 in Equation (32) will be negative if $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}\left[5\lambda_j^{*2}-2\lambda_j^*k-3k^2\right]}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2\left[2\lambda_j^*+k\right]}{\lambda_j^*(\lambda_j^*+k)^4}$, *then* $MSE(\hat{\beta}_{RPMKL}) < MSE(\hat{\beta}_{RPKL})$. That means, the RPMKL estimator is better than the RPKL estimator if $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}\left(5\lambda_j^{*2}-2\lambda_j^*k-3k^2\right)}{(\lambda_j^*+k)^4} < \sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2\left(2\lambda_j^*+k\right)}{\lambda_j^*(\lambda_j^*+k)^4}$. □

2.5. Selection of Biasing Parameters

Following Månsson and Shukur [10], Mansson et al. [11], Lukman et al. [12], and Abonazel and Dawoud [26], we suggest the following biasing parameters for the PRR, RPRR, PL, RPL, PKL, RPKL, PMKL, and RPMKL estimators.

- The suggested $\hat{k}$ for the PRR estimator is:

$$\hat{k}_{PRR} = \max\left(\sqrt{\hat{\alpha}_j^2}\right)_{j=1}^{p+1}. \tag{33}$$

- The suggested $\hat{k}$ for the RPRR estimator is:

$$\hat{k}_{RPRR} = \max\left(\sqrt{\hat{\alpha}_j^{*2}}\right)_{j=1}^{p+1}. \tag{34}$$

- The suggested $\hat{d}$ for the PL estimator is:

$$\hat{d}_{PL} = \max\left(0, \frac{\hat{\alpha}_j^2 - 1}{\hat{\alpha}_j^2 + \frac{1}{\lambda_j}}\right)_{j=1}^{p+1}. \tag{35}$$

- The suggested $\hat{d}$ for the RPL estimator is:

$$\hat{d}_{RPL} = \max\left(0, \frac{\hat{\alpha}_j^{*2} - 1}{\hat{\alpha}_j^{*2} + \frac{1}{\lambda_j^*}}\right)_{j=1}^{p+1}. \tag{36}$$

- The two suggested $\hat{k}$ for the PKL estimator are $\hat{k}_{PKL.1} = \hat{k}_{PRR}$ and

$$\hat{k}_{PKL.2} = \left(\min\left(\frac{1}{2\hat{\alpha}_j^2 + \frac{1}{\lambda_j}}\right)_{j=1}^{p+1}\right)^{1/2}. \tag{37}$$

- The two suggested $\hat{k}$ for the RPKL estimator are $\hat{k}_{RPKL.1} = \hat{k}_{RPRR}$ and

$$\hat{k}_{RPKL.2} = \left(\min\left(\frac{1}{2\hat{\alpha}_j^{*2} + \frac{1}{\lambda_j^*}}\right)_{j=1}^{p+1}\right)^{1/2}. \tag{38}$$

Here, we discuss obtaining the optimal value of k by minimizing

$$MSE\left(\hat{\beta}_{PMKL}\right) = \sum_{j=1}^{p+1} \frac{\lambda_j(\lambda_j - k)^2}{(\lambda_j + k)^4} + k^2 \sum_{j=1}^{p+1} \frac{(k + 3\lambda_j)^2 \alpha_j^2}{(\lambda_j + k)^4}. \tag{39}$$

Differentiating $MSE\left(\hat{\beta}_{PMKL}\right)$ with respect to k and setting $\left(\partial MSE\left(\hat{\beta}_{PMKL}\right)/\partial k\right) = 0$, we have:

$$k = \frac{-\left(3\lambda_j\alpha_j^2 + 1\right) \mp \sqrt{9\lambda_j^2\left(\alpha_j^2\right)^2 + 10\lambda_j\alpha_j^2 + 1}}{2\alpha_j^2} \text{ or } k = 3\lambda_j. \tag{40}$$

Since $k > 0$, we take:

$$k = \frac{\left(\sqrt{9\lambda_j^2\left(\alpha_j^2\right)^2 + 10\lambda_j\alpha_j^2 + 1}\right) - \left(3\lambda_j\alpha_j^2 + 1\right)}{2\alpha_j^2} \text{ or } k = 3\lambda_j. \tag{41}$$

Then, we replace the unknown parameter α_j^2 with its estimator for the optimal value of k in (41) in order to use it in the application. Therefore, we have:

$$k = \frac{\left(\sqrt{9\lambda_j^2\left(\hat{\alpha}_j^2\right)^2 + 10\lambda_j\hat{\alpha}_j^2 + 1}\right) - \left(3\lambda_j\hat{\alpha}_j^2 + 1\right)}{2\hat{\alpha}_j^2} \text{ or } k = 3\lambda_j. \tag{42}$$

- Three $\hat{k}$ are suggested for the PMKL estimator: $\hat{k}_{PMKL.1} = \hat{k}_{PRR}$, $\hat{k}_{PMKL.2} = \hat{k}_{PKL.2}$, and

$$\hat{k}_{PMKL.3} = \left(\min \left\{ \frac{\left(\sqrt{9\lambda_j^2 (\hat{\alpha}_j^2)^2 + 10\lambda_j \hat{\alpha}_j^2 + 1} \right) - \left(3\lambda_j \hat{\alpha}_j^2 + 1 \right)}{2\hat{\alpha}_j^2}, 3\lambda_j \right\}_{j=1}^{p+1} \right)^{1/2} \tag{43}$$

- The three suggested $\hat{k}$ for the RPMKL estimator are $\hat{k}_{RPMKL.1} = \hat{k}_{RPRR}$, $\hat{k}_{RPMKL.2} = \hat{k}_{RPKL.2}$ and

$$\hat{k}_{RPMKL.3} = \left(\min \left\{ \frac{\left(\sqrt{9\lambda_j^{*2} (\hat{\alpha}_j^{*2})^2 + 10\lambda_j^* \hat{\alpha}_j^{*2} + 1} \right) - \left(3\lambda_j^* \hat{\alpha}_j^{*2} + 1 \right)}{2\hat{\alpha}_j^{*2}}, 3\lambda_j^* \right\}_{j=1}^{p+1} \right)^{1/2}. \tag{44}$$

3. Simulation Study

We conducted a Monte Carlo simulation study to examine the performance of non-robust estimators (PML, PRR, PL, PKL, and PMKL) and robust estimators (RPML, RPRR, RPL, RPKL, and RPMKL). The program of the simulation study was written in the R programming language.

3.1. Simulation Design

The simulated data were implemented with the following settings: y_i was generated from a Poisson distribution with mean equal to $\mu_i = \exp(x_i\beta)$; for $i = 1, 2, \ldots, n$. The sample sizes (n) were set to 75, 100, 200, and 300. The number of explanatory variables (p) was set to 2 and 6. $\beta\prime\beta = 1$, and $\beta_2 = \ldots = \beta_{p+1}$, as in Kaçıranlar and Dawoud [28] and Abonazel and Dawoud [26]. The intercept was defined as $\beta_1 = -1$, 0, and 1, as in Lukman et al. [12] and Abonazel and Dawoud [26]. The explanatory variables (X), as in Abonazel and Dawoud [26], were generated from a multivariate normal distribution $MVN(0, \Sigma_X)$, where $\text{diag}(\Sigma_X) = 1$ and off–diag $(\Sigma_X) = \rho$; ρ = 0.90, 0.95, and 0.99, where ρ was the correlation coefficient between the explanatory variables. Some outliers were generated with different percentages ($\tau = 0\%$, 10%, and 20%) in the model by randomly replacing some values due to the selected percentage $\tau\%$ from the response variable. As in Abonazel and Saber [22], Abonazel et al. [24], Dawoud and Abonazel [29], Awwad et al. [30], and Abonazel and Dawoud [26], the outliers were generated from a Poisson distribution with mean equal to $(Q_3 + 10 \times IQR)$, where Q_3 was the third quartile of $\mu_i(\beta)$ values, and IQR was the interquartile range of $\mu_i(\beta)$ values.

We used the average MSE (AMSE) measure for verification, which was computed as:

$$\text{AMSE}(\hat{\beta}) = \frac{1}{1000} \sum_{l=1}^{1000} (\hat{\beta}_l - \beta)' (\hat{\beta}_l - \beta), \tag{45}$$

where $\hat{\beta}_l$ was the estimated vector at the l-th experiment of the simulation (the experiment was replicated 1000 times), and β was the true parameter vector.

3.2. Simulation Results

The simulation results are summarized in Tables A1–A9 in Appendix A. In each table, the best value of the AMSE is highlighted in **bold**. According to the simulation results, we concluded the following:

1. The PML performance was the worst of the given estimators in the existence of both outliers and multicollinearity problems, as expected;
2. The estimators' AMSE values were increased in the case of the multicollinearity degree (ρ), explanatory variables number (p), and outlier percentage (τ);

3. The estimators' AMSE values were decreased in the case of the sample size (n) being increased, when the other factors were fixed;
4. When only the multicollinearity problem existed in the model (as in Tables A1–A3 or when no outliers existed, i.e., $\tau = 0\%$), the non-robust estimators (PML, PRR, PL, PKL, and PMKL) were better than the corresponding robust estimators (RPML, RPRR, RPL, RPKL, and RPMKL) for all ρ, p, and n values;
5. When both problems existed in the model (as in Tables A4–A9 or when outliers existed, i.e., $\tau > 0\%$), the RPML, RPRR, RPL, RPKL, and RPMKL estimators were better than the PML, PRR, PL, PKL, and PMKL estimators, respectively, for all ρ, p, and n values;
6. For n = 75, 100, $\beta_1 = -1$, τ = 0% and ρ = 0.90, 0.95, the PKL.1 AMSE values were the lowest among all models, and for n = 75, 100, $\beta_1 = -1$, τ = 10% and ρ = 0.90, the RPKL.1 AMSE values were the lowest among all models;
7. When $\tau = 0\%$, the PMKL estimator, and in particular, the PMKL.1 was the best, followed by the PKL, and in particular, the PKL.1 in most situations;
8. In addition, when $\tau > 0\%$, the RPMKL estimator was the best, particularly the xRPMKL.1, followed by the RPKL estimator, particularly the RPKL.1 in most situations;
9. Finally, the PMKL estimator achieved the best performance among all given estimators when only the multicollinearity problem existed in the model. If both outliers and multicollinearity problems existed in the model, the RPMKL estimator achieved the best performance among all given estimators in most situations.

4. Empirical Applications

In this section, we consider two real life applications to evaluate the performance of the proposed robust estimators.

4.1. Aircraft Damage Data

To evaluate the performance of the proposed robust estimators, we first considered the aircraft damage dataset. Myers et al. [31] discussed these data in detail. The dataset provided information related to two aircrafts, that is, the McDonnell Douglas A-4 Skyhawk and the Grumman A-6 Itruder. The sample size of the dataset was n = 30 with one dependent (y) and three explanatory variables (X's): y denoted the number of locations where damage was inflicted on the aircraft, X1 was the type of aircraft, X2 was the bomb load measured in tons, and X3 was the representation of aircraft experience in months.

Myers et al. [31] indicated the existence of severe multicollinearity in the data set. However, the eigenvalues of this data were 4.3333, 374.8961, and 2085.2251. In addition, the condition number (CN) was 219.37 > 30, showing a multicollinearity issue among the explanatory variables. On the other hand, for outliers, there were three outlier values (1, 29, and 30) in the data, as shown in Figure 1. Therefore, these data suffered from the problems of multicollinearity and outliers, together.

