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Non-Canonical Functional Differential Equation of Fourth-Order: New Monotonic Properties and Their Applications in Oscillation Theory

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Abstract: In the present article, we iteratively deduce new monotonic properties of a class from the positive solutions of fourth-order delay differential equations. We discuss the non-canonical case in which there are possible decreasing positive solutions. Then, we find iterative criteria that exclude the existence of these positive decreasing solutions. Using these new criteria and based on the comparison and Riccati substitution methods, we create sufficient conditions to ensure that all solutions of the studied equation oscillate. In addition to having many applications in various scientific domains, the study of the oscillatory and non-oscillatory features of differential equation solutions is a theoretically rich field with many intriguing issues. Finally, we show the importance of the results by applying them to special cases of the studied equation.

Keywords: delay differential equation; higher-order; oscillatory; nonoscillatory; non-canonical case

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1. Introduction

In this work, we study the asymptotic behavior of solutions to the fourth-order delay differential equation of the form

$$(h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^{\alpha})' + q(\mathfrak{r})\Phi^{\alpha}(\tau(\mathfrak{r})) = 0, \tag{1}$$

where $\mathfrak{r} \geq \mathfrak{r}_o$. Through the paper, the next conditions are satisfied:

(V1) $\alpha > 0$ is a quotient of odd positive integers;
(V2) $h, q, \tau \in C([\mathfrak{r}_o, \infty), (0, \infty))$, $\tau(\mathfrak{r}) < \mathfrak{r}$, $\lim_{\mathfrak{r}\to\infty} \tau(\mathfrak{r}) = \infty$, and

$$\eta(\mathfrak{r}_o) = \int_{\mathfrak{r}_o}^{\infty} h^{-1/\alpha}(v)\mathrm{d}v < \infty. \tag{2}$$

By a solution of (1), we mean a function $\Phi \in C([\mathfrak{r}_*, \infty), \mathbb{R})$, $\mathfrak{r}_* \geq \mathfrak{r}_o$ such that $\Phi(\mathfrak{r})$ satisfies (1) on $[\mathfrak{r}_*, \infty)$. In what follows, we suppose that solutions of (1) exist and can be continued indefinitely to the right. Furthermore, we consider only solutions $\Phi(\mathfrak{r})$ of (1) that satisfy $\sup\{|\Phi(\mathfrak{r})| : \mathfrak{r}_* \leq \mathfrak{r}\} > 0$ for all $\mathfrak{r} \geq \mathfrak{r}_*$, and we tacitly assume that (1) possesses such solutions.

Definition 1. *A solution Φ of (1) is said to be non-oscillatory if, essentially, it is positive or negative; otherwise, it is said to be oscillatory. If all of its solutions oscillate, the equation itself is called oscillatory.*

The delay differential equations are a subclass of functional differential equations. The concept of delay in systems is proposed as a key role in modeling when representing the time taken to complete some hidden operations. Examples of the delay in the predator–prey model occur when the predator birth rate is affected by previous levels of predator or prey rather than only current levels. With the rapid development of communication technologies, sending measured signals to the remote-control center has become increasingly simple. However, the main problem facing engineers is the inevitable time delay between the measurement and the signal received by the controller, and this time delay must be taken into account at the design stage to avoid risks of experimental instability and potential damage, see [1,2].

Differential equations of the fourth-order delay can be found in the mathematical models of numerous biological, chemical, and physical phenomena. Examples of such applications include elastic problems and soil settlement. One model that can be represented by a fourth-order oscillatory equation with delay is the oscillatory traction of a muscle, which occurs when the muscle is under an inertial load [3].

One of two things is necessarily required to explain natural phenomena and problems that use differential equations in their modeling: either finding solutions to these equations or studying the properties of these solutions. However, the equations resulting from the modeling of natural phenomena are often non-linear differential equations that are difficult to find a closed-form solution to, and this has strongly stimulated the study of the qualitative behavior of these models. From here, strong interest has emerged in the study of the qualitative theory of differential equations, one of the most important branches of which is the theory of oscillation. Obtaining lower bounds for the separation between succeeding zeros, taking into account the number of zeros, studying the laws of distribution of the zeros, and establishing the conditions for the existence of oscillatory (non-oscillatory) solutions and/or convergence to zeroconstitute the essence of oscillation theory, see [4].

Finding sufficient conditions for the oscillatory and non-oscillatory properties of second and higher-order differential equations has been a persistent area of research over the last few years, see [5–7]. Among the numerous papers dealing with this subject, we refer in particular to the following.

Onose [8] focused on the oscillation of fourth-order functional differential equations

$$\left(h(\mathfrak{r})\Phi''(\mathfrak{r})\right)'' + f(\Phi(\tau(\mathfrak{r})),\mathfrak{r}) = 0$$

and

$$\left(h(\mathfrak{r})\Phi''(\mathfrak{r})\right)'' + q(\mathfrak{r})f(\Phi(\tau(\mathfrak{r}))) = \tau(\mathfrak{r}),$$

under the canonical case. The oscillation and non-oscillation of the fourth and higher-order differential equations have been the focus of the attention of numerous authors since this paper was first published.

Wu [9] and Kamo and Usami [10] studied the oscillatory of a fourth-order differential equation

$$\left(h(\mathfrak{r})\left|\Phi''(\mathfrak{r})\right|^{\alpha-1}\Phi''(\mathfrak{r})\right)'' + q|\Phi(\mathfrak{r})|^{\beta-1}\Phi(\mathfrak{r}) = 0,$$

when the noncanonical holds and the constants α and β are positive.

Grace et al. [11] focused on the oscillatory behavior of the fourth-order differential equation of the form

$$\left(h(\Phi')^{\alpha}\right)'''(\mathfrak{r}) + q(\mathfrak{r})f(\Phi(g(\mathfrak{r}))) = 0,$$

under the noncanonical case.

