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Some Inequalities for Certain *p*-Valent Functions Connected with the Combination Binomial Series and Confluent Hypergeometric Function

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Abstract: The present paper deals with a new differential operator denoted by $F_{p,t}^{\delta,n,b,c,m,\beta}$, whose certain properties are deduced by using well-known earlier studies regarding differential inequalities and the Caratheodory function. The new introduced operator is defined by making use of a linear combination of the binomial series and confluent hypergeometric function. In addition, by using special values of the parameters, we establish certain results concretized in specific corollaries, which provide useful inequalities. Studying these properties by using various types of operators is a technique that is widely used.

Keywords: *p*-valent; confluent hypergeometric function; binomial series

MSC: 30C45

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1. Introduction, Definition, and Preliminaries

When L. de Branges employed hypergeometric functions in the proof of the famous Bieberbach conjecture [1], interest in the study of hypergeometric functions and their connection to the theory of univalent functions resurfaced. Recently, the Confluent (Kummer) hypergeometric function was studied from numerous angles. An analytical study on Mittag–Leffler confluent hypergeometric functions was made in [2] using a fractional integral operator. Conditions related to univalence of the Confluent (Kummer) hypergeometric function were established in [3] and its applications on certain classes of univalent functions are shown in [4]. These operators have been shown to be particularly beneficial in a wide range of applicability by modeling various phenomena and processes.

In order to develop the present study, the usual definitions are used.

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$\mathcal{F}(\zeta) = \zeta^p + \sum_{k=p+1}^{\infty} a_k \zeta^k, \quad (p \in \mathbb{N} = \{1, 2, \dots\},\ \zeta \in \Delta), \tag{1}$$

which are analytic and p-valent in the unit disc $\Delta = \{\zeta : |\zeta| < 1\}$ and let the function $\Omega \in \mathcal{A}_p$ be given by:

$$\Omega(\zeta) := \zeta^p + \sum_{k=p+1}^{\infty} \psi_k \zeta^k, \ \ \zeta \in \Delta. \tag{2}$$

The Hadamard (or convolution) product of $\mathcal{F}$ and Ω is defined by:

$$(\mathcal{F} * \Omega)(\zeta) := \zeta^p + \sum_{k=p+1}^{\infty} a_k \psi_k \zeta^k, \ \ \zeta \in \Delta.$$

The confluent hypergeometric function of the first kind is given by the power series

$$\begin{aligned} F(b;c;\zeta) &= 1 + \frac{b}{c}\zeta + \frac{b\,(b+1)}{c\,(c+1)}\frac{\zeta^2}{2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k\,(1)_k}\zeta^k, \quad (b \in \mathbb{C},\ c \in \mathbb{C} \setminus \{0,-1,-2,\dots\}), \end{aligned}$$

where $(b)_k$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} 1, & \text{if} \quad k = 0, \\ b(b+1)\dots(b+k-1), & \text{if} \quad k \in \mathbb{N} = \{1,2,\dots\}, \end{cases}$$

which is convergent for all finite values of ζ (see [5]). It can be written otherwise as:

$$F(b;c;m) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k\,(1)_k} m^k, \quad (b \in \mathbb{C},\ c \in \mathbb{C} \setminus \{0,-1,-2,\dots\}),$$

which is convergent for $b, c, m > 0$.

Very recently, Porwal and Kumar [4] (see also [6–8]) introduced the confluent hypergeometric distribution (CHD) whose probability mass function is

$$P(k) = \frac{(b)_k}{(c)_k\, k! F(b;c;m)} m^k, \ (b,c,m > 0,\ k = 0,1,2,\dots).$$

We introduce a series $\mathcal{I}_p(b;c;m;\zeta^p)$ whose coefficients are probabilities of the confluent hypergeometric distribution

$$\mathcal{I}_p(b;c;m;\zeta^p) = \zeta^p + \sum_{k=p+1}^{\infty} \frac{(b)_{k-p} m^{k-p}}{(c)_{k-p}\,(k-p)! F(b;c;m)} \zeta^k, \ (b,c,m > 0), \tag{3}$$

and defined a linear operator $Q_p^{b;c;m}\mathcal{F} : \mathcal{A}_p \to \mathcal{A}_p$ as follows

$$\begin{aligned} Q_p^{b;c;m}\mathcal{F}(\zeta) &= \mathcal{I}_p(b;c;m;\zeta^p) * \mathcal{F}(\zeta) \\ &= \zeta^p + \sum_{k=p+1}^{\infty} \frac{(b)_{k-p} m^{k-p}}{(c)_{k-p}\,(k-p)! F(b;c;m)} a_k\, \zeta^k, \ (b,c,m > 0). \end{aligned}$$