Since this section aims to assess the superiority of the new robust estimators using a real-life application, the efficiency of the considered estimators was assessed through the MSE. The estimated coefficients and MSEs of the ten estimators on these data are presented in Table 1. Table 1 shows the estimates of the regression coefficients and the estimated MSE values for the different estimators. From Table 1, we can note that:

1. The PML estimator performed the worst among all given estimators;
2. The robust estimators achieved a better performance than the corresponding non-robust estimators;
3. The PMKL performed better in general, followed by the PKL, and then the other non-robust estimators. Additionally, the RPMKL achieved a better performance in general, followed by the RPKL and then the other robust estimators;
4. Finally, the RPMKL, particularly the RPMKL.1 estimator was the best, which had the lowest AMSE value.

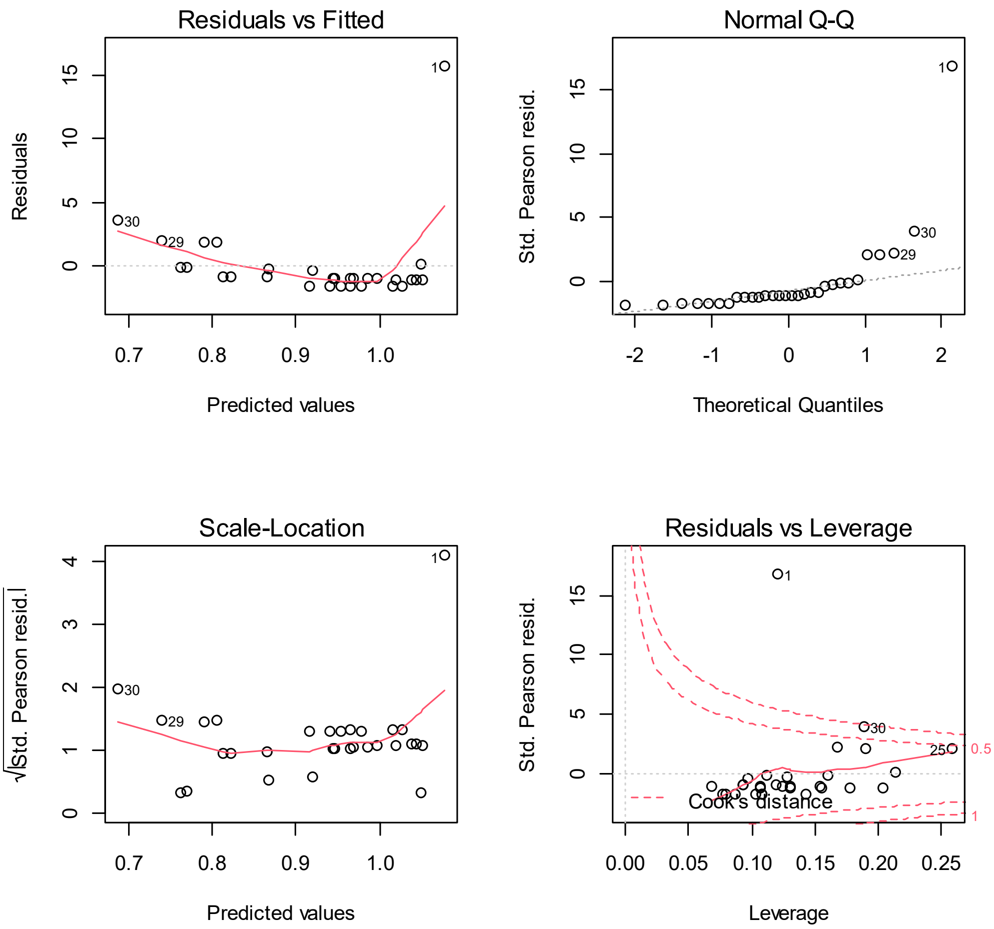

Figure 1. Residuals analysis of the aircraft damage data.

Table 1. Estimated coefficients and MSEs of the aircraft damage data.

	Non-Robust Estimators							
	PML	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**
Intercept	0.868	0.624	0.588	0.381	0.418	0.283	0.318	0.372
X1	−0.038	−0.095	−0.103	−0.153	−0.145	−0.166	−0.161	−0.151
X2	−0.024	−0.010	−0.008	0.003	0.001	0.008	0.006	0.004
X3	0.003	0.005	0.006	0.007	0.007	0.008	0.008	0.007
MSE	1.239	0.540	0.482	0.209	0.240	0.135	0.159	0.202
	Robust Estimators							
	RPML	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
Intercept	−0.882	−0.485	−0.454	−0.087	−0.222	−0.053	−0.143	−0.194
X1	−0.024	0.048	0.052	0.119	0.100	0.106	0.101	0.097
X2	0.218	0.198	0.196	0.178	0.184	0.177	0.181	0.183
X3	−0.009	−0.012	−0.012	−0.015	−0.014	−0.015	−0.015	−0.014
MSE	0.623	0.344	0.322	0.146	0.180	**0.088**	0.140	0.172

We verified the theoretical results through the aircraft damage data, as follows:

1. Since the condition $\sum_{j=1}^{p+1} \frac{\left((1-d)\alpha_j^{*2}-2\right)}{(\lambda_j^*+1)^2} = -0.083 < \sum_{j=1}^{p+1} \frac{(1+d)}{\lambda_j^*(\lambda_j^*+1)^2} = 0.067$ was satisfied, then the RPL estimator was better than the RPML estimator;
2. Since the condition $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.089 < \sum_{j=1}^{p+1} \frac{1}{(\lambda_j^*+k)^2} = 0.141$ was satisfied, then the RPKL estimator was better than the RPML estimator;
3. Since the condition $\sum_{j=1}^{p+1} \frac{k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} = 0.431 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+k)^4-\lambda_j^{*2}(\lambda_j^*-k)^2}{\lambda_j^*(\lambda_j^*+k)^4} = 0.536$ was satisfied, then the RPMKL estimator was better than the RPML estimator;
4. Since the condition $\sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.392 < \sum_{j=1}^{p+1} \frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.400$ was satisfied, then the RPL estimator was better than the RPRR estimator;
5. Since the condition $\sum_{j=1}^{p+1} \frac{3k\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.249 < \sum_{j=1}^{p+1} \frac{2\lambda_j^*-k}{\lambda_j^*(\lambda_j^*+k)^2} = 0.252$ was satisfied, then the RPKL estimator was better than the RPRR estimator;
6. Since the condition $\sum_{j=1}^{p+1} \frac{k\lambda_j^*\alpha_j^{*2}\left(2\lambda_j^*+k\right)}{(\lambda_j^*+k)^4} = 0.098 < \sum_{j=1}^{p+1} \frac{\lambda_j^{*2}}{(\lambda_j^*+k)^4} = 0.111$ was satisfied, then the RPMKL estimator was better than the RPRR estimator;
7. Since the condition $\sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} = 0.386 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.392$ was satisfied, then the RPKL estimator was better than the RPL estimator;
8. Since the condition $\sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} = 0.172 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.302$ was satisfied, then the RPMKL estimator was better than the RPL estimator;
9. Since the condition $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}\left(5\lambda_j^{*2}-2\lambda_j^*k-3k^2\right)}{(\lambda_j^*+k)^4} = 0.007 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2\left(2\lambda_j^*+k\right)}{\lambda_j^*(\lambda_j^*+k)^4} = 0.132$ was satisfied, then the RPMKL estimator was better than the RPKL estimator.

Figure 2 presents the MSE values of the non-robust estimators against k and d from 0 to 1. The MSE values of the PKL and PMKL estimators were smaller than the MSE values of the other estimators. However, the PMKL estimator had the smallest MSE values. Figure 3 presents the MSE values of the robust estimators against k and d from 0 to 1. Additionally, we noted that the MSE values of the RPKL and RPMKL estimators were smaller than the MSE values of the other estimators, and the RPMKL estimator had the smallest MSE values.

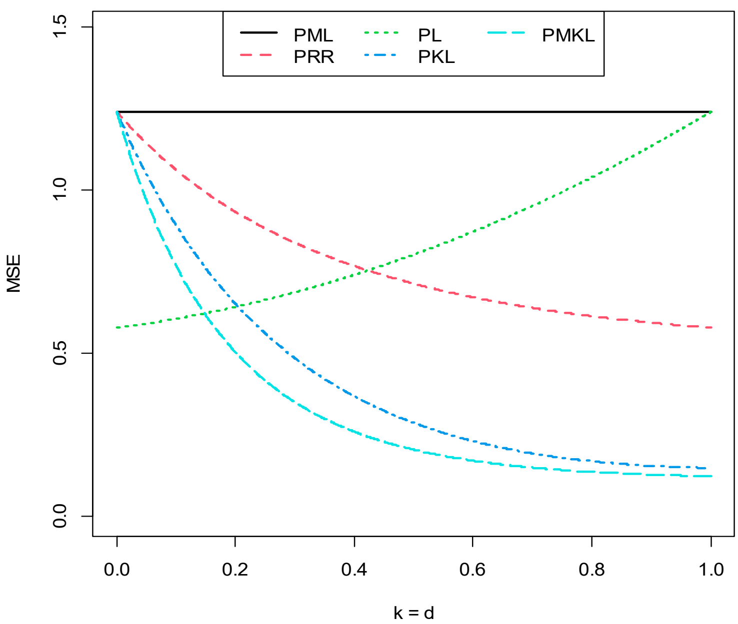

Figure 2. MSE values of the non-robust estimators against k and d of the aircraft damage data.

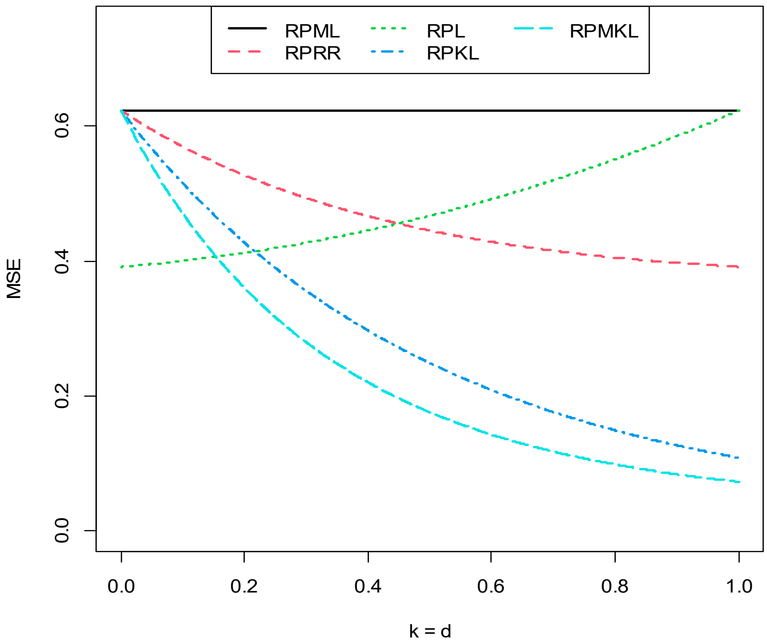

Figure 3. MSE values of the robust estimators against k and d of the aircraft damage data.

4.2. Somerville Lake Data

We used a dataset from Cameron and Trivedi [32] to estimate a recreation demand function. These data were from a survey on the number of recreational boating trips in Somerville Lake, East Texas, in 1980. The sample size was 201 observations with one response variable (y) and three explanatory variables (X's): y was the number of recreational boating trips in Somerville Lake in 1980, X1 was the cost of visiting Lake Conroe, X2 was cost of visiting Somerville Lake, and X3 was cost of visiting Houston Lake. These data were used by Månsson and Kibria [33] and Abonazel and Dawoud [26].