Zhang et al. [12] and Baculikova et al. [13] studied the oscillatory behavior of the higher-order differential equation

$$\left(h(\mathfrak{r})\left(\Phi^{n-1}(\mathfrak{r})\right)^{\alpha}\right)' + q(\mathfrak{r})f(\Phi(\tau(\mathfrak{r}))) = 0. \tag{3}$$

Ref. [12] provided some oscillation criteria for Equation (3), in which $f(\Phi) = \Phi^{\beta}$ and β is a quotient of odd positive integers. In [13], various techniques have been used in investigating higher-order differential equations. In the case where $n = 4$ and $f(\Phi) = \Phi^{\alpha}$, by the Riccatti technique, Zhang et al. [14] established some new criteria for the oscillation of all solutions of the fourth-order differential Equation (3).

Theorem 1. *Ref. [12] Let $n \geq 2$. Suppose that (2) holds. Further, assume that for some constant $\lambda_0 \in (0,1)$, the differential equation*

$$\Phi'(\mathfrak{r}) + q(\mathfrak{r})\left(\frac{\lambda_0 \tau^{n-1}(\mathfrak{r})}{(n-1)!h^{1/\alpha}(\tau(\mathfrak{r}))}\right)^{\beta}\Phi^{\beta/\alpha}(\tau(\mathfrak{r})) = 0, \tag{4}$$

is oscillatory. If

$$\limsup_{\mathfrak{r}\to\infty}\int_{\mathfrak{r}_0}^{\mathfrak{r}}\left(M^{\beta-\alpha}q(\jmath)\left(\frac{\lambda_1\tau^{n-2}(\jmath)}{(n-2)!}\right)^{\beta}\eta^{\alpha}(\jmath) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\eta(\jmath)h^{1/\alpha}(\jmath)}\right)\mathrm{d}\jmath = \infty, \tag{5}$$

for some constant $\lambda_1 \in (0,1)$ and for every constant $M > 0$, then every solution of (6) is oscillatory or tends to zero.

Zhang et al. [15] suggested some new oscillation criteria for an even-order delay differential equation

$$\left(h(\mathfrak{r})\left(\Phi^{n-1}(\mathfrak{r})\right)^{\alpha}\right)' + q(\mathfrak{r})\Phi^{\beta}(\tau(\mathfrak{r})) = 0, \tag{6}$$

in the noncanonical case with $n \geq 4$, where β is a quotient of odd positive integers.

Theorem 2. *Ref. [15] Let $n \geq 4$ be even, $(V1)$, $(V2)$, and (2). Suppose that differential Equation (4) is oscillatory for some constant $\lambda_0 \in (0,1)$. If (5) and*

$$\limsup_{\mathfrak{r}\to\infty}\int_{\mathfrak{r}_o}^{\mathfrak{r}}\left[M^{\beta-\alpha}q(\jmath)H^{\alpha}(\jmath) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{(H'(\jmath))^{\alpha+1}}{H(\jmath)\eta_1^{\alpha}(\jmath)}\right]\mathrm{d}\jmath = \infty,$$

hold for some constants $\lambda_1 \in (0,1)$ and for every constant $M > 0$, then (6) is oscillatory, where

$$H(\mathfrak{r}) = \int_{\mathfrak{r}}^{\infty}(\jmath - \mathfrak{r})\eta(\jmath)\mathrm{d}\jmath.$$

By using a generalized Riccatti substitution, in the case $f(\Phi) = q\Phi^{\beta}$ where q is a nonnegative function and β is a quotient of odd positive integers, Moaaz and Muhib [16] provided a new criterion for the oscillation of solutions of fourth-order quasi-linear differential equations

$$\left(h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^{\alpha}\right)' + f(\mathfrak{r}, \Phi(\sigma(\mathfrak{r}))) = 0. \tag{7}$$

Theorem 3. *Ref. [16] Suppose that $\alpha \geq 1$ and the differential equation*

$$\Phi'(\mathfrak{r}) + q(\mathfrak{r})\left(\frac{\lambda_0\tau^3(\mathfrak{r})}{3!h^{1/\alpha}(\tau(\mathfrak{r}))}\right)^{\beta}\Phi^{\beta/\alpha}(\tau(\mathfrak{r})) = 0, \tag{8}$$

oscillates where $\lambda_0 \in (0,1)$. If there is a positive function $\gamma \in C^1([\mathfrak{r}_o,\infty),(0,\infty))$ such that

$$\limsup_{\mathfrak{r}\to\infty}\int_{\mathfrak{r}_0}^{\mathfrak{r}}\left(\varphi(\jmath) - \frac{h(\jmath)\gamma(\jmath)}{(\alpha+1)^{\alpha+1}}\left(\frac{\gamma'(\jmath)}{\gamma(\jmath)} + \frac{(1+\alpha)}{h^{1/\alpha}(\jmath)\eta(\jmath)}\right)^{\alpha+1}\right)\mathrm{d}\jmath = \infty, \tag{9}$$

holds for any positive constants c_1 and c_2 and for $\lambda_1 \in (0,1)$, where

$$\varphi(\mathfrak{r}) = \gamma(\mathfrak{r})q\left(\frac{\lambda_1}{2!}\tau^2\right)^{\beta} + (1-\alpha)\frac{\gamma(\mathfrak{r})}{h^{1/\alpha}(\mathfrak{r})\eta^{\alpha+1}(\mathfrak{r})}$$

then every solution of (6) is oscillatory or tends to zero.

Theorem 4. *Ref. [16] Suppose that Equation (8) oscillates where $\lambda_o \in (0,1)$. If there is a function $\gamma \in C^1([\mathfrak{r}_o,\infty),(0,\infty))$ such that*

$$\limsup_{\mathfrak{r}\to\infty} \frac{\eta^{\alpha}(\mathfrak{r})}{\gamma(\mathfrak{r})} \int_{\mathfrak{r}_o}^{\mathfrak{r}} \left(\gamma(\jmath)q(\jmath)\left(\frac{\lambda_1}{2!}\tau^2(\jmath)\right)^{\alpha} - \frac{h(\jmath)(\gamma'(\jmath))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\gamma^{\alpha}(\jmath)}\right)\mathrm{d}\jmath > 1, \tag{10}$$

then every solution of (7) is oscillatory or converges to zero as $\mathfrak{r} \to \infty$ for $\lambda_1 \in (0,1)$.