Making use of the binomial series,

$$(1-\delta)^t = \sum_{i=0}^{t} \binom{t}{i} (-1)^i\, \delta^i \quad (t \in \mathbb{N}).$$

For $\mathcal{F} \in \mathcal{A}_p$, we introduced the linear differential operator as follows:

$$\mathcal{D}_{p,t}^{\delta,0,b,c,m}\mathcal{F}(\zeta) = Q_p^{b;c;m}\mathcal{F}(\zeta),$$

$$
\begin{aligned}
\mathcal{D}_{p,t}^{\delta,1,b,c,m}\mathcal{F}(\zeta) = &\ \mathcal{D}_{p,t}^{\delta,b;c;m}\mathcal{F}(\zeta) = \left[1-(1-\delta)^t\right]Q_p^{b;c;m}\mathcal{F}(\zeta) + \tfrac{\zeta}{p}(1-\delta)^t\left(Q_p^{b;c;m}\mathcal{F}\right)'(\zeta) \\
= &\ \zeta^p + \sum_{k=p+1}^{\infty}\left[1+\left(\tfrac{k}{p}-1\right)c^t(\delta)\right]\left[\tfrac{(b)_{k-p}m^{k-p}}{(c)_{k-p}\,(k-p)!F(b;c;m)}\right]a_k\,\zeta^k \\
&\ \cdot \\
&\ \cdot \\
&\ \cdot \\
\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta) = &\ \mathcal{D}_{p,t}^{\delta,b;c;m}\left(\mathcal{D}_{p,t}^{\delta,n-1,b,c,m}\mathcal{F}(\zeta)\right) \qquad (4)\\
= &\ \left[1-(1-\delta)^t\right]\mathcal{D}_{p,t}^{\delta,n-1,b,c,m}\mathcal{F}(\zeta) + \tfrac{\zeta}{p}(1-\delta)^t\left(\mathcal{D}_{p,t}^{\delta,n-1,b,c,m}\mathcal{F}(\zeta)\right)' \\
= &\ \zeta^p + \sum_{k=p+1}^{\infty}\left[1+\left(\tfrac{k}{p}-1\right)c^t(\delta)\right]^n\left[\tfrac{(b)_{k-p}m^{k-p}}{(c)_{k-p}\,(k-p)!F(b;c;m)}\right]a_k\,\zeta^k \\
= &\ \zeta^p + \sum_{k=p+1}^{\infty}\psi_k a_k\,\zeta^k, \\
&\ (\delta>0;\ b,c,m>0;\ t\in\mathbb{N};\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}),
\end{aligned}
$$

where

$$\psi_k = \left[1+\left(\frac{k}{p}-1\right)c^t(\delta)\right]^n\left[\frac{(b)_{k-p}m^{k-p}}{(c)_{k-p}\,(k-p)!F(b;c;m)}\right], \qquad (5)$$

and

$$c^t(\delta) = \sum_{i=1}^{t}\binom{t}{i}(-1)^i\,\delta^i \quad (t\in\mathbb{N}). \qquad (6)$$

From (4), we obtain that

$$c^t(\delta)\,\zeta\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)' = p\mathcal{D}_{p,t}^{\delta,n+1,b,c,m}\mathcal{F}(\zeta) - p\left[1-c^t(\delta)\right]\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta). \qquad (7)$$

Definition 1. *We define a function $F_{p,t}^{\delta,n,b,c,m,\beta}$ as follows*

$$
\begin{aligned}
F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta) = &\ (1-\beta)\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta) + \beta\mathcal{D}_{p,t}^{\delta,n+1,b,c,m}\mathcal{F}(\zeta) \\
&\ (\mathcal{F}\in\mathcal{A}_p;\ p,t\in\mathbb{N};\ n\in\mathbb{N}_0;\ \delta>0;\ b,c,m>0 \text{ and } \beta\in\mathbb{C}).
\end{aligned} \qquad (8)
$$