To investigate the existence of multicollinearity, Abonazel and Dawoud [26] showed that the condition number (CN) was 24.56, and the values of variance inflation factor (VIF) of the explanatory variables were 41.6, 13.2, and 25.03, respectively, and all coefficients of correlations between three explanatory variables were greater than 0.90. All these measures (correlations, VIF, and CN) confirmed the presence of severe multicollinearity. On the other hand, for outliers, there were 22 outlier values in the data as shown in Figure 4. Therefore, these data suffered from the problems of multicollinearity and outliers, together. The estimated coefficients and MSEs of the ten estimators on these data are presented in Table 2. Table 2 shows the estimates of the regression parameters and the estimated MSE values for the different estimators. From Table 2, we can note the following:

1. The PML performed worst among all given estimators;
2. The robust estimators achieved a better performance than the corresponding non-robust estimators;
3. The PMKL achieved a better performance in general, followed by the PKL, and then the other non-robust estimators. Additionally, the RPMKL achieved a better performance in general, followed by the RPKL, and then the other robust estimators;
4. Finally, the RPMKL, particularly the RPMKL.1 estimator was the best, which had the lowest AMSE value.

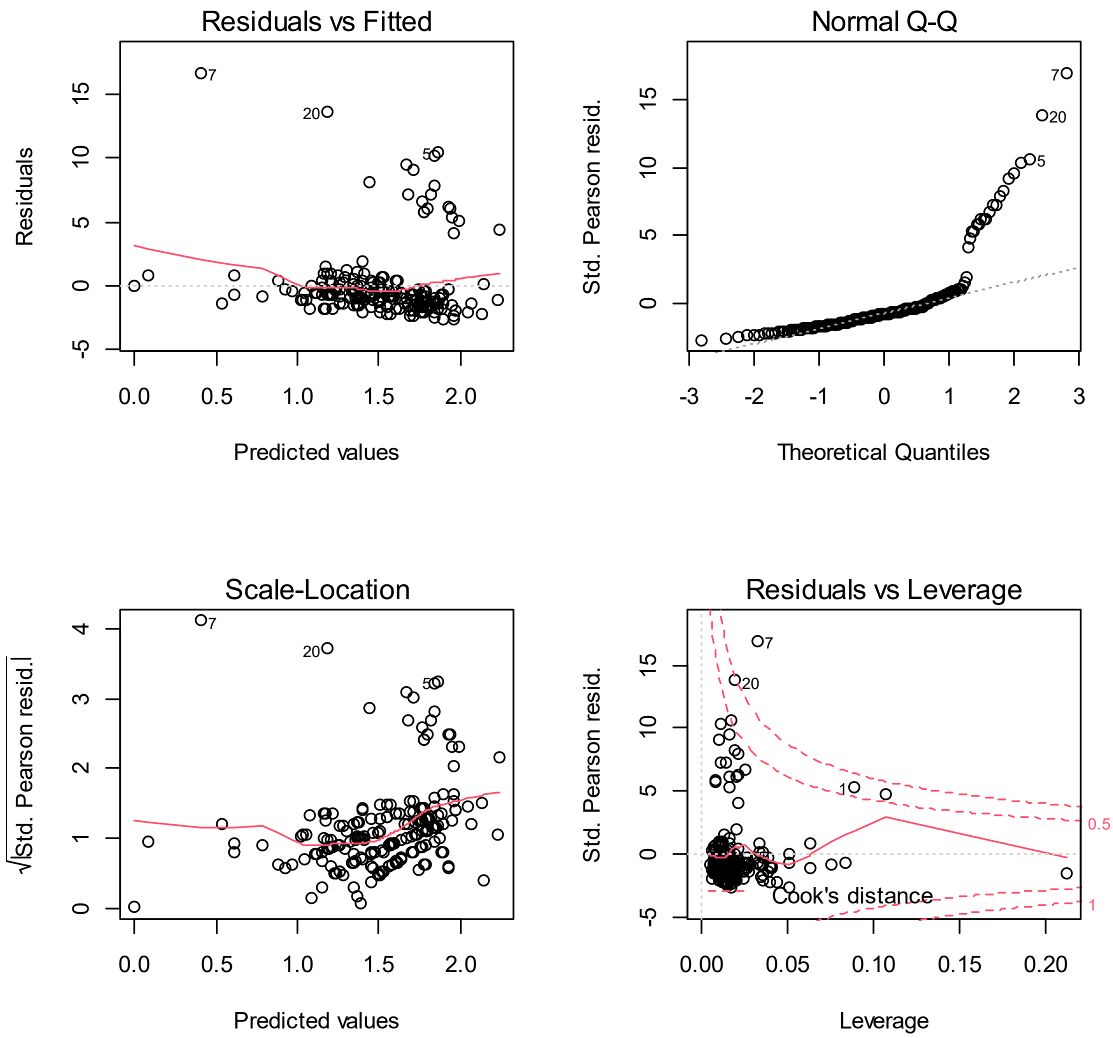

Figure 4. Residuals analysis of the Somerville Lake data.

Table 2. Estimated coefficients and MSEs of the Somerville Lake data.

	Non-Robust Estimators							
	PML	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**
Intercept	1.804	1.812	1.805	1.820	1.809	1.805	1.811	1.810
X1	−1.450	−0.808	−1.395	−0.166	−1.201	−0.214	−1.106	−1.159
X2	2.284	1.556	2.231	0.827	2.069	0.616	1.976	2.027
X3	−1.308	−1.193	−1.309	−1.077	−1.337	−0.786	−1.337	−1.337
MSE	1.059	0.756	0.945	1.061	0.670	0.630	0.631	0.650
	Robust Estimators							
	RPML	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
Intercept	1.348	1.345	1.347	1.342	1.347	1.334	1.344	1.345
X1	−0.774	−0.497	−0.594	−0.219	−0.411	−0.197	−0.342	−0.387
X2	0.923	0.684	0.772	0.446	0.638	0.351	0.552	0.602
X3	−0.483	−0.509	−0.505	−0.536	−0.550	−0.450	−0.526	−0.534
MSE	0.683	0.335	0.309	0.188	0.436	**0.111**	0.203	0.241

Through Somerville Lake data, we verified the theoretical results as follows:

1. Since the condition $\sum_{j=1}^{p+1} \frac{\left((1-d)\alpha_j^{*2}-2\right)}{(\lambda_j^*+1)^2} = -0.042 < \sum_{j=1}^{p+1} \frac{(1+d)}{\lambda_j^*(\lambda_j^*+1)^2} = 0.089$ was satisfied, then the RPL estimator was better than the RPML estimator;
2. Since the condition $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.066 < \sum_{j=1}^{p+1} \frac{1}{(\lambda_j^*+k)^2} = 0.123$ was satisfied, then the RPKL estimator was better than the RPML estimator;
3. Since the condition $\sum_{j=1}^{p+1} \frac{k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} = 0.438 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+k)^4-\lambda_j^{*2}(\lambda_j^*-k)^2}{\lambda_j^*(\lambda_j^*+k)^4} = 0.614$ was satisfied, then the RPMKL estimator was better than the RPML estimator;
4. Since the condition $\sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.453 < \sum_{j=1}^{p+1} \frac{\lambda_j^*+k^2\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.555$ was satisfied, then the RPL estimator was better than the RPRR estimator;
5. Since the condition $\sum_{j=1}^{p+1} \frac{3k\alpha_j^{*2}}{(\lambda_j^*+k)^2} = 0.079 < \sum_{j=1}^{p+1} \frac{2\lambda_j^*-k}{\lambda_j^*(\lambda_j^*+k)^2} = 0.455$ was satisfied, then the RPKL estimator was better than the RPRR estimator;
6. Since the condition $\sum_{j=1}^{p+1} \frac{k\lambda_j^*\alpha_j^{*2}\left(2\lambda_j^*+k\right)}{(\lambda_j^*+k)^4} = 0.063 < \sum_{j=1}^{p+1} \frac{\lambda_j^{*2}}{(\lambda_j^*+k)^4} = 0.129$ was satisfied, then the RPMKL estimator was better than the RPRR estimator;
7. Since the condition $\sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2+4k^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+k)^2} = 0.373 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.453$ was satisfied, then the RPKL estimator was better than the RPL estimator;
8. Since the condition $\sum_{j=1}^{p+1} \frac{\lambda_j^*(\lambda_j^*-k)^2+k^2(k+3\lambda_j^*)^2\alpha_j^{*2}}{(\lambda_j^*+k)^4} = 0.241 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*+d)^2+(1-d)^2\lambda_j^*\alpha_j^{*2}}{\lambda_j^*(\lambda_j^*+1)^2} = 0.429$ was satisfied, then the RPMKL estimator was better than the RPL estimator;
9. Since the condition $\sum_{j=1}^{p+1} \frac{k\alpha_j^{*2}\left(5\lambda_j^{*2}-2\lambda_j^*k-3k^2\right)}{(\lambda_j^*+k)^4} = 0.029 < \sum_{j=1}^{p+1} \frac{(\lambda_j^*-k)^2\left(2\lambda_j^*+k\right)}{\lambda_j^*(\lambda_j^*+k)^4} = 0.166$ was satisfied, then the RPMKL estimator was better than the RPKL estimator.

We checked the efficiency of our proposed estimators for $0 < k = d < 1$ through the Somerville Lake data. Figures 5 and 6 present the MSE values of the non-robust and robust estimators, respectively. As in the first application, these figures indicated that the PKL and PMKL estimators had smaller MSE values than the other estimators, and the PMKL estimator had the smallest MSE values.

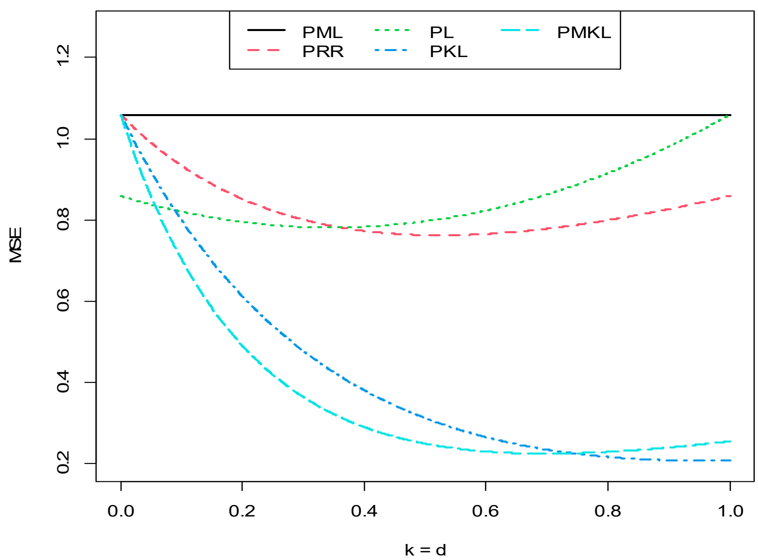

Figure 5. MSE values of the non-robust estimators against k and d of the Somerville Lake data.