Theorem 5. *Ref. [16] Suppose that $\alpha \geq 1$ and the differential Equation (8) is oscillatory or some constant $\lambda_o \in (0,1)$. If there is a function $\gamma \in C^1([\mathfrak{r}_o,\infty),(0,\infty))$ such that (9) and*

$$\limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_o}^{\mathfrak{r}} \left[\psi(\jmath) - \frac{\gamma(\jmath)}{(\alpha+1)^{(\alpha+1)}\eta_1^{\alpha}(\mathfrak{r})}\left(\frac{\gamma'(\jmath)}{\gamma(\jmath)} + \frac{(1+\alpha)\eta_1(\jmath)}{\eta_2(\jmath)}\right)^{\alpha+1}\right]\mathrm{d}\jmath = \infty, \tag{11}$$

holds for $\lambda_1 \in (0,1)$, where

$$\psi(\mathfrak{r}) = q\gamma(\mathfrak{r}) + (1-\alpha)\gamma(\mathfrak{r})\eta_1(\mathfrak{r})/\eta_2^{\alpha+1}(\mathfrak{r}).$$

Then, (7) is oscillatory .

Elabbasy et al. [17] considered the even-order neutral differential equation with several delays

$$(h(\mathfrak{r})(z^{(n-1)}(\mathfrak{r}))^{\alpha})' + \sum_{i=1}^{k} q_i(\mathfrak{r})f(\Phi(\tau_i(\mathfrak{r}))) = 0,$$

where $z(\mathfrak{r}) = \Phi(\mathfrak{r}) + p(\mathfrak{r})\Phi(\tau(\mathfrak{r}))$ and $n \geq 4$ with the noncanonical operator. Moaaz et al. [18] studied the fourth-order delay differential equation of the form

$$(h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^{\alpha})' + f(\mathfrak{r},\Phi(\sigma(\mathfrak{r}))) = 0,$$

under the noncanonical case.

Lemma 1. *Ref. [19] Let $f \in C^n([\mathfrak{r}_o,\infty),(0,\infty))$. If the derivative $f^{(n)}(\mathfrak{r})$ is eventually of one sign for all large $\mathfrak{r}$, then there is a $\mathfrak{r}_\Phi$ such that $\mathfrak{r}_\Phi \geq \mathfrak{r}_o$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for $f^{(n)}(\mathfrak{r}) \geq 0$, or $n+l$ odd for $f^{(n)}(\mathfrak{r}) \leq 0$ such that*

$$l > 0 \text{ implies } f^{(k)}(\mathfrak{r}) > 0 \text{ for } \mathfrak{r} \geq \mathfrak{r}_\Phi,\ k = 0,1,\ldots,l-1,$$

and

$$l \leq n-1 \text{ implies } (-1)^{l+k}f^{(k)}(\mathfrak{r}) > 0 \text{ for } \mathfrak{r} \geq \mathfrak{r}_\Phi,\ k = l,l+1,\ldots,n-1.$$

Lemma 2. *Ref. [12] Let α be a ratio of two odd positive integers. Then,*

$$Lv^{(\alpha+1)/\alpha} - Kv \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{K^{\alpha+1}}{L^{\alpha}},\ L > 0 \tag{12}$$

and

$$A^{(\alpha+1)/\alpha} - (A-B)^{(\alpha+1)/\alpha} \leq \frac{1}{\alpha}B^{1/\alpha}[(1+\alpha)A - B],\quad \alpha \geq 1,\ AB \geq 0. \tag{13}$$

The main purpose of this work is to test the oscillation of solutions of a fourth-order delay differential Equation (1). This paper is organized as follows: In Section 2, we create new properties that help us achieve more effective terms in the oscillation of the studied equation. In Section 3, we apply the Riccati substitution in the general form and the comparison method to obtain criteria that excluded decreasing solutions. In Section 4, by combining the results known in the literature and the results we obtained, we set criteria that ensure the oscillation of the studied equation and offer an illustrative example to show our results. Finally, in Section 5, we conclude the article with a summary.

2. New Monotonic Properties

It is well known that positive solutions of delay differential equations must be categorized based on the sign of their derivatives when investigating their oscillatory behavior. Now, we assume that Φ is an eventually positive solution of (1). From the differential equation in (1) and taking into account that $q(\mathfrak{t}) > 0$, we have that $h(\mathfrak{t})(\Phi'''(\mathfrak{t}))^{\alpha}$ is a nonincreasing function. Furthermore, according to Lemma 1, we obtain the following three cases, eventually:

$$\begin{array}{ll} \text{Case (1)}: & \Phi'(\mathfrak{t}) > 0, \Phi'''(\mathfrak{t}) > 0 \text{ and } \Phi^{(4)}(\mathfrak{t}) < 0; \\ \text{Case (2)}: & \Phi'(\mathfrak{t}) > 0, \Phi''(\mathfrak{t}) > 0 \text{ and } \Phi'''(\mathfrak{t}) < 0; \\ \text{Case (3)}: & \Phi'(\mathfrak{t}) < 0, \Phi''(\mathfrak{t}) > 0 \text{ and } \Phi'''(\mathfrak{t}) < 0, \end{array}$$

for $\mathfrak{t} \geq \mathfrak{t}_1$, where $\mathfrak{t}_1$ is sufficiently large. For convenience, we will symbolize the set of all eventually positive solutions of the Equation (1) by $^+$and the set of all solutions with satisfying case (3) by $^+_3$.

In order to prove our main results, we define the following:

$$\eta_i(\mathfrak{t}) = \int_{\mathfrak{t}}^{\infty} \eta_{i-1}(\jmath)\mathrm{d}\jmath \ \text{ for } \ i = 1, 2.$$

and

$$\beta_* = \liminf_{\mathfrak{t}\to\infty} \frac{1}{\alpha} q(\mathfrak{t}) \eta_1^{-1}(\mathfrak{t}) \eta_2^{\alpha+1}(\mathfrak{t}).$$

In addition, we put

$$\mu_* = \liminf_{\mathfrak{t}\to\infty} \frac{\eta_2(\tau(\mathfrak{t}))}{\eta_2(\mathfrak{t})}.$$

It is useful to note that in view of (V2), $\mu_* \geq 1$. In the proofs, we will often use that there is $\mathfrak{t}_1 \geq \mathfrak{t}_o$ sufficiently large such that, for arbitrary $\beta \in (0, \beta_*)$ and $\mu \in [1, \mu_*)$, we have

$$q(\mathfrak{t}) \eta_1^{-1}(\mathfrak{t}) \eta_2^{\alpha+1}(\mathfrak{t}) \geq \alpha\beta, \tag{14}$$

and

$$\frac{\eta_2(\tau(\mathfrak{t}))}{\eta_2(\mathfrak{t})} \geq \mu.$$

on $[\mathfrak{t}_1, \infty)$.