Remark 1. *Putting $n=0$ in Definition 1, we define a function $E_p^{b,c,m,\beta}$ as follows*

$$
\begin{aligned}
E_p^{b,c,m,\beta}(\zeta) = &\ (1-\beta)Q_p^{b;c;m}\mathcal{F}(\zeta) + \beta\tfrac{\zeta}{p}\left(Q_p^{b;c;m}\mathcal{F}(\zeta)\right)' \\
&\ (\mathcal{F}\in\mathcal{A}_p;\ p\in\mathbb{N};\ b,c,m>0 \text{ and } \beta\in\mathbb{C}).
\end{aligned} \qquad (9)
$$

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that:
$p,t\in\mathbb{N}$; $n\in\mathbb{N}_0$; $\delta>0$; $b,c,m>0$ and $\beta\in\mathbb{C}$.
In order to prove our main results, we recall here the following lemma.

Lemma 1 ([9,10]). *Let $\Phi(u,v)$ be a complex valued function, $\Phi: D\to\mathbb{C}$, $D\subset\mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u=u_1+iu_2$, $v=v_1+iv_2$. Suppose that $\Phi(u,v)$ satisfies the following conditions:*

(i) $\Phi(u,v)$ *is continuous in* D;
(ii) $(1,0)\in D$ *and* $\Re\{\Phi(1,0)\}>0$;

(iii) $\Re\{\Phi(iu_2, v_1)\} \le 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \le -\dfrac{(1+u_2^2)}{2}$.

Let $q(\zeta) = 1 + q_1\zeta + q_2\zeta^2 + \ldots$ *be regular in the unit disc* Δ *such that* $(q(\zeta), \zeta q'(\zeta)) \in D$ *for all* $\zeta \in \Delta$. *If*

$$\Re\left\{\Phi(q(\zeta), \zeta q'(\zeta))\right\} > 0 \quad (\zeta \in \Delta),$$

then

$$\Re\{q(\zeta)\} > 0 \quad (\zeta \in \Delta).$$

Applying Lemma 1, we derive the following theorem.

Theorem 2. *Let a function* $F_{p,t}^{\delta,n,b,c,m,\beta}$ *be defined by (8) and* $\mathcal{F} \in \mathcal{A}_p$. *If*

$$\Re\left\{\frac{F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)}{\zeta^p}\right\} > \alpha \quad (0 \le \alpha < 1;\ \Re(\beta) \ge 0),$$

then

$$\Re\left\{\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p}\right\} > \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)} \quad (\zeta \in \Delta),$$

where $c^t(\delta)$ *is given by (6).*

Proof. Defining the function $q(\zeta)$ by

$$\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p} = \gamma + (1-\gamma)q(\zeta) \tag{10}$$

with

$$\gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)},$$

we see that $q(\zeta) = 1 + q_1\zeta + q_2\zeta^2 + \ldots$ is regular in the unit disc Δ. Then by using (7), we have

$$\begin{aligned} \frac{F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)}{\zeta^p} &= (1-\beta)\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p} + \beta\frac{\mathcal{D}_{p,t}^{\delta,n+1,b,c,m}\mathcal{F}(\zeta)}{\zeta^p} \\ &= \gamma + (1-\gamma)q(\zeta) + \frac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta). \end{aligned} \tag{11}$$

It follows from (10) and (11) that

$$\begin{aligned} &\Re\left\{\frac{F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)}{\zeta^p} - \alpha\right\} \\ = \ &\Re\left\{\gamma - \alpha + (1-\gamma)q(\zeta) + \frac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta)\right\} > 0. \end{aligned} \tag{12}$$

Let

$$\Phi(u, v) = \gamma - \alpha + (1-\gamma)u + \frac{\beta c^t(\delta)}{p}(1-\gamma)v \tag{13}$$

with $q(\zeta) = u = u_1 + i\,u_2$ and $\zeta q'(\zeta) = v = v_1 + i\,v_2$. Then