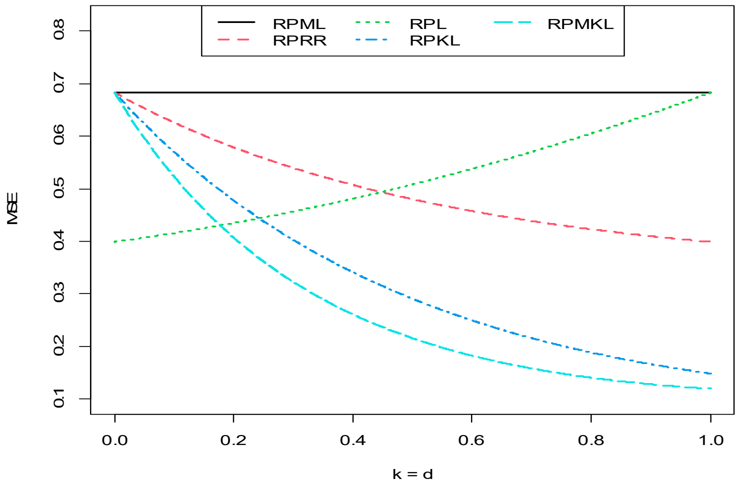

Figure 6. MSE values of the robust estimators against k and d of the Somerville Lake data.

5. Conclusions

In the PRM model, the problem of outliers causes the PML to be inefficient. In this case, the RPML is considered to reduce the effects of the outlier values. Additionally, the PML performed the worst in the case of the existence of the multicollinearity problem. Therefore, some proposed biased regression estimators in the PRM model were considered to handle multicollinearity, such as the PRR, PL, PKL, and PMKL (Månsson and Shukur [10], Månsson et al. [11], Lukman et al. [12], and Aladeitan et al. [13], respectively). Following Abonazel and Dawoud [26]'s provision of the RPRR estimator for dealing with both outliers and multicollinearity problems simultaneously in the PRM, we proposed, in this paper, new robust versions of three one-parameter biased estimators for the PRM, namely, the RPL, RPKL, and RPMKL estimators. Theoretically, the necessary and sufficient conditions were obtained for the proposed estimators (RPL, RPKL, and RPMKL). Then, a simulation study and two applications were conducted to investigate the proposed estimators' performances, and the other estimators' performances. The results indicated that the proposed estimators' performances were better than the others, especially RPMKL in most situations where both outliers and multicollinearity problems occurred in the model. Finally, if outliers and multicollinearity problems occurred together in the PRM, we recommend that practitioners use the RPMKL and RPKL estimators to estimate the regression parameters of the model. For future work, we aim to develop new estimators to deal with both outliers and multicollinearity problems simultaneously in the PRM, such as the robust jackknife Liu, robust jackknife Kibria–Lukman, and robust jackknife modified Kibria–Lukman estimators as an extension to Abonazel and Dawoud [26] and Arum et al. [27].

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. AMSE of different estimators when τ = 0% and ρ = 0.90.

		Non-Robust Estimators								Robust Estimators							
p	n	PML	PRR	PL	PKL.1	PKL.2	PMKL.1	PMKL.2	PMKL.3	RPML	RPRR	RPL	RPKL.1	RPKL.2	RPMKL.1	RPMKL.2	RPMKL.3
$\beta_1 = -1$																	
2	75	0.3623	0.1756	0.2830	**0.1090**	0.2078	0.1359	0.1701	0.1897	0.7051	0.2091	0.4876	0.1912	0.2728	0.2390	0.2066	0.2410
	100	0.1933	0.1304	0.1708	**0.1049**	0.1488	0.1157	0.1352	0.1422	0.3022	0.1732	0.2531	0.1347	0.2055	0.1635	0.1788	0.1925
	200	0.0958	0.0794	0.0898	0.0689	0.0840	**0.0656**	0.0797	0.0820	0.1408	0.1073	0.1275	0.0862	0.1152	0.0810	0.1061	0.1109
	300	0.0445	0.0409	0.0431	0.0383	0.0418	**0.0370**	0.0408	0.0413	0.0687	0.0604	0.0652	0.0544	0.0622	0.0512	0.0597	0.0610
6	75	0.7520	0.4799	0.5620	0.2970	0.4489	**0.2230**	0.3606	0.4072	1.1707	0.6060	0.8324	0.3197	0.6071	0.2366	0.4662	0.5402
	100	0.2071	0.1833	0.1896	0.1665	0.1809	**0.1579**	0.1733	0.1771	0.2714	0.2313	0.2447	0.2059	0.2315	0.1963	0.2215	0.2263
	200	0.1174	0.1095	0.1113	0.1026	0.1081	**0.0972**	0.1042	0.1063	0.1657	0.1513	0.1541	0.1388	0.1477	0.1293	0.1405	0.1444
	300	0.0918	0.0882	0.0876	0.0848	0.0851	**0.0818**	0.0822	0.0838	0.1293	0.1216	0.1214	0.1145	0.1175	0.1086	0.1125	0.1152
$\beta_1 = 0$																	
2	75	0.0930	0.0829	0.0824	0.0735	0.0773	**0.0658**	0.0708	0.0743	0.1304	0.1101	0.1093	0.0918	0.0990	0.0784	0.0872	0.0936
	100	0.0573	0.0540	0.0536	0.0508	0.0515	**0.0480**	0.0489	0.0503	0.0778	0.0716	0.0709	0.0659	0.0669	0.0611	0.0625	0.0649
	200	0.0304	0.0292	0.0291	0.0280	0.0285	**0.0269**	0.0276	0.0281	0.0473	0.0443	0.0442	0.0415	0.0427	0.0389	0.0406	0.0418
	300	0.0174	0.0170	0.0170	0.0167	0.0168	**0.0164**	0.0166	0.0167	0.0248	0.0241	0.0240	0.0233	0.0236	0.0226	0.0230	0.0234
6	75	0.1373	0.1290	0.1287	0.1211	0.1247	**0.1141**	0.1191	0.1221	0.1779	0.1649	0.1645	0.1526	0.1582	0.1421	0.1496	0.1543
	100	0.0882	0.0848	0.0847	0.0814	0.0832	**0.0784**	0.0808	0.0821	0.1164	0.1104	0.1103	0.1046	0.1076	0.0994	0.1035	0.1057
	200	0.0437	0.0427	0.0427	0.0417	0.0422	**0.0408**	0.0415	0.0419	0.0580	0.0562	0.0561	0.0544	0.0553	0.0528	0.0541	0.0548
	300	0.0162	0.0161	0.0161	0.0159	0.0160	**0.0158**	0.0159	0.0160	0.0221	0.0219	0.0219	0.0217	0.0218	0.0215	0.0216	0.0217
$\beta_1 = 1$																	
2	75	0.0591	0.0531	0.0565	0.0476	0.0542	**0.0432**	0.0520	0.0532	0.0799	0.0690	0.0753	0.0591	0.0710	0.0516	0.0670	0.0692
	100	0.0225	0.0218	0.0222	0.0212	0.0219	**0.0207**	0.0217	0.0218	0.0338	0.0323	0.0331	0.0308	0.0325	0.0295	0.0319	0.0322
	200	0.0102	0.0101	0.0102	0.0099	0.0101	**0.0098**	0.0100	0.0101	0.0161	0.0157	0.0159	0.0153	0.0157	0.0150	0.0156	0.0157
	300	0.0063	0.0063	0.0063	0.0062	0.0063	**0.0062**	0.0063	0.0063	0.0088	0.0087	0.0088	0.0086	0.0088	0.0085	0.0087	0.0087
6	75	0.1006	0.0945	0.0971	0.0886	0.0945	**0.0835**	0.0918	0.0933	0.1349	0.1244	0.1289	0.1146	0.1245	0.1062	0.1198	0.1224
	100	0.0265	0.0262	0.0263	0.0259	0.0262	**0.0257**	0.0261	0.0262	0.0341	0.0336	0.0338	0.0331	0.0336	0.0327	0.0334	0.0335
	200	0.0106	0.0106	0.0106	0.0105	0.0106	**0.0105**	0.0105	0.0106	0.0133	0.0133	0.0133	0.0132	0.0133	0.0131	0.0132	0.0132
	300	0.0092	0.0092	0.0092	0.0092	0.0092	**0.0091**	0.0092	0.0092	0.0128	0.0127	0.0127	0.0126	0.0127	0.0125	0.0126	0.0127

Table A2. AMSE of different estimators when τ = 0% and ρ = 0.95.

		Non-Robust Estimators								Robust Estimators							
p	n	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	0.6385	0.1948	0.4356	**0.0981**	0.2383	0.1287	0.1714	0.2069	1.2700	0.1884	0.7694	0.4252	0.2751	0.2675	0.1902	0.2376
	100	0.3149	0.1711	0.2566	**0.0999**	0.2000	0.1053	0.1668	0.1843	0.5010	0.2081	0.3789	0.1074	0.2580	0.1451	0.2017	0.2311
	200	0.1618	0.1174	0.1427	0.0851	0.1252	**0.0718**	0.1119	0.1190	0.2443	0.1518	0.2026	0.0915	0.1646	0.0762	0.1392	0.1528
	300	0.0782	0.0672	0.0729	0.0580	0.0686	**0.0518**	0.0646	0.0668	0.1189	0.0937	0.1067	0.0735	0.0965	0.0621	0.0879	0.0926
6	**75**	0.6138	0.4405	0.4746	0.3125	0.3992	**0.2522**	0.3360	0.3696	0.7978	0.5136	0.5929	0.3321	0.4753	0.2674	0.3913	0.4355
	100	0.6059	0.4418	0.4889	0.3150	0.4188	**0.2501**	0.3569	0.3899	0.7670	0.5170	0.5935	0.3376	0.4886	0.2627	0.4047	0.4491
	200	0.3477	0.2925	0.3068	0.2445	0.2835	**0.2103**	0.2578	0.2717	0.4461	0.3592	0.3809	0.2861	0.3440	0.2383	0.3056	0.3262
	300	0.0810	0.0780	0.0786	0.0752	0.0773	**0.0728**	0.0757	0.0766	0.1112	0.1057	0.1069	0.1007	0.1047	0.0963	0.1017	0.1033
$\beta_1 = 0$																	
2	**75**	0.1704	0.1327	0.1315	0.1002	0.1131	**0.0792**	0.0939	0.1043	0.2626	0.1791	0.1855	0.1135	0.1477	0.0812	0.1149	0.1325
	100	0.0842	0.0751	0.0746	0.0666	0.0699	**0.0597**	0.0640	0.0672	0.1294	0.1086	0.1077	0.0900	0.0970	0.0763	0.0849	0.0915
	200	0.0441	0.0416	0.0415	0.0391	0.0403	**0.0369**	0.0385	0.0395	0.0819	0.0731	0.0727	0.0649	0.0684	0.0581	0.0627	0.0658
	300	0.0267	0.0257	0.0256	0.0247	0.0251	**0.0239**	0.0244	0.0248	0.0422	0.0398	0.0397	0.0375	0.0384	0.0354	0.0366	0.0376
6	**75**	0.1358	0.1261	0.1260	0.1169	0.1217	**0.1089**	0.1154	0.1188	0.1571	0.1441	0.1440	0.1319	0.1382	0.1216	0.1300	0.1345
	100	0.1337	0.1222	0.1220	0.1114	0.1169	**0.1024**	0.1097	0.1136	0.1833	0.1630	0.1628	0.1442	0.1537	0.1296	0.1417	0.1482
	200	0.0570	0.0557	0.0557	0.0544	0.0551	**0.0531**	0.0541	0.0547	0.0848	0.0820	0.0819	0.0792	0.0807	0.0767	0.0787	0.0798
	300	0.0511	0.0499	0.0499	0.0487	0.0493	**0.0476**	0.0485	0.0490	0.0702	0.0680	0.0679	0.0658	0.0669	0.0638	0.0654	0.0662
$\beta_1 = 1$																	
2	**75**	0.0599	0.0539	0.0573	0.0483	0.0549	**0.0438**	0.0527	0.0539	0.0850	0.0729	0.0797	0.0619	0.0749	0.0538	0.0704	0.0729
	100	0.0521	0.0470	0.0497	0.0422	0.0476	**0.0382**	0.0455	0.0467	0.0735	0.0630	0.0686	0.0534	0.0643	0.0462	0.0603	0.0625
	200	0.0164	0.0160	0.0163	0.0156	0.0161	**0.0152**	0.0159	0.0160	0.0258	0.0246	0.0253	0.0235	0.0249	0.0225	0.0244	0.0247
	300	0.0098	0.0097	0.0097	0.0095	0.0097	**0.0094**	0.0096	0.0096	0.0156	0.0151	0.0154	0.0147	0.0152	0.0143	0.0150	0.0151
6	**75**	0.0677	0.0654	0.0663	0.0633	0.0654	**0.0613**	0.0643	0.0649	0.0872	0.0834	0.0848	0.0797	0.0832	0.0765	0.0814	0.0824
	100	0.0992	0.0922	0.0949	0.0856	0.0920	**0.0801**	0.0887	0.0905	0.1324	0.1205	0.1250	0.1094	0.1200	0.1005	0.1145	0.1175
	200	0.0276	0.0273	0.0274	0.0269	0.0273	**0.0266**	0.0271	0.0272	0.0378	0.0372	0.0374	0.0365	0.0372	0.0359	0.0368	0.0370
	300	0.0171	0.0170	0.0170	0.0168	0.0170	**0.0167**	0.0169	0.0169	0.0240	0.0237	0.0238	0.0234	0.0237	0.0232	0.0236	0.0236

Table A3. AMSE of different estimators when τ = 0% and ρ = 0.99.