Below, we define a sequence that is used to improve the monotonic properties of the positive solutions of (1).

Definition 2. *We define sequence $\{\beta_n\}$ as $\beta_o = \sqrt[\alpha]{\beta_*}$ and*

$$\beta_n = \frac{\beta_o \mu_*^{\beta_{n-1}}}{\sqrt[\alpha]{1 - \beta_{n-1}}}, \quad n \in \mathbb{N}. \tag{15}$$

Remark 1. *By induction, it is easy to see that if, for any $n \in \mathbb{N}$, $\beta_i < 1$, for $i = 0,1,2,\ldots,n$. Then, β_{n+1} exists and*

$$\beta_{n+1} = \ell_n \beta_n > \beta_n, \tag{16}$$

where ℓ_n is defined by

$$\ell_o = \frac{\mu_*^{\beta_o}}{\sqrt[\alpha]{1-\beta_o}},$$

and

$$\ell_{n+1} = \mu_*^{\beta_o(\ell_n - 1)} \sqrt[\alpha]{\frac{1-\beta_n}{1-\ell_n\beta_n}}, \ \ n \in \mathbb{N}_o.$$

Lemma 3. *Assume that $\Phi \in C([\mathfrak{r}_o,\infty),(0,\infty))$ is a solution of (1) and Case (3) holds. If*

$$\int_{\mathfrak{r}_o}^{\infty} \left(\frac{1}{h(v)} \int_{\mathfrak{r}_1}^{v} q(\jmath) \mathrm{d}\jmath \right)^{1/\alpha} \mathrm{d}v = \infty, \tag{17}$$

then $\Phi(\mathfrak{r})$ converges to zero and $\Phi(\mathfrak{r})/\eta_2(\mathfrak{r})$ is eventually nondecreasing.

Proof. Assume that $\Phi \in {}^+$ and satisfies case (3). Then, we obtain that $\lim_{\mathfrak{r}\to\infty} \Phi(\mathfrak{r}) = \delta \geq 0$. We claim that $\lim\limits_{\mathfrak{r}\to\infty} \Phi(\mathfrak{r}) = 0$. Assume the contrary that $\delta > 0$. Thus, there is $\mathfrak{r}_1 \geq \mathfrak{r}_o$ such that $\Phi(\tau(\mathfrak{r})) \geq \delta$ for $\mathfrak{r} \geq \mathfrak{r}_1$, and hence

$$\left(h(\mathfrak{r}) \left(\Phi'''(\mathfrak{r}) \right)^{\alpha} \right)' = -q(\mathfrak{r}) \Phi^{\alpha}(\tau(\mathfrak{r})) \leq -\delta^{\alpha} q(\mathfrak{r}),$$

for $\mathfrak{r} \geq \mathfrak{r}_1$. Integrating the above inequality twice from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$\Phi'''(\mathfrak{r}) \leq -\delta \left(\frac{1}{h(\mathfrak{r})} \int_{\mathfrak{r}_1}^{\mathfrak{r}} q(\jmath) \mathrm{d}\jmath \right)^{1/\alpha}$$

and

$$\Phi''(\mathfrak{r}) \leq \Phi''(\mathfrak{r}_1) - \delta \int_{\mathfrak{r}_1}^{\mathfrak{r}} \left(\frac{1}{h(v)} \int_{\mathfrak{r}_1}^{v} q(\jmath) \mathrm{d}\jmath \right)^{1/\alpha} \mathrm{d}v.$$

Letting $\mathfrak{r} \to \infty$ and using (17), we obtain that $\lim_{\mathfrak{r}\to\infty} \Phi''(\mathfrak{r}) = -\infty$, which contradicts $\Phi''(\mathfrak{r}) > 0$. Thus, the proof is complete. Using the fact that $h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})$ is nonincreasing, we see that

$$\Phi''(\mathfrak{r}) \geq -\int_{\mathfrak{r}}^{\infty} h^{-1/\alpha}(\jmath) h^{1/\alpha}(\jmath) \Phi'''(\jmath) \mathrm{d}\jmath \geq -h^{1/\alpha}(\mathfrak{r}) \Phi'''(\mathfrak{r}) \eta(\mathfrak{r}). \tag{18}$$

Now, we have

$$\begin{aligned} \left(\frac{\Phi''(\mathfrak{r})}{\eta(\mathfrak{r})} \right)' &= \frac{\eta(\mathfrak{r})\Phi'''(\mathfrak{r}) + h^{-1/\alpha}(\mathfrak{r})\Phi''(\mathfrak{r})}{\eta^2(\mathfrak{r})} \\ &= \frac{1}{h^{1/\alpha}(\mathfrak{r})\eta^2(\mathfrak{r})} \left[h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta(\mathfrak{r}) + \Phi''(\mathfrak{r}) \right] \\ &\geq 0. \end{aligned} \tag{19}$$

Thus, we obtain

$$\Phi'(\mathfrak{r}) \leq -\int_{\mathfrak{r}}^{\infty} \eta(\jmath) \frac{\Phi''(\jmath)}{\eta(\jmath)} \mathrm{d}\jmath \leq -\frac{\Phi''(\mathfrak{r})}{\eta(\mathfrak{r})} \eta_1(\mathfrak{r}), \tag{20}$$

which implies

$$\begin{aligned}\left(\frac{\Phi'(\mathfrak{r})}{\eta_1(\mathfrak{r})}\right)' &= \frac{\eta_1(\mathfrak{r})\Phi''(\mathfrak{r})+\eta(\mathfrak{r})\Phi'(\mathfrak{r})}{\eta_1^2(\mathfrak{r})}\\ &= \frac{1}{\eta_1^2(\mathfrak{r})}\left[\Phi''(\mathfrak{r})\eta_1(\mathfrak{r})+\Phi'(\mathfrak{r})\eta(\mathfrak{r})\right]\\ &\leq 0. \qquad (21)\end{aligned}$$