(i) $\Phi(u, v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1, 0) \in D$ and $\Re\{\Phi(1, 0)\} = 1 - \alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}\Re\{\Phi(iu_2, v_1)\} &= \gamma - \alpha + \frac{c^t(\delta)(1-\gamma)v_1}{p}\Re(\beta) \\ &\leq \gamma - \alpha - \frac{c^t(\delta)(1-\gamma)(1+u_2^2)}{2p}\Re(\beta) \leq 0.\end{aligned}$$

Therefore, the function $\Phi(u, v)$ satisfies the conditions in Lemma 1. Thus, we have $\Re\{q(\zeta)\} > 0$ $(\zeta \in \Delta)$, that is,

$$\Re\left\{\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p}\right\} > \gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)}.$$

This completes the proof of Theorem 2. □

Putting $n = 0$ in Theorem 2, we obtain the following corollary:

Corollary 1. *Let a function $E_p^{b,c,m,\beta}$ be defined by (9) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{E_p^{b,c,m,\beta}(\zeta)}{\zeta^p}\right\} > \alpha \quad (0 \leq \alpha < 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{Q_p^{b;c;m}\mathcal{F}(\zeta)}{\zeta^p}\right\} > \frac{2p\alpha + \Re(\beta)}{2p + \Re(\beta)} \quad (\zeta \in \Delta).$$

Theorem 3. *Let a function $F_{p,t}^{\delta,n,b,c,m,\beta}$ be defined by (8) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)}{\zeta^p}\right\} < \alpha \quad (\alpha > 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p}\right\} < \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)} \quad (\zeta \in \Delta),$$

where $c^t(\delta)$ is given by (6).

Proof. Defining the function $q(\zeta)$ by

$$\frac{\mathcal{D}_p^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p} = \gamma + (1-\gamma)q(\zeta)$$

with

$$\gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)} > 1.$$

Then we oserve that $q(\zeta) = 1 + q_1\zeta + q_2\zeta^2 + \ldots$ is regular in the unit disc Δ, and

$$\begin{aligned}&\Re\left\{\alpha - \frac{F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)}{\zeta^p}\right\} \\ &= \Re\left\{\alpha - \gamma - (1-\gamma)q(\zeta) - \frac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta)\right\} > 0.\end{aligned} \tag{14}$$

Let

$$\Phi(u,v) = \alpha - \gamma - (1-\gamma)u - \frac{\beta c^t(\delta)}{p}(1-\gamma)v, \tag{15}$$

with $q(z) = u = u_1 + iu_2$ and $\zeta q'(\zeta) = v = v_1 + iv_2$. Then it follows from (15) that

(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1,0) \in D$ and $\Re\{\Phi(1,0)\} = \alpha - 1 > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned} \Re\{\Phi(iu_2,v_1)\} &= \alpha - \gamma - \frac{c^t(\delta)v_1(1-\gamma)}{p}\Re(\beta) \\ &\leq \alpha - \gamma + \frac{c^t(\delta)(1-\gamma)(1+u_2^2)}{2p}\Re(\beta) \leq 0. \end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1. Thus, we have $\Re\{q(\zeta)\} > 0$ $(\zeta \in \Delta)$, that is,

$$\Re\left\{\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)}{\zeta^p}\right\} < \gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)}.$$

This completes the proof of Theorem 3. □

Putting $n = 0$ in Theorem 3, we obtain the following corollary:

Corollary 2. *Let a function $E_p^{b,c,m,\beta}$ be defined by (9) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{E_p^{b,c,m,\beta}(\zeta)}{\zeta^p}\right\} < \alpha \quad (\alpha > 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{Q_p^{b;c;m}\mathcal{F}(\zeta)}{\zeta^p}\right\} < \frac{2p\alpha + \Re(\beta)}{2p + \Re(\beta)} \quad (\zeta \in \Delta).$$

Theorem 4. *Let a function $F_{p,t}^{\delta,n,b,c,m,\beta}$ be defined by (8) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)'}{p\zeta^{p-1}}\right\} > \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)} \quad (\zeta \in \Delta),$$

where $c^t(\delta)$ is given by (6).