		Non-Robust Estimators								Robust Estimators							
p	n	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	2.8112	0.2068	1.4652	1.7103	0.1921	**0.1067**	0.1176	0.1656	5.8602	0.1187	3.0773	5.0631	0.3039	0.4151	0.1466	0.1912
	100	1.2386	0.1603	0.7362	0.2671	0.2330	**0.1088**	0.1466	0.1964	2.0404	0.1396	1.1556	0.8643	0.2485	0.2000	0.1532	0.2120
	200	0.5435	0.1938	0.3658	0.0382	0.2096	**0.0296**	0.1433	0.1790	0.8062	0.1704	0.4910	0.0586	0.2073	0.0372	0.1303	0.1728
	300	0.4803	0.1791	0.3300	0.0339	0.1915	**0.0228**	0.1307	0.1634	0.7983	0.1821	0.4972	0.0376	0.2137	0.0246	0.1316	0.1766
6	**75**	3.4382	0.6860	2.0765	0.6910	0.9546	**0.1682**	0.6493	0.8144	4.4014	0.6426	2.6343	1.2442	1.0647	0.2037	0.7109	0.9032
	100	1.9867	0.7275	1.2835	0.2553	0.7674	**0.1555**	0.5462	0.6645	2.4733	0.7243	1.5556	0.3228	0.8417	0.1604	0.5850	0.7230
	200	0.9501	0.5947	0.7236	0.3408	0.5697	**0.2138**	0.4574	0.5177	1.2346	0.6768	0.8858	0.3230	0.6475	0.2382	0.4958	0.5771
	300	0.5228	0.4042	0.4452	0.3031	0.3938	**0.2390**	0.3449	0.3714	0.7143	0.5088	0.5804	0.3429	0.4908	0.2530	0.4132	0.4551
$\beta_1 = 0$																	
2	**75**	0.7911	0.2046	0.4163	0.0395	0.1954	**0.0192**	0.1206	0.1613	1.1938	0.1715	0.6235	0.1752	0.2334	0.0291	0.1412	0.1929
	100	0.3656	0.2057	0.2249	0.0960	0.1550	**0.0402**	0.1094	0.1336	0.5637	0.2382	0.3232	0.0658	0.1930	0.0608	0.1286	0.1628
	200	0.3148	0.1879	0.2021	0.0955	0.1452	**0.0367**	0.1039	0.1259	0.4406	0.2058	0.2556	0.0660	0.1588	0.0599	0.1056	0.1339
	300	0.1144	0.0937	0.0926	0.0751	0.0816	**0.0617**	0.0696	0.0761	0.1827	0.1348	0.1346	0.0945	0.1101	0.0706	0.0873	0.0996
6	**75**	2.4153	0.7634	1.4138	0.2240	0.7896	**0.1183**	0.5303	0.6681	3.0941	0.7216	1.7867	0.3617	0.8801	0.1223	0.5800	0.7410
	100	0.6968	0.4988	0.5114	0.3412	0.4257	**0.2586**	0.3446	0.3881	0.9751	0.6182	0.6784	0.3637	0.5324	0.2609	0.4153	0.4779
	200	0.2928	0.2491	0.2487	0.2096	0.2291	**0.1804**	0.2045	0.2179	0.4072	0.3266	0.3287	0.2569	0.2933	0.2112	0.2531	0.2749
	300	0.1721	0.1597	0.1595	0.1477	0.1540	**0.1372**	0.1458	0.1503	0.2392	0.2161	0.2158	0.1943	0.2056	0.1760	0.1911	0.1990
$\beta_1 = 1$																	
2	**75**	0.3711	0.1777	0.2794	0.0579	0.1900	**0.0169**	0.1413	0.1676	0.5337	0.1848	0.3700	0.0260	0.2082	0.0326	0.1408	0.1773
	100	0.1290	0.0984	0.1142	0.0721	0.1011	**0.0504**	0.0901	0.0961	0.2071	0.1328	0.1706	0.0757	0.1375	0.0558	0.1137	0.1266
	200	0.0825	0.0695	0.0764	0.0576	0.0711	**0.0487**	0.0661	0.0688	0.1153	0.0902	0.1035	0.0683	0.0930	0.0538	0.0838	0.0888
	300	0.0375	0.0343	0.0359	0.0312	0.0345	**0.0285**	0.0332	0.0339	0.0569	0.0497	0.0534	0.0430	0.0504	0.0377	0.0475	0.0491
6	**75**	0.7294	0.4563	0.5612	0.2677	0.4394	**0.1988**	0.3588	0.4023	0.8980	0.5137	0.6678	0.2738	0.4993	0.2012	0.3988	0.4530
	100	0.2923	0.2529	0.2675	0.2167	0.2506	**0.1887**	0.2327	0.2425	0.3601	0.3016	0.3232	0.2491	0.2979	0.2108	0.2722	0.2862
	200	0.1961	0.1768	0.1849	0.1587	0.1768	**0.1435**	0.1681	0.1729	0.2695	0.2355	0.2498	0.2040	0.2354	0.1791	0.2205	0.2287
	300	0.0732	0.0703	0.0713	0.0675	0.0701	**0.0648**	0.0686	0.0694	0.1024	0.0969	0.0989	0.0916	0.0966	0.0867	0.0938	0.0953

Table A4. AMSE of different estimators when τ = 10% and ρ = 0.90.

		Non-Robust Estimators								Robust Estimators							
p	n	PML	PRR	PL	PKL.1	PKL.2	PMKL.1	PMKL.2	PMKL.3	RPML	RPRR	RPL	RPKL.1	RPKL.2	RPMKL.1	RPMKL.2	RPMKL.3
$\beta_1 = -1$																	
2	**75**	2.8151	2.7129	2.7431	2.6275	2.6802	2.5729	2.6299	2.6567	0.9038	0.5328	0.7127	**0.4211**	0.5754	0.4326	0.5256	0.5513
	100	2.2468	2.2345	2.2309	2.2226	2.2146	2.2118	2.2004	2.2081	0.3264	0.3129	0.3164	**0.3053**	0.3125	0.3066	0.3125	0.3122
	200	2.1981	2.1922	2.1897	2.1864	2.1804	2.1809	2.1724	2.1768	0.2510	0.2454	0.2459	0.2413	0.2436	**0.2403**	0.2424	0.2430
	300	2.1846	2.1817	2.1802	2.1789	2.1748	2.1761	2.1702	2.1728	0.2223	0.2236	0.2223	0.2224	0.2226	**0.2221**	0.2235	0.2230
6	**75**	5.5013	5.2493	5.3696	5.0253	5.2672	4.8591	5.1647	5.2204	1.2307	0.8870	1.0395	0.6818	0.9021	**0.6150**	0.8098	0.8582
	100	5.2272	5.1188	5.1460	5.0159	5.0918	4.9269	5.0299	5.0637	0.9436	0.7822	0.8198	0.6574	0.7423	**0.5915**	0.6798	0.7127
	200	6.3219	6.2995	6.3091	6.2771	6.3001	6.2551	6.2893	6.2952	0.2018	0.2022	0.2017	0.2018	0.2021	**0.2017**	0.2027	0.2023
	300	3.8474	3.8389	3.8369	3.8304	3.8290	3.8221	3.8199	3.8249	0.1943	0.1933	0.1934	0.1937	0.1938	**0.1929**	0.1945	0.1941
$\beta_1 = 0$																	
2	**75**	1.7949	1.7536	1.7623	1.7142	1.7433	1.6796	1.7197	1.7326	0.3308	0.2725	0.2781	0.2234	0.2490	**0.1938**	0.2220	0.2365
	100	2.3420	2.3188	2.3243	2.2960	2.3146	2.2742	2.3014	2.3087	0.1601	0.1507	0.1502	0.1419	0.1451	**0.1348**	0.1390	0.1423
	200	2.0699	2.0629	2.0645	2.0559	2.0615	2.0490	2.0573	2.0596	0.0915	0.0901	0.0900	0.0888	0.0893	**0.0876**	0.0883	0.0889
	300	2.1435	2.1390	2.1396	2.1345	2.1377	2.1301	2.1349	2.1365	0.0742	0.0738	0.0738	0.0734	0.0736	**0.0731**	0.0733	0.0734
6	**75**	5.4692	5.3284	5.4449	5.1935	5.4073	5.0744	5.3773	5.3938	0.6037	0.5385	0.5492	0.4796	0.5214	**0.4349**	0.4872	0.5058
	100	4.6669	4.6116	4.6458	4.5569	4.6269	4.5042	4.6073	4.6181	0.2507	0.2408	0.2405	0.2312	0.2359	**0.2225**	0.2291	0.2328
	200	4.3847	4.3664	4.3809	4.3482	4.3758	4.3302	4.3713	4.3738	0.1186	0.1167	0.1167	0.1149	0.1158	**0.1131**	0.1145	0.1152
	300	3.2519	3.2432	3.2482	3.2346	3.2450	3.2259	3.2416	3.2435	0.0746	0.0741	0.0741	0.0735	0.0738	**0.0730**	0.0734	0.0736
$\beta_1 = 1$																	
2	**75**	2.3046	2.2816	2.3015	2.2587	2.2962	2.2363	2.2920	2.2943	0.0760	0.0714	0.0743	0.0671	0.0726	**0.0631**	0.0710	0.0719
	100	1.9057	1.8913	1.9031	1.8768	1.8992	1.8626	1.8960	1.8977	0.0579	0.0552	0.0568	0.0525	0.0558	**0.0500**	0.0548	0.0553
	200	2.6469	2.6372	2.6457	2.6275	2.6436	2.6178	2.6420	2.6429	0.0365	0.0354	0.0361	0.0343	0.0357	**0.0333**	0.0352	0.0355
	300	2.3276	2.3220	2.3269	2.3165	2.3257	2.3109	2.3248	2.3253	0.0302	0.0296	0.0299	0.0289	0.0297	**0.0283**	0.0294	0.0296
6	**75**	6.4395	6.3180	6.4335	6.1987	6.4162	6.0861	6.4046	6.4110	0.2974	0.2815	0.2876	0.2663	0.2810	**0.2525**	0.2733	0.2775
	100	6.7408	6.6906	6.7390	6.6408	6.7332	6.5921	6.7294	6.7315	0.1205	0.1166	0.1182	0.1128	0.1166	**0.1092**	0.1147	0.1158
	200	3.2133	3.2004	3.2122	3.1875	3.2098	3.1747	3.2081	3.2091	0.0432	0.0426	0.0428	0.0420	0.0426	**0.0414**	0.0422	0.0424
	300	4.2628	4.2545	4.2624	4.2461	4.2612	4.2378	4.2604	4.2608	0.0349	0.0345	0.0347	0.0341	0.0345	**0.0337**	0.0343	0.0344

Table A5. AMSE of different estimators when τ = 10% and ρ = 0.95.