This leads to

$$\Phi(\mathfrak{r}) \geq -\int_{\mathfrak{r}}^{\infty}\eta_1(\jmath)\frac{\Phi'(\jmath)}{\eta_1(\jmath)}\mathrm{d}\jmath \geq -\frac{\Phi'(\mathfrak{r})}{\eta_1(\mathfrak{r})}\eta_2(\mathfrak{r}), \qquad (22)$$

hence

$$\begin{aligned}\left(\frac{\Phi(\mathfrak{r})}{\eta_2(\mathfrak{r})}\right)' &= \frac{\eta_2(\mathfrak{r})\Phi'(\mathfrak{r})+\eta_1(\mathfrak{r})\Phi(\mathfrak{r})}{\eta_2^2(\mathfrak{r})}\\ &= \frac{1}{\eta_2^2(\mathfrak{r})}\left[\eta_2(\mathfrak{r})\Phi'(\mathfrak{r})+\eta_1(\mathfrak{r})\Phi(\mathfrak{r})\right]\\ &\geq 0. \qquad (23)\end{aligned}$$

This completes the proof. □

Lemma 4. *Let $\beta_* > 0$ and $\mu_* < \infty$. If $\Phi \in C([\mathfrak{r}_o,\infty),(0,\infty))$ is a solution of (1) and Case (3) holds, then for any $n \in \mathbb{N}_o$*

$$\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\beta_n}(\mathfrak{r})}\right)' < 0.$$

Proof. Assume that $\Phi \in {}^+$ and satisfies case (3) on $[\mathfrak{r}_1,\infty)$ where $\mathfrak{r}_1 \geq \mathfrak{r}_o$ such that $\Phi(\tau(\mathfrak{r})) > 0$ and (14) holds for $\mathfrak{r} \geq \mathfrak{r}_1$. Integrating (1) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$\begin{aligned}h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &= h(\mathfrak{r}_1)\left(\Phi'''(\mathfrak{r}_1)\right)^{\alpha} - \int_{\mathfrak{r}_1}^{\mathfrak{r}} q(\jmath)\Phi^{\alpha}(\tau(\jmath))\mathrm{d}\jmath\\ &\leq h(\mathfrak{r}_1)\left(\Phi'''(\mathfrak{r}_1)\right)^{\alpha} - \Phi^{\alpha}(\mathfrak{r})\int_{\mathfrak{r}_1}^{\mathfrak{r}} q(\jmath)\mathrm{d}\jmath.\end{aligned}$$

By using (14) in the above inequality, we obtain

$$\begin{aligned}h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &\leq h(\mathfrak{r}_1)\left(\Phi'''(\mathfrak{r}_1)\right)^{\alpha} - \beta\Phi^{\alpha}(\mathfrak{r})\int_{\mathfrak{r}_1}^{\mathfrak{r}}\frac{\alpha}{\eta_1^{-1}(\jmath)\eta_2^{\alpha+1}(\jmath)}\mathrm{d}\jmath\\ &\leq h(\mathfrak{r}_1)\left(\Phi'''(\mathfrak{r}_1)\right)^{\alpha} - \beta\frac{\Phi^{\alpha}(\mathfrak{r})}{\eta_2^{\alpha}(\mathfrak{r})} + \beta\frac{\Phi^{\alpha}(\mathfrak{r})}{\eta_2^{\alpha}(\mathfrak{r}_1)}.\end{aligned}$$

From Lemma 3, we have that $\lim_{\mathfrak{r}\to\infty}\Phi(\mathfrak{r}) = 0$. Hence, there is a $\mathfrak{r}_2 \in [\mathfrak{r}_1,\infty)$ such that

$$h(\mathfrak{r}_1)\left(\Phi'''(\mathfrak{r}_1)\right)^{\alpha} + \beta\frac{\Phi^{\alpha}(\mathfrak{r})}{\eta_2^{\alpha}(\mathfrak{r}_1)} < 0,$$

for $\mathfrak{r} \geq \mathfrak{r}_2$. Thus,

$$h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} < -\beta\frac{\Phi^{\alpha}(\mathfrak{r})}{\eta_2^{\alpha}(\mathfrak{r})}$$

or

$$h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta_2(\mathfrak{r}) < -\sqrt[\alpha]{\beta}\Phi(\mathfrak{r}) = -\varepsilon_o\beta_o\Phi(\mathfrak{r}), \qquad (24)$$

where $\varepsilon_o = \sqrt[\alpha]{\beta}/\beta_o$ is an arbitrary constant from $(0,1)$. Note that,

$$h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta(\mathfrak{r}) \geq \int_{\mathfrak{r}}^{\infty} h^{-1/\alpha}(\jmath)h^{1/\alpha}(\jmath)\Phi'''(\jmath)\mathrm{d}\jmath \geq -\Phi''(\mathfrak{r}),$$

then,

$$\Phi''(\mathfrak{r}) \geq -h^{1/\alpha}(\mathfrak{r})\eta(\mathfrak{r})\Phi'''(\mathfrak{r}).$$

By repeating this step twice over $[\mathfrak{r},\infty)$, we obtain

$$\Phi'(\mathfrak{r}) \leq h^{1/\alpha}(\mathfrak{r})\eta_1(\mathfrak{r})\Phi'''(\mathfrak{r}) \tag{25}$$

and

$$\Phi(\mathfrak{r}) \geq -h^{1/\alpha}(\mathfrak{r})\eta_2(\mathfrak{r})\Phi'''(\mathfrak{r}).$$

From (24) and (25), we obtain

$$\frac{\Phi'(\mathfrak{r})}{\eta_1(\mathfrak{r})} \leq h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})$$

and

$$\frac{\Phi'(\mathfrak{r})}{\eta_1(\mathfrak{r})} \leq -\sqrt[\alpha]{\beta}\frac{\Phi(\mathfrak{r})}{\eta_2(\mathfrak{r})},$$

hence,

$$\eta_2(\mathfrak{r})\Phi'(\mathfrak{r}) + \sqrt[\alpha]{\beta}\eta_1(\mathfrak{r})\Phi(\mathfrak{r}) \leq 0.$$