Proof. Replace $\mathcal{F}(\zeta)$ by $\frac{\zeta\mathcal{F}'(\zeta)}{p}$ in the proof of Theorem 2, we define the function $q(\zeta)$ by

$$\frac{\mathcal{D}_{p,t}^{\delta,n,b,c,m}\left(\frac{\zeta\mathcal{F}'(\zeta)}{p}\right)}{\zeta^p} = \gamma + (1-\gamma)q(\zeta), \tag{16}$$

is equivalent to

$$\frac{1}{p}\frac{\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)'}{\zeta^{p-1}} = \gamma + (1-\gamma)q(\zeta), \tag{17}$$

with

$$\gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)},$$

we see that $q(\zeta) = 1 + q_1\zeta + q_2\zeta^2 + \ldots$ is regular in the unit disc Δ. Then by using (7) and replace $\mathcal{F}(\zeta)$ by $\dfrac{\zeta\mathcal{F}'(\zeta)}{p}$, we have

$$\begin{aligned}\frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}} &= \frac{(1-\beta)}{p}\frac{\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)'}{\zeta^{p-1}} + \frac{\beta}{p}\frac{\left(\mathcal{D}_{p,t}^{\delta,n+1,b,c,m}\mathcal{F}(\zeta)\right)'}{\zeta^{p-1}} \\ &= \gamma + (1-\gamma)q(\zeta) + \frac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta).\end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned}&\Re\left\{\frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}} - \alpha\right\} \\ = \quad &\Re\left\{\gamma - \alpha + (1-\gamma)q(\zeta) + \frac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta)\right\} > 0.\end{aligned} \tag{19}$$

Let

$$\Phi(u,v) = \gamma - \alpha + (1-\gamma)u + \frac{\beta c^t(\delta)}{p}(1-\gamma)v$$

with $q(\zeta) = u = u_1 + i\,u_2$ and $\zeta q'(\zeta) = v = v_1 + i\,v_2$. Then

(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;

(ii) $(1,0) \in D$ and $\Re\{\Phi(1,0)\} = 1 - \alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\dfrac{(1+u_2^2)}{2}$,

$$\begin{aligned}\Re\{\Phi(iu_2,v_1)\} &= \gamma - \alpha + \frac{c^t(\delta)(1-\gamma)v_1}{p}\Re(\beta) \\ &\leq \gamma - \alpha - \frac{c^t(\delta)(1-\gamma)(1+u_2^2)}{2p}\Re(\beta) \leq 0.\end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1. Thus, we have $\Re\{q(\zeta)\} > 0$ $(\zeta \in \Delta)$, that is,

$$\Re\left\{\frac{1}{p}\frac{\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)'}{\zeta^{p-1}}\right\} > \gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)}.$$

This completes the proof of Theorem 4. □

Putting $n = 0$ in Theorem 4, we obtain the following corollary:

Corollary 3. *Let a function $E_p^{b,c,m,\beta}$ be defined by (9) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{\left(E_p^{b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{\left(Q_p^{b;c;m}\mathcal{F}(\zeta)\right)'}{p\zeta^{p-1}}\right\} > \frac{2p\alpha + \Re(\beta)}{2p + \Re(\beta)} \quad (\zeta \in \Delta).$$

Using the same technique as in the proof of Theorem 3 (or putting $\dfrac{\zeta\mathcal{F}'(\zeta)}{p}$ instead of $\mathcal{F}(\zeta)$ in Theorem 3, respectively), we obtain the following result.

Theorem 5. *Let a function $F_{p,t}^{\delta,n,b,c,m,\beta}$ be defined by (8) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{\frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}}\right\} < \alpha \quad (\alpha > 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{\frac{\left(\mathcal{D}_{p,t}^{\delta,n,b,c,m}\mathcal{F}(\zeta)\right)'}{p\zeta^{p-1}}\right\} < \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)} \quad (\zeta \in \Delta),$$

where $c^t(\delta)$ is given by (6).