		Non-Robust Estimators								Robust Estimators							
p	n	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	3.3888	2.9505	3.2041	2.6709	3.0320	2.5727	2.9161	2.9774	1.5794	0.5247	1.1028	0.5514	0.6926	**0.4499**	0.5854	0.6438
	100	2.4924	2.4343	2.4437	2.3816	2.4025	2.3415	2.3668	2.3860	0.4541	0.3679	0.3928	0.3092	0.3569	**0.2935**	0.3349	0.3462
	200	2.3163	2.2968	2.2937	2.2784	2.2716	2.2623	2.2530	2.2630	0.3950	0.3516	0.3584	0.3171	0.3378	**0.2998**	0.3211	0.3299
	300	2.3972	2.3887	2.3854	2.3804	2.3722	2.3728	2.3611	2.3671	0.2877	0.2759	0.2760	0.2659	0.2698	**0.2601**	0.2642	0.2671
6	**75**	8.7786	7.3917	8.5135	6.3566	8.1630	5.8484	7.8967	8.0413	2.0824	1.2287	1.6898	0.7803	1.3789	**0.6353**	1.1791	1.2848
	100	6.3576	5.9041	6.1668	5.5085	6.0037	5.2361	5.8496	5.9332	1.1873	0.9021	0.9790	0.6929	0.8492	**0.5980**	0.7486	0.8018
	200	6.0540	5.8437	5.9603	5.6410	5.8834	5.4593	5.8022	5.8467	0.5489	0.5044	0.5025	0.4642	0.4752	**0.4336**	0.4471	0.4623
	300	4.7909	4.7654	4.7629	4.7401	4.7434	4.7156	4.7204	4.7330	0.3169	0.3093	0.3083	0.3021	0.3030	**0.2959**	0.2970	0.3002
$\beta_1 = 0$																	
2	**75**	2.5781	2.4219	2.5088	2.2843	2.4546	2.1883	2.4010	2.4302	0.4467	0.3166	0.3491	0.2193	0.2911	**0.1772**	0.2462	0.2702
	100	3.4023	3.3403	3.3758	3.2807	3.3535	3.2274	3.3302	3.3430	0.2428	0.2103	0.2123	0.1816	0.1961	**0.1614**	0.1789	0.1882
	200	2.4161	2.4023	2.4069	2.3887	2.4013	2.3757	2.3940	2.3980	0.1311	0.1253	0.1249	0.1198	0.1218	**0.1151**	0.1178	0.1200
	300	2.2667	2.2600	2.2619	2.2533	2.2591	2.2467	2.2553	2.2574	0.0870	0.0853	0.0852	0.0835	0.0842	**0.0820**	0.0830	0.0837
6	**75**	8.2479	7.9403	8.1862	7.6465	8.1023	7.3883	8.0317	8.0705	0.8803	0.7381	0.7894	0.6195	0.7353	**0.5444**	0.6782	0.7091
	100	5.1983	5.0077	5.1493	4.8252	5.0904	4.6644	5.0385	5.0670	0.5980	0.5291	0.5362	0.4664	0.5060	**0.4187**	0.4686	0.4889
	200	3.8541	3.8144	3.8402	3.7752	3.8267	3.7374	3.8132	3.8206	0.1726	0.1663	0.1662	0.1601	0.1634	**0.1544**	0.1591	0.1614
	300	4.6406	4.6173	4.6342	4.5941	4.6267	4.5713	4.6198	4.6236	0.1052	0.1035	0.1035	0.1019	0.1028	**0.1003**	0.1016	0.1022
$\beta_1 = 1$																	
2	**75**	2.5475	2.4686	2.5351	2.3927	2.5147	2.3253	2.4988	2.5075	0.1578	0.1345	0.1476	0.1133	0.1382	**0.0972**	0.1296	0.1343
	100	2.6955	2.6308	2.6869	2.5684	2.6713	2.5123	2.6595	2.6660	0.1280	0.1104	0.1198	0.0942	0.1126	**0.0818**	0.1058	0.1095
	200	2.0196	2.0091	2.0180	1.9987	2.0153	1.9885	2.0132	2.0144	0.0499	0.0478	0.0490	0.0458	0.0482	**0.0439**	0.0474	0.0478
	300	2.3524	2.3453	2.3514	2.3383	2.3498	2.3313	2.3485	2.3492	0.0365	0.0354	0.0360	0.0344	0.0356	**0.0334**	0.0352	0.0354
6	**75**	8.0775	7.8114	8.0650	7.5573	8.0280	7.3356	8.0036	8.0170	0.3448	0.3131	0.3257	0.2837	0.3124	**0.2603**	0.2982	0.3060
	100	8.2781	7.9770	8.2600	7.6855	8.2128	7.4207	8.1805	8.1983	0.2696	0.2501	0.2568	0.2316	0.2486	**0.2157**	0.2390	0.2442
	200	5.3931	5.3683	5.3921	5.3437	5.3888	5.3194	5.3867	5.3879	0.0659	0.0647	0.0651	0.0635	0.0646	**0.0623**	0.0640	0.0644
	300	3.4826	3.4676	3.4811	3.4526	3.4780	3.4378	3.4757	3.4770	0.0349	0.0346	0.0347	0.0342	0.0345	**0.0339**	0.0343	0.0344

Table A6. AMSE of different estimators when τ = 10% and ρ = 0.99.

		Non-Robust Estimators								Robust Estimators							
p	n	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	7.6983	2.7794	6.5951	4.2084	4.5580	2.5064	3.9256	4.2663	6.9211	0.6871	4.2272	5.6295	0.9199	**0.5899**	0.6382	0.7954
	100	4.3229	2.7826	3.9236	2.3063	3.4104	2.2172	3.1490	3.2879	1.7406	0.3845	1.0938	0.4547	0.5418	**0.2652**	0.4087	0.4825
	200	4.6199	3.2958	4.2886	2.6769	3.8690	2.5881	3.6292	3.7569	1.5089	0.4168	0.9975	0.3099	0.5630	**0.2296**	0.4303	0.5022
	300	2.5914	2.2930	2.4496	2.0744	2.3270	1.9817	2.2385	2.2855	0.8450	0.4257	0.6006	0.2222	0.4356	**0.1984**	0.3536	0.3971
6	**75**	17.8758	5.4743	16.2241	6.2223	12.1308	3.4949	10.4488	11.3519	9.6092	1.0961	6.9501	3.7993	3.1357	**0.5142**	2.2387	2.7283
	100	30.9101	9.9383	29.8493	4.4672	25.7472	3.0131	23.6760	24.7998	8.3612	1.3260	6.2391	1.8421	3.3919	**0.3936**	2.4744	2.9681
	200	7.8012	6.1385	7.4009	4.9089	6.9229	4.3489	6.5624	6.7572	2.2209	1.3036	1.6985	0.7118	1.3379	**0.5087**	1.0812	1.2171
	300	19.4374	12.9020	19.1773	8.2888	18.3326	6.5513	17.8176	18.0994	1.5589	1.1003	1.2336	0.7473	1.0387	**0.5783**	0.8733	0.9616
$\beta_1 = 0$																	
2	**75**	5.5753	3.3358	5.2472	2.4257	4.6740	2.2674	4.3441	4.5219	1.4670	0.3390	0.9263	0.2141	0.4875	**0.1159**	0.3461	0.4225
	100	6.2799	3.7634	5.9554	2.6836	5.3436	2.5355	4.9898	5.1808	1.1138	0.3057	0.6889	0.1134	0.3770	**0.0782**	0.2639	0.3248
	200	3.4328	2.7999	3.2704	2.3457	3.0916	2.1684	2.9572	3.0298	0.7277	0.3324	0.4826	0.1240	0.3288	**0.0895**	0.2446	0.2892
	300	2.9434	2.8158	2.8904	2.6999	2.8464	2.6125	2.8032	2.8268	0.2995	0.2283	0.2364	0.1696	0.2037	**0.1366**	0.1730	0.1894
6	**75**	19.9384	13.9564	19.6483	10.1563	18.7188	8.9046	18.1655	18.4679	2.9218	1.1515	2.2673	0.6577	1.6813	**0.3661**	1.3663	1.5343
	100	25.2893	17.1104	25.0190	11.3518	24.0558	9.0782	23.4773	23.7941	2.0787	1.2655	1.6641	0.7496	1.3674	**0.5643**	1.1556	1.2688
	200	12.7306	11.0780	12.5925	9.6043	12.2904	8.5403	12.0804	12.1956	0.9712	0.7804	0.8150	0.6174	0.7356	**0.5126**	0.6496	0.6960
	300	6.4412	6.0621	6.3591	5.7088	6.2493	5.4212	6.1585	6.2082	0.7052	0.5906	0.5940	0.4882	0.5425	**0.4148**	0.4807	0.5141
$\beta_1 = 1$																	
2	**75**	4.9323	3.7959	4.8396	2.9747	4.6181	2.6396	4.4770	4.5540	0.4141	0.2352	0.3335	0.1112	0.2554	**0.0713**	0.2057	0.2325
	100	4.2485	3.4853	4.1805	2.8835	4.0329	2.5785	3.9343	3.9882	0.3903	0.2242	0.3181	0.1078	0.2461	**0.0688**	0.1998	0.2247
	200	3.1450	2.9440	3.1207	2.7617	3.0744	2.6251	3.0410	3.0593	0.2124	0.1584	0.1880	0.1131	0.1651	**0.0865**	0.1466	0.1566
	300	2.8173	2.7504	2.8089	2.6864	2.7931	2.6308	2.7813	2.7877	0.1114	0.0967	0.1048	0.0832	0.0989	**0.0728**	0.0933	0.0963
6	**75**	33.1927	27.2297	33.1204	22.4099	32.7336	19.6590	32.5085	32.6323	1.4230	0.9664	1.2587	0.6317	1.1145	**0.4815**	1.0027	1.0632
	100	15.6867	13.0540	15.5832	10.7895	15.2426	9.3298	15.0286	15.1461	0.8285	0.6600	0.7379	0.5144	0.6681	**0.4192**	0.6040	0.6388
	200	14.6044	13.9161	14.5745	13.2549	14.4838	12.6678	14.4242	14.4570	0.2346	0.2194	0.2249	0.2049	0.2186	**0.1920**	0.2112	0.2152
	300	10.0555	9.8791	10.0494	9.7061	10.0290	9.5432	10.0158	10.0230	0.1360	0.1310	0.1329	0.1261	0.1309	**0.1215**	0.1284	0.1298

Table A7. AMSE of different estimators when τ = 20% and ρ = 0.90.