Therefore,

$$\begin{aligned}
\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)' &= \frac{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})\Phi'(\mathfrak{r}) + \sqrt[\alpha]{\beta}\eta_2^{\sqrt[\alpha]{\beta}-1}(\mathfrak{r})\eta_1(\mathfrak{r})\Phi(\mathfrak{r})}{\eta_2^{2\sqrt[\alpha]{\beta}}(\mathfrak{r})} \\
&= \frac{\eta_2^{\sqrt[\alpha]{\beta}-1}\left[\sqrt[\alpha]{\beta}\eta_1(\mathfrak{r})\Phi(\mathfrak{r}) + \eta_2(\mathfrak{r})\Phi'(\mathfrak{r})\right]}{\eta_2^{2\sqrt[\alpha]{\beta}}(\mathfrak{r})} \\
&= \frac{1}{\eta_2^{\sqrt[\alpha]{\beta}+1}(\mathfrak{r})}\left[\sqrt[\alpha]{\beta}\eta_1(\mathfrak{r})\Phi(\mathfrak{r}) + \eta_2(\mathfrak{r})\Phi'(\mathfrak{r})\right] \\
&\leq 0.
\end{aligned}$$

Integrating (1) from $\mathfrak{r}_2$ to $\mathfrak{r}$ and using that $\Phi(\mathfrak{r})/\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})$ is decreasing, we have

$$\begin{aligned}
h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \int_{\mathfrak{r}_2}^{\mathfrak{r}} q(\jmath)\eta_2^{\alpha\sqrt[\alpha]{\beta}}(\tau(\jmath))\frac{\Phi^{\alpha}(\tau(\jmath))}{\eta_2^{\alpha\sqrt[\alpha]{\beta}}(\tau(\jmath))}\mathrm{d}\jmath \\
&\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \left(\frac{\Phi(\tau(\mathfrak{r}))}{\eta_2^{\sqrt[\alpha]{\beta}}(\tau(\mathfrak{r}))}\right)^{\alpha}\int_{\mathfrak{r}_2}^{\mathfrak{r}} q(\jmath)\eta_2^{\alpha\sqrt[\alpha]{\beta}}(\tau(\jmath))\mathrm{d}\jmath,
\end{aligned}$$

hence,

$$h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} \leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\int_{\mathfrak{r}_2}^{\mathfrak{r}} q(\jmath)\left(\frac{\eta_2(\tau(\jmath))}{\eta_2(\jmath)}\right)^{\alpha\sqrt[\alpha]{\beta}}\eta_2^{\alpha\sqrt[\alpha]{\beta}}(\jmath)\mathrm{d}\jmath.$$

It is clear that from (14), we have

$$\begin{aligned} h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \beta\left(\frac{\Phi(\mathfrak{r})}{\eta^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\int_{\mathfrak{r}_2}^{\mathfrak{r}}\frac{\alpha\left(\frac{\eta_2(\tau(\jmath))}{\eta_2(\jmath)}\right)^{\alpha\sqrt[\alpha]{\beta}}}{\eta_1(\jmath)\eta_2^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(\jmath)}\mathrm{d}\jmath \\ &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\int_{\mathfrak{r}_2}^{\mathfrak{r}}\frac{\alpha\left(1-\sqrt[\alpha]{\beta}\right)}{\eta_1(\jmath)\eta_2^{\alpha+1-\alpha\sqrt[\alpha]{\beta}}(\jmath)}\mathrm{d}\jmath, \end{aligned}$$

which implies that,

$$\begin{aligned} h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \\ &\quad \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\left(\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r})} - \frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r}_2)}\right). \end{aligned} \tag{26}$$

Now, we claim that $\lim_{\mathfrak{r}\to\infty}\Phi(\mathfrak{r})/\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r}) = 0$. It is enough to show that there is $\epsilon > 0$ such that $\Phi(\mathfrak{r})/\eta_2^{\sqrt[\alpha]{\beta}+\epsilon}(\mathfrak{r})$ is eventually decreasing. Since $\eta_2(\mathfrak{r})$ tends to zero, there is a constant

$$\ell \in \left(\frac{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}}{\mu^{\sqrt[\alpha]{\beta}}}, 1\right)$$

and a $\mathfrak{r}_3 \geq \mathfrak{r}_2$ such that

$$\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r})} - \frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r}_2)} > \ell^{\alpha}\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r})}, \tag{27}$$

for $\mathfrak{r} \geq \mathfrak{r}_3$. By using (27) in (26), we obtain

$$h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} \leq -\frac{\ell^{\alpha}\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2(\mathfrak{r})}\right)^{\alpha},$$

its mean ,

$$h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r}) \leq -\left(\sqrt[\alpha]{\beta}+\epsilon\right)\frac{\Phi(\mathfrak{r})}{\eta_2(\mathfrak{r})}, \tag{28}$$

where

$$\epsilon = \sqrt[\alpha]{\beta}\left(\frac{\ell\mu^{\sqrt[\alpha]{\beta}}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}} - 1\right) > 0.$$