Proof. From (16) and (17), we have

$$\begin{aligned} &\Re\left\{\alpha - \frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}}\right\} \\ =\ &\Re\left\{\alpha - \gamma - (1-\gamma)q(\zeta) - \tfrac{\beta c^t(\delta)}{p}(1-\gamma)\zeta q'(\zeta)\right\} > 0. \end{aligned} \tag{20}$$

Let

$$\Phi(u,v) = \alpha - \gamma - (1-\gamma)u - \frac{\beta c^t(\delta)}{p}(1-\gamma)v, \tag{21}$$

with $q(z) = u = u_1 + iu_2$ and $\zeta q'(\zeta) = v = v_1 + iv_2$. Then it follows from (21) that
(i) $\Phi(u,v)$ is continuous in $D = \mathbb{C}^2$;
(ii) $(1,0) \in D$ and $\Re\{\Phi(1,0)\} = \alpha - 1 > 0$;
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\dfrac{(1+u_2^2)}{2}$,

$$\begin{aligned} \Re\{\Phi(iu_2,v_1)\} &= \alpha - \gamma - \frac{c^t(\delta)v_1(1-\gamma)}{p}\Re(\beta) \\ &\leq \alpha - \gamma + \frac{c^t(\delta)(1-\gamma)(1+u_2^2)}{2p}\Re(\beta) \leq 0. \end{aligned}$$

Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1. Thus, we have $\Re\{q(\zeta)\} > 0$ $(\zeta \in \Delta)$, that is,

$$\Re\left\{ \frac{\left(F_{p,t}^{\delta,n,b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}} \right\} < \gamma = \frac{2p\alpha + c^t(\delta)\Re(\beta)}{2p + c^t(\delta)\Re(\beta)}.$$

This completes the proof of Theorem 5. □

Putting $n = 0$ in Theorem 5, we obtain the following corollary:

Corollary 4. *Let a function $E_p^{b,c,m,\beta}$ be defined by (9) and $\mathcal{F} \in \mathcal{A}_p$. If*

$$\Re\left\{ \frac{\left(E_p^{b,c,m,\beta}(\zeta)\right)'}{p\zeta^{p-1}} \right\} < \alpha \quad (\alpha > 1;\ \Re(\beta) \geq 0),$$

then

$$\Re\left\{ \frac{\left(Q_p^{b;c;m}\mathcal{F}(\zeta)\right)'}{p\zeta^{p-1}} \right\} < \frac{2p\alpha + \Re(\beta)}{2p + \Re(\beta)} \quad (\zeta \in \Delta).$$

3. Conclusions

In the present paper, we mainly obtain some properties of p-valent functions $F_{p,t}^{\delta,n,b,c,m,\beta}$ involving the combination binomial series and confluent hypergeometric function in the open unit disc. Several consequences of the results are also pointed out as corollaries. Many interesting outcomes of the study conducted using the theories of differential subordination and superordination are due to the use of operators. We intend to work further by generalizing these results using fractional operators.

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References

1. Branges, L.D. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
2. Ghanim, F.; Al-Janaby, H.F. An analytical study on Mittag-Leffler confluent hypergeometric functions with fractional integral operator. *Math. Methods Appl. Sci.* **2020**, *44*, 3605–3614. [CrossRef]
3. Oros, G.I. New Conditions for Univalence of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 82. [CrossRef]
4. Porwal, S.; Kumar, S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions. *Afr. Mat.* **2017**, *28*, 1–8. [CrossRef]
5. Rainville, E.D. *Special Functions*; The Macmillan Co.: New York, NY, USA, 1960.
6. El-Deeb, S.M. On unified subclass of complex order connected with q-confluent hypergeometric distribution. *Surv. Math. Appl.* **2022**, *16*, 327–337.
7. El-Deeb, S.M.; El-Matary, B.M. Subclasses of bi-univalent functions associated with q-confluent hypergeometric distribution based upon the Horadam polynomials. *Adv. Theory Nonlinear Anal.* **2021**, *1*, 82–93. [CrossRef]

8. Porwal, S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions. *Kyungpook Math. J.* **2018**, *58*, 495–505.
9. Miller, S.S.; Mocanu, P.T. Second order differential inequalities in the complex plane. *J. Math. Anal. Appl.* **1978**, *65*, 289–305. [CrossRef]
10. Miller, S.S. Differential inequalities and Caratheodory function. *Bull. Am. Math. Soc.* **1975**, *8*, 79–81. [CrossRef]