		Non-Robust Estimators								Robust Estimators							
p	n	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	4.2811	4.2152	4.2269	4.1550	4.1848	4.1073	4.1432	4.1658	1.5777	1.1351	1.3947	0.9430	1.2395	**0.9152**	1.1629	1.2030
	100	3.6474	3.6154	3.6129	3.5850	3.5813	3.5587	3.5525	3.5681	1.2085	1.0481	1.1126	0.9365	1.0470	**0.8952**	1.0057	1.0272
	200	4.8739	4.8636	4.8626	4.8533	4.8559	4.8431	4.8471	4.8520	0.7430	0.7349	0.7343	0.7275	0.7286	**0.7220**	0.7247	0.7267
	300	4.1308	4.1266	4.1251	4.1224	4.1199	4.1182	4.1145	4.1175	0.7172	0.7168	0.7172	0.7166	0.7186	**0.7165**	0.7205	0.7194
6	**75**	11.9761	11.8639	11.9647	11.7544	11.9414	11.6521	11.9243	11.9337	1.9450	1.7390	1.8645	1.5778	1.7979	**1.4826**	1.7393	1.7708
	100	9.4021	9.3403	9.3917	9.2795	9.3754	9.2213	9.3623	9.3695	1.6099	1.5104	1.5518	1.4272	1.5069	**1.3709**	1.4656	1.4878
	200	7.0164	6.9899	7.0040	6.9636	6.9941	6.9379	6.9830	6.9891	0.7382	0.7274	0.7259	0.7172	0.7170	0.7081	**0.7079**	0.7128
	300	6.5249	6.5112	6.5181	6.4976	6.5128	6.4841	6.5068	6.5101	0.6546	0.6533	0.6529	0.6520	0.6515	0.6509	**0.6501**	0.6509
$\beta_1 = 0$																	
2	**75**	3.9084	3.8727	3.8969	3.8375	3.8849	3.8036	3.8734	3.8797	0.3923	0.3741	0.3753	0.3569	0.3666	**0.3425**	0.3555	0.3616
	100	4.2398	4.2123	4.2339	4.1850	4.2259	4.1586	4.2191	4.2229	0.3862	0.3711	0.3713	0.3567	0.3638	**0.3446**	0.3541	0.3594
	200	4.2903	4.2788	4.2880	4.2673	4.2848	4.2559	4.2820	4.2835	0.2921	0.2893	0.2892	0.2866	0.2878	**0.2840**	0.2858	0.2869
	300	3.9659	3.9590	3.9644	3.9520	3.9624	3.9451	3.9607	3.9616	0.2726	0.2712	0.2711	0.2699	0.2704	**0.2686**	0.2694	0.2700
6	**75**	6.9747	6.8894	6.9671	6.8055	6.9505	6.7259	6.9385	6.9451	1.7863	1.6435	1.7121	1.5114	1.6611	**1.4051**	1.6041	1.6352
	100	6.0737	6.0142	6.0663	5.9555	6.0528	5.8988	6.0425	6.0482	0.7269	0.6967	0.6985	0.6678	0.6844	**0.6423**	0.6651	0.6756
	200	4.9827	4.9663	4.9804	4.9499	4.9766	4.9337	4.9735	4.9752	0.2703	0.2682	0.2681	0.2661	0.2671	**0.2641**	0.2656	0.2664
	300	9.5978	9.5800	9.5969	9.5623	9.5943	9.5447	9.5926	9.5935	0.2053	0.2045	0.2044	0.2036	0.2040	**0.2028**	0.2034	0.2038
$\beta_1 = 1$																	
2	**75**	3.7759	3.7493	3.7743	3.7227	3.7702	3.6965	3.7673	3.7689	0.1495	0.1430	0.1462	0.1366	0.1437	**0.1306**	0.1409	0.1424
	100	3.1071	3.0877	3.1056	3.0685	3.1022	3.0496	3.0997	3.1011	0.1947	0.1834	0.1911	0.1724	0.1871	**0.1627**	0.1835	0.1855
	200	3.6246	3.6155	3.6240	3.6063	3.6225	3.5973	3.6215	3.6220	0.1054	0.1035	0.1046	0.1016	0.1038	**0.0997**	0.1031	0.1035
	300	4.0732	4.0668	4.0728	4.0605	4.0718	4.0542	4.0711	4.0715	0.0923	0.0912	0.0919	0.0901	0.0914	**0.0891**	0.0910	0.0912
6	**75**	8.5310	8.4687	8.5294	8.4069	8.5231	8.3468	8.5192	8.5214	0.6294	0.6068	0.6192	0.5849	0.6115	**0.5648**	0.6029	0.6076
	100	9.3580	9.3198	9.3572	9.2817	9.3539	9.2441	9.3518	9.3529	0.4154	0.4049	0.4090	0.3946	0.4048	**0.3848**	0.3996	0.4024
	200	7.7307	7.7142	7.7302	7.6977	7.7286	7.6813	7.7276	7.7281	0.1410	0.1392	0.1398	0.1374	0.1391	**0.1357**	0.1381	0.1387
	300	5.8651	5.8569	5.8648	5.8486	5.8638	5.8404	5.8632	5.8635	0.0689	0.0685	0.0686	0.0680	0.0684	**0.0676**	0.0682	0.0683

Table A8. AMSE of different estimators when τ = 20% and ρ = 0.95.

		Non-Robust Estimators								Robust Estimators							
p	**n**	**PML**	**PRR**	**PL**	**PKL.1**	**PKL.2**	**PMKL.1**	**PMKL.2**	**PMKL.3**	**RPML**	**RPRR**	**RPL**	**RPKL.1**	**RPKL.2**	**RPMKL.1**	**RPMKL.2**	**RPMKL.3**
$\beta_1 = -1$																	
2	**75**	4.7600	4.4950	4.6341	4.2936	4.5278	4.1895	4.4398	4.4870	2.4622	1.1336	2.0155	1.0328	1.5290	**0.8965**	1.3477	1.4451
	100	3.9945	3.8575	3.9059	3.7424	3.8340	3.6696	3.7725	3.8055	1.7513	1.1295	1.4869	0.8688	1.2596	**0.8417**	1.1482	1.2069
	200	4.1293	4.1104	4.1081	4.0919	4.0911	4.0749	4.0735	4.0831	0.9183	0.8506	0.8623	0.7935	0.8225	**0.7589**	0.7934	0.8088
	300	4.4117	4.4024	4.4001	4.3931	4.3910	4.3843	4.3811	4.3865	0.8111	0.7927	0.7915	0.7760	0.7762	**0.7635**	0.7653	0.7710
6	**75**	15.8352	14.2632	15.6967	12.8762	15.4018	11.8915	15.1954	15.3085	4.5094	2.8839	4.0424	1.9197	3.5358	**1.5812**	3.1921	3.3762
	100	9.3047	9.0463	9.2266	8.7990	9.1461	8.5807	9.0700	9.1117	2.3672	2.0308	2.1942	1.7475	2.0613	**1.5663**	1.9398	2.0053
	200	9.9485	9.8540	9.9277	9.7614	9.8997	9.6740	9.8757	9.8889	1.2990	1.2189	1.2271	1.1454	1.1761	**1.0876**	1.1260	1.1531
	300	7.4891	7.4515	7.4711	7.4143	7.4568	7.3781	7.4409	7.4497	0.9795	0.9669	0.9640	0.9547	0.9519	0.9435	**0.9397**	0.9464
$\beta_1 = 0$																	
2	**75**	4.3483	4.2668	4.3298	4.1894	4.3050	4.1225	4.2843	4.2957	0.6594	0.5545	0.5894	0.4669	0.5472	**0.4148**	0.5062	0.5283
	100	5.0517	4.9934	5.0395	4.9370	5.0226	4.8858	5.0084	5.0162	0.5562	0.4812	0.4982	0.4168	0.4660	**0.3756**	0.4328	0.4507
	200	4.1082	4.0891	4.1032	4.0703	4.0972	4.0524	4.0918	4.0948	0.3477	0.3344	0.3341	0.3218	0.3275	**0.3112**	0.3189	0.3236
	300	4.9098	4.8996	4.9082	4.8894	4.9055	4.8793	4.9034	4.9046	0.2687	0.2656	0.2655	0.2626	0.2640	**0.2597**	0.2618	0.2630
6	**75**	13.1473	12.9369	13.1402	12.7339	13.1158	12.5513	13.1002	13.1088	2.7049	2.2216	2.6016	1.8759	2.5123	**1.6813**	2.4261	2.4731
	100	11.6828	11.5088	11.6756	11.3410	11.6529	11.1905	11.6381	11.6462	1.2241	1.0895	1.1433	0.9680	1.0939	**0.8776**	1.0383	1.0686
	200	9.5354	9.4580	9.5291	9.3813	9.5148	9.3069	9.5046	9.5102	0.4676	0.4570	0.4567	0.4465	0.4518	**0.4366**	0.4442	0.4483
	300	7.2326	7.2071	7.2303	7.1817	7.2252	7.1567	7.2216	7.2236	0.3417	0.3371	0.3370	0.3326	0.3348	**0.3282**	0.3314	0.3333
$\beta_1 = 1$																	
2	**75**	3.5202	3.4757	3.5170	3.4321	3.5092	3.3910	3.5038	3.5068	0.2255	0.2063	0.2169	0.1881	0.2094	**0.1727**	0.2019	0.2060
	100	4.2360	4.2033	4.2341	4.1710	4.2292	4.1399	4.2258	4.2276	0.2063	0.1927	0.1995	0.1797	0.1940	**0.1682**	0.1883	0.1914
	200	4.0112	3.9965	4.0103	3.9819	4.0080	3.9676	4.0064	4.0072	0.1453	0.1402	0.1432	0.1351	0.1413	**0.1305**	0.1393	0.1404
	300	3.6987	3.6917	3.6982	3.6847	3.6970	3.6778	3.6962	3.6967	0.1103	0.1084	0.1094	0.1065	0.1087	**0.1047**	0.1080	0.1084
6	**75**	8.7521	8.5429	8.7456	8.3411	8.7224	8.1601	8.7078	8.7158	1.9775	1.5545	1.9041	1.2453	1.8251	**1.0817**	1.7574	1.7944
	100	7.9852	7.8788	7.9818	7.7743	7.9698	7.6756	7.9622	7.9664	0.5264	0.5006	0.5110	0.4757	0.5004	**0.4531**	0.4882	0.4949
	200	6.8966	6.8638	6.8953	6.8312	6.8914	6.7990	6.8888	6.8902	0.1741	0.1725	0.1730	0.1708	0.1723	**0.1692**	0.1714	0.1719
	300	6.8789	6.8641	6.8785	6.8493	6.8768	6.8346	6.8758	6.8764	0.0924	0.0917	0.0919	0.0910	0.0916	**0.0903**	0.0912	0.0914

Table A9. AMSE of different estimators when τ = 20% and ρ = 0.99.