Thus, from (28),

$$\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}+\epsilon}(\mathfrak{r})}\right)' \leq 0,$$

for $\mathfrak{r} \geq \mathfrak{r}_3$, and hence the claim is valid. Therefore, for $\mathfrak{r}_4 \in [\mathfrak{r}_3, \infty)$,

$$h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} + \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r}_2)} < 0,$$

for $\mathfrak{r} \geq \mathfrak{r}_4$. By using the above inequality in (26), we have

$$\begin{aligned} h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r})} \\ &\quad + \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r}_2)} \\ &\leq h(\mathfrak{r}_2)\left(\Phi'''(\mathfrak{r}_2)\right)^{\alpha} - \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2(\mathfrak{r})}\right)^{\alpha} \\ &\quad + \frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\sqrt[\alpha]{\beta}}(\mathfrak{r})}\right)^{\alpha}\frac{1}{\eta_2^{\alpha(1-\sqrt[\alpha]{\beta})}(\mathfrak{r}_2)} \end{aligned}$$

hence,

$$h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} < -\frac{\beta}{1-\sqrt[\alpha]{\beta}}\mu^{\alpha\sqrt[\alpha]{\beta}}\Phi^{\alpha}(\mathfrak{r}),$$

or

$$h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r}) < -\frac{\sqrt[\alpha]{\beta}}{\sqrt[\alpha]{1-\sqrt[\alpha]{\beta}}}\mu^{\sqrt[\alpha]{\beta}}\Phi(\mathfrak{r}) = -\epsilon_1\beta_1\Phi(\mathfrak{r}),$$

for $\mathfrak{r} \geq \mathfrak{r}_4$, where

$$\epsilon_1 = \sqrt[\alpha]{\frac{\beta(1-\sqrt[\alpha]{\beta_*})}{\beta_*(1-\sqrt[\alpha]{\beta})}}\frac{\mu^{\sqrt[\alpha]{\beta}}}{\mu_*^{\sqrt[\alpha]{\beta_*}}}$$

is an arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\mu \to \mu_*$. Hence,

$$\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\epsilon_1\beta_1}(\mathfrak{r})}\right) < 0,$$

for $\mathfrak{r} \geq \mathfrak{r}_4$. One can show that through induction, for any $n \in \mathbb{N}_o$ and $\mathfrak{r}$ large enough,

$$\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\epsilon_n\beta_n}(\mathfrak{r})}\right)' < 0,$$

where ϵ_n given by

$$\epsilon_o = \sqrt[\alpha]{\frac{\beta}{\beta_*}}$$

$$\epsilon_{n+1} = \epsilon_o\sqrt[\alpha]{\frac{1-\beta_n}{1-\epsilon_n\beta_n}}\frac{\mu^{\epsilon_n\beta_n}}{\mu_*^{\beta_n}}, \quad n \in \mathbb{N}_o$$

is an arbitrary constant from $(0,1)$ approaching 1 if $\beta \to \beta_*$ and $\mu \to \mu_*$. Finally, we claim that from any $n \in \mathbb{N}_o$

$$\left(\frac{\Phi(\mathfrak{r})}{\eta_2^{\epsilon_{n+1}\beta_{n+1}}(\mathfrak{r})}\right)' < 0$$

implies that from (16) and the fact that ϵ_{n+1} is arbitrary close to 1,

$$\epsilon_{n+1}\beta_{n+1} > \beta_n.$$

Hence, for $\mathfrak{r}$ large enough,

$$h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta_2(\mathfrak{r}) < -\epsilon_{n+1}\beta_{n+1}\Phi(\mathfrak{r}) < -\beta_n\Phi(\mathfrak{r}).$$

So, for any $n \in \mathbb{N}_o$ and $\mathfrak{t}$ large enough,

$$\left(\frac{\Phi(\mathfrak{t})}{\eta_2^{\beta_n}(\mathfrak{t})}\right)' < 0.$$

The proof is complete. □

3. Nonexistence of Solutions in the Class $_3^+$

Theorem 6. *Suppose that* $(V1)$ *and* $(V2)$ *hold. If*

$$\limsup_{\mathfrak{t}\to\infty} \int_{\mathfrak{t}_o}^{\mathfrak{t}} \left[H^\alpha(\jmath) q(\jmath) \frac{\eta_2^{\alpha\beta_n}(\tau(\jmath))}{\eta_2^{\alpha\beta_n}(\jmath)} - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{(H'(\jmath))^{\alpha+1}}{H(\jmath)\eta_1^\alpha(\jmath)} \right] \mathrm{d}\jmath = \infty, \tag{29}$$

then $_3^+ = \varnothing$. *Where*

$$H(\mathfrak{t}) = \int_{\mathfrak{t}}^{\infty} (\jmath - \mathfrak{t}) \eta(\jmath) \mathrm{d}\jmath.$$

Proof. Consider the case where (1) has a nonoscillatory solution. We can suppose that $\Phi \in {}^+$ eventually without losing generality. Assume that Φ satisfies case (3). Since $h(\mathfrak{t})(\Phi'''(\mathfrak{t}))^\alpha$ is nonincreasing, we obtain

$$h^{1/\alpha}(\jmath)\Phi'''(\jmath) \le h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t}), \quad \jmath \ge \mathfrak{t} \ge \mathfrak{t}_1. \tag{30}$$

By dividing (30) by $h^{1/\alpha}(\jmath)$ and integrating the resulting inequality from $\mathfrak{t}$ to ℓ, we obtain

$$\Phi''(\ell) \le \Phi''(\mathfrak{t}) + h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t}) \int_{\mathfrak{t}}^{\ell} h^{1/\alpha}(\jmath) \mathrm{d}\jmath.$$

Letting $\ell \to \infty$, we have

$$0 \le \Phi''(\mathfrak{t}) + h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t})\eta(\mathfrak{t}),$$

which produces

$$\Phi''(\mathfrak{t}) \ge -\eta(\mathfrak{t}) h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t}). \tag{31}$$

Integrating (31) from $\mathfrak{t}$ to ∞, yields

$$-\Phi'(\mathfrak{t}) \ge -h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t}) \int_{\mathfrak{t}}^{\infty} \eta(\jmath) \mathrm{d}\jmath. \tag{32}$$

Again, integrating (32) from $\mathfrak{t}$ to ∞, we obtain

$$\Phi(\mathfrak{t}) \ge -h^{1/\alpha}(\mathfrak{t})\Phi'''(\mathfrak{t}) \int_{\mathfrak{t}}^{\infty} (\jmath - \mathfrak{t})\eta(\jmath) \mathrm{d}\jmath.$$

Now, define the function ω by

$$\omega(\mathfrak{t}) := \frac{h(\mathfrak{t})(\Phi'''(\mathfrak{t}))^\alpha}{(\Phi(\mathfrak{t}))^\alpha}, \qquad \mathfrak{t} \ge \mathfrak{t}_1. \tag{33}$$