		Non-Robust Estimators								Robust Estimators							
p	n	PML	PRR	PL	PKL.1	PKL.2	PMKL.1	PMKL.2	PMKL.3	RPML	RPRR	RPL	RPKL.1	RPKL.2	RPMKL.1	RPMKL.2	RPMKL.3
$\beta_1 = -1$																	
2	**75**	8.2256	4.5239	7.4637	4.7939	6.1482	4.0340	5.6287	5.9065	9.6714	1.0014	7.1708	7.5424	2.4720	**0.8927**	1.8154	2.1930
	100	6.3867	4.0095	5.7921	3.7622	4.9507	3.5104	4.5881	4.7811	5.9421	0.9001	4.3733	3.7132	1.9181	**0.8936**	1.4715	1.7250
	200	4.9827	4.5520	4.8073	4.2494	4.6503	4.1339	4.5354	4.5968	2.2739	0.9270	1.7488	0.7129	1.2302	**0.6307**	1.0344	1.1388
	300	4.2691	4.0711	4.1582	3.9128	4.0661	3.8284	3.9940	4.0325	1.6416	0.9744	1.3242	0.6983	1.0622	**0.6690**	0.9398	1.0042
6	**75**	23.4522	14.0226	22.7194	10.4698	20.6796	9.5688	19.5589	20.1665	15.3713	2.2675	13.2129	6.2161	8.5896	**1.1221**	7.0071	7.8596
	100	20.9071	12.1156	20.2108	8.5468	18.3749	7.6725	17.3405	17.9015	14.9886	2.5761	13.1327	3.9930	8.8957	**1.0292**	7.2259	8.1218
	200	13.3872	10.3492	13.0478	8.0855	12.4323	7.0585	12.0113	12.2405	4.7162	2.3540	3.9190	1.1830	3.0868	**0.9309**	2.6083	2.8626
	300	12.6973	11.6510	12.5282	10.7154	12.2676	10.0420	12.0676	12.1770	2.5631	1.9641	2.2348	1.4880	1.9864	**1.2389**	1.7791	1.8899
$\beta_1 = 0$																	
2	**75**	6.3345	5.4254	6.2082	4.7938	5.9754	4.5631	5.8245	5.9065	1.4397	0.6212	1.1090	0.3030	0.8141	**0.2579**	0.6623	0.7431
	100	5.3272	4.7492	5.2243	4.3130	5.0595	4.1183	4.9468	5.0081	1.2861	0.6264	0.9760	0.3171	0.7348	**0.2711**	0.6019	0.6725
	200	5.7303	5.5492	5.7049	5.3865	5.6588	5.2681	5.6253	5.6437	0.8218	0.5881	0.6807	0.4193	0.5895	**0.3532**	0.5179	0.5561
	300	4.2193	4.1240	4.1935	4.0362	4.1621	3.9673	4.1357	4.1502	0.6632	0.5241	0.5618	0.4121	0.5043	**0.3529**	0.4510	0.4796
6	**75**	17.2258	10.0177	16.6082	6.2957	15.0366	5.4328	14.1336	14.6238	13.0179	2.2961	11.2937	3.3488	7.7460	**0.6167**	6.3708	7.1103
	100	24.1181	19.0853	23.9625	15.0367	23.4112	12.7783	23.0740	23.2590	6.1646	3.4790	5.6738	2.2571	4.9396	**1.8599**	4.5242	4.7479
	200	16.9612	16.2111	16.9280	15.5013	16.8258	14.8985	16.7592	16.7958	1.3386	1.1783	1.2242	1.0321	1.1597	**0.9201**	1.0843	1.1252
	300	10.1440	9.7122	10.1095	9.3090	10.0269	8.9802	9.9701	10.0013	1.3752	1.1803	1.2202	1.0058	1.1377	**0.8793**	1.0423	1.0940
$\beta_1 = 1$																	
2	**75**	6.2201	4.3197	6.0989	3.2088	5.7411	2.9611	5.5326	5.6463	1.7794	0.4542	1.3864	0.1698	0.8624	**0.1026**	0.6519	0.7650
	100	5.7088	5.3243	5.6864	4.9868	5.6238	4.7575	5.5830	5.6054	0.5920	0.3872	0.5235	0.2355	0.4444	**0.1736**	0.3900	0.4194
	200	5.7302	5.5733	5.7229	5.4274	5.7004	5.3102	5.6858	5.6939	0.3718	0.3010	0.3404	0.2397	0.3112	**0.2000**	0.2863	0.2999
	300	5.0166	4.9712	5.0143	4.9273	5.0076	4.8874	5.0032	5.0056	0.1929	0.1778	0.1859	0.1637	0.1798	**0.1520**	0.1738	0.1771
6	**75**	39.8806	34.2196	39.8184	29.3812	39.4801	26.1974	39.2827	39.3913	2.9709	2.2537	2.8184	1.7068	2.6438	**1.4275**	2.5056	2.5807
	100	24.4446	22.7553	24.4089	21.1706	24.2585	19.8582	24.1665	24.2171	1.7611	1.4258	1.6420	1.1440	1.5330	**0.9669**	1.4366	1.4890
	200	11.9006	11.2627	11.8796	10.6617	11.8049	10.1589	11.7577	11.7836	0.9173	0.8057	0.8547	0.7035	0.8087	**0.6246**	0.7615	0.7872
	300	12.2014	11.9531	12.1946	11.7105	12.1692	11.4840	12.1532	12.1620	0.4140	0.3936	0.4015	0.3738	0.3931	**0.3558**	0.3832	0.3886

References

1. Hadi, A.S. A modification of a method for the detection of outliers in multivariate samples. *J. R. Stat. Soc. Ser. B (Methodol.)* **1994**, *56*, 393–396. [CrossRef]
2. Lawrence, K.D. *Robust Regression: Analysis and Applications*; Routledge: Oxfordshire, UK, 2019.
3. Wu, J.; Asar, Y.; Arashi, M. On the restricted almost unbiased Liu estimator in the logistic regression model. *Commun. Stat. Theory Methods* **2018**, *47*, 4389–4401. [CrossRef]
4. Arashi, M.; Saleh, A.M.E.; Kibria, B.G. *Theory of Ridge Regression Estimation with Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2019.
5. Algamal, Z.Y.; Lukman, A.F.; Abonazel, M.R.; Awwad, F.A. Performance of the Ridge and Liu Estimators in the zero-inflated Bell Regression Model. *J. Math.* **2022**, *2022*, 9503460. [CrossRef]
6. Rasheed, H.A.; Sadik, N.J.; Algamal, Z.Y. Jackknifed Liu-type estimator in the Conway-Maxwell Poisson regression model. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 3153–3168.
7. Majid, A.; Amin, M.; Akram, M.N. On the Liu estimation of Bell regression model in the presence of multicollinearity. *J. Stat. Comput. Simul.* **2022**, *92*, 262–282. [CrossRef]
8. Asar, Y.; Algamal, Z. A New Two-parameter Estimator for the Gamma Regression Model. *Stat. Optim. Inf. Comput.* **2022**, *10*, 750–761. [CrossRef]
9. Jabur, D.M.; Rashad, N.K.; Algamal, Z.Y. Jackknifed Liu-type estimator in the negative binomial regression model. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 2675–2684.
10. Månsson, K.; Shukur, G. A Poisson ridge regression estimator. *Econ. Model.* **2011**, *28*, 1475–1481. [CrossRef]
11. Månsson, K.; Kibria, B.G.; Sjolander, P.; Shukur, G. Improved Liu estimators for the Poisson regression model. *Int. J. Stat. Probab.* **2012**, *1*, 2–6. [CrossRef]
12. Lukman, A.F.; Adewuyi, E.; Månsson, K.; Kibria, B.G. A new estimator for the multicollinear Poisson regression model: Simulation and application. *Sci. Rep.* **2021**, *11*, 3732. [CrossRef]
13. Aladeitan, B.B.; Adebimpe, O.; Lukman, A.F.; Oludoun, O.; Abiodun, O.E. Modified Kibria-Lukman (MKL) estimator for the Poisson Regression Model: Application and simulation. *F1000Res.* **2021**, *10*, 548. [CrossRef] [PubMed]
14. Amin, M.; Akram, M.N.; Kibria, B.G. A new adjusted Liu estimator for the Poisson regression model. *Concurr. Comput. Pract. Exp.* **2021**, *33*, e6340. [CrossRef]
15. Yehia, E.G. On the Restricted Poisson Ridge Regression Estimator. *Sci. J. Appl. Math. Stat.* **2021**, *9*, 106. [CrossRef]
16. Jadhav, N.H. A new linearized ridge Poisson estimator in the presence of multicollinearity. *J. Appl. Stat.* **2022**, *49*, 2016–2034. [CrossRef] [PubMed]
17. Cantoni, E.; Ronchetti, E. Robust inference for generalized linear models. *J. Am. Stat. Assoc.* **2001**, *96*, 1022–1030. [CrossRef]
18. Hall, D.B.; Shen, J. Robust estimation for zero-inflated Poisson regression. *Scand. J. Stat.* **2010**, *37*, 237–252. [CrossRef]
19. Chen, W.; Shi, J.; Qian, L.; Azen, S.P. Comparison of robustness to outliers between robust poisson models and log-binomial models when estimating relative risks for common binary outcomes: A simulation study. *BMC Med. Res. Methodol.* **2014**, *14*, 82. [CrossRef]
20. Hosseinian, S.; Morgenthaler, S. Weighted maximum likelihood estimates in Poisson regression. In Proceedings of the International Conference on Robust Statistics, Valladolid, Spain, 27 June–1 July 2011.
21. Chen, W.; Qian, L.; Shi, J.; Franklin, M. Comparing performance between log-binomial and robust Poisson regression models for estimating risk ratios under model misspecification. *BMC Med. Res. Methodol.* **2018**, *18*, 63. [CrossRef]
22. Abonazel, M.R.; Saber, O. A comparative study of robust estimators for Poisson regression model with outliers. *J. Stat. Appl. Probab.* **2020**, *9*, 279–286.
23. Marazzi, A. Improving the efficiency of robust estimators for the generalized linear model. *Stats* **2021**, *4*, 88–107. [CrossRef]
24. Abonazel, M.R.; El-Sayed, S.M.; Saber, O.M. Performance of robust count regression estimators in the case of overdispersion, zero inflated, and outliers: Simulation study and application to German health data. *Commun. Math. Biol. Neurosci.* **2021**, *2021*, 55.
25. Hosseinian, S. Robust Inference for Generalized Linear Models: Binary and Poisson Regression. Ph.D. Thesis, EPFL, Lausanne, Switzerland, 2009.
26. Abonazel, M.R.; Dawoud, I. Developing Robust Ridge Estimators for Poisson Regression Model. *Concurr. Comput. Pract. Exp.* **2022**, 34. [CrossRef]
27. Arum, K.C.; Ugwuowo, F.I.; Oranye, H.E. Robust Modified Jackknife Ridge Estimator for the Poisson Regression Model with Multicollinearity and outliers. *Sci. Afr.* **2022**, *17*, e01386. [CrossRef]
28. Kaçıranlar, S.; Dawoud, I. On the performance of the Poisson and the negative binomial ridge predictors. *Commun. Stat. Simul. Comput.* **2018**, *47*, 1751–1770. [CrossRef]
29. Dawoud, I.; Abonazel, M.R. Robust Dawoud-Kibria estimator for handling multicollinearity and outliers in the linear regression model. *J. Stat. Comput. Simul.* **2021**, *91*, 3678–3692. [CrossRef]
30. Awwad, F.A.; Dawoud, I.; Abonazel, M.R. Development of robust Özkale-Kaçiranlar and Yang-Chang estimators for regression models in the presence of multicollinearity and outliers. *Concurr. Comput. Pract. Exp.* **2022**, *34*, e6779. [CrossRef]
31. Myers, R.H.; Montgomery, D.C.; Vining, G.G.; Robinson, T.J. *Generalized Linear Models: With Applications in Engineering and the Sciences*; John Wiley & Sons: Hoboken, NJ, USA, 2012; Volume 791.

32. Cameron, A.C.; Trivedi, P.K. *Regression Analysis of Count Data*, 2nd ed.; Econometric Society Monograph No. 53; Cambridge University Press: Cambridge, UK, 2013.
33. Månsson, K.; Kibria, B.G. Estimating the unrestricted and restricted Liu estimators for the Poisson regression model: Method and application. *Comput. Econ.* **2021**, *58*, 311–326. [CrossRef]