Then, we see that $\omega(\mathfrak{t}) < 0$ for $\mathfrak{t} \ge \mathfrak{t}_1$. Differentiating (33), we obtain

$$\omega'(\mathfrak{t}) = \frac{(h(\mathfrak{t})(\Phi'''(\mathfrak{t}))^\alpha)'}{(\Phi(\mathfrak{t}))^\alpha} - \alpha \frac{h(\mathfrak{t})(\Phi'''(\mathfrak{t}))^\alpha \Phi'(\mathfrak{t})}{(\Phi(\mathfrak{t}))^{\alpha+1}}.$$

It follows from (1) and (32) that

$$\omega'(\mathfrak{t}) \le -q(\mathfrak{t}) \frac{\Phi^\alpha(\tau(\mathfrak{t}))}{(\Phi(\mathfrak{t}))^\alpha} - \alpha\omega^{1+1/\alpha}(\mathfrak{t}) \int_{\mathfrak{t}}^{\infty} \eta(\jmath) \mathrm{d}\jmath.$$

$$\omega'(\mathfrak{r}) \leq -q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\mathfrak{r})}{(\Phi(\mathfrak{r}))^{\alpha}}\frac{\Phi^{\alpha}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} - \alpha\omega^{1+1/\alpha}(\mathfrak{r})\int_{\mathfrak{r}}^{\infty}\eta(\jmath)\mathrm{d}\jmath.$$

Lemma 4 yields

$$-\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{(\Phi(\tau(\mathfrak{r})))^{\alpha}} \geq -\frac{\eta_2^{\alpha\beta_n}(\mathfrak{r})}{(\Phi(\mathfrak{r}))^{\alpha}},$$

hence,

$$\omega'(\mathfrak{r}) \leq -q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} - \alpha\omega^{1+1/\alpha}(\mathfrak{r})\int_{\mathfrak{r}}^{\infty}\eta(\jmath)\mathrm{d}\jmath. \tag{34}$$

Multiplying (34) by $H^{\alpha}(\mathfrak{r})$ and integrating the resulting inequality from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$\begin{gathered}H^{\alpha}(\mathfrak{r})\omega(\mathfrak{r}) - H^{\alpha}(\mathfrak{r}_1)\omega(\mathfrak{r}_1) - \alpha\int_{\mathfrak{r}_1}^{\mathfrak{r}} H'(\jmath)H^{\alpha-1}(\jmath)\omega(\jmath)\mathrm{d}\jmath + \int_{\mathfrak{r}_1}^{\mathfrak{r}} q(\jmath)\frac{\eta_2^{\alpha\beta_n}(\tau(\jmath))}{\eta_2^{\alpha\beta_n}(\jmath)}H^{\alpha}(\jmath)\mathrm{d}\jmath \\ +\alpha\int_{\mathfrak{r}_1}^{\mathfrak{r}}\omega^{1+1/\alpha}(\jmath)\eta_1(\jmath)H^{\alpha}(\jmath)\mathrm{d}\jmath \leq 0.\end{gathered}$$

By using the inequality (12) with $K = -H'(\jmath)H^{\alpha-1}(\jmath)$, $L = \eta_1(\jmath)H^{\alpha}(\jmath)$, and $v = -\omega(\jmath)$, we obtain

$$\int_{\mathfrak{r}_1}^{\mathfrak{r}}\left[q(\jmath)\frac{\eta_2^{\alpha\beta_n}(\tau(\jmath))}{\eta_2^{\alpha\beta_n}(\jmath)}H^{\alpha}(\jmath) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(H'(\jmath))^{\alpha+1}}{H(\jmath)\eta_1^{\alpha}(\jmath)}\right]\mathrm{d}\jmath \leq H^{\alpha}(\mathfrak{r}_1)\omega(\mathfrak{r}_1) + 1,$$

we obtain a contradiction with (29) by taking the lim sup on both sides of this inequality. The proof is now complete. □

Theorem 7. *Suppose that $\alpha \geq 1$. If there is a function $\gamma \in C^1([\mathfrak{r}_o,\infty),(0,\infty))$ such that*

$$\limsup_{\mathfrak{r}\to\infty}\int_{\mathfrak{r}_o}^{\mathfrak{r}}\left[\psi(\jmath) - \frac{\gamma(\jmath)}{(\alpha+1)^{(\alpha+1)}\eta_1^{\alpha}(\mathfrak{r})}\left(\frac{\gamma'(\jmath)}{\gamma(\jmath)} + \frac{(1+\alpha)\eta_1(\jmath)}{\eta_2(\jmath)}\right)^{\alpha+1}\right]\mathrm{d}\jmath = \infty, \tag{35}$$

where

$$\psi(\mathfrak{r}) = \gamma(\mathfrak{r})q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} + (1-\alpha)\gamma(\mathfrak{r})\eta_1(\mathfrak{r})/\eta_2^{\alpha+1}(\mathfrak{r}).$$

Then ${}_3^+ = \varnothing$.

Proof. Consider the case where (1) has a nonoscillatory solution. We can suppose that $\Phi \in {}^+$ eventually without losing generality. Assume that Φ satisfies case (3). Since $h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^{\alpha}$ is non-increasing, we obtain

$$\begin{aligned}\Phi''(v) - \Phi''(\mathfrak{r}) &= \int_{\mathfrak{r}}^{v}\frac{1}{h^{1/\alpha}(\zeta)}\left(h(\zeta)(\Phi'''(\zeta))^{\alpha}\right)^{1/\alpha}\mathrm{d}\zeta \\ &\leq h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\int_{\mathfrak{r}}^{v}\frac{1}{h^{1/\alpha}(\zeta)}\mathrm{d}\zeta.\end{aligned}$$

Letting $v \to \infty$, we have

$$\Phi''(\mathfrak{r}) \geq -h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta(\mathfrak{r}). \tag{36}$$

Integrating (36) from $\mathfrak{r}$ to ∞ yields

$$-\Phi'(\mathfrak{r}) \geq -h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta_1(\mathfrak{r}). \tag{37}$$

Again, integrating (37) from $\mathfrak{r}$ to ∞, we obtain

$$\Phi(\mathfrak{r}) \geq -h^{1/\alpha}(\mathfrak{r})\Phi'''(\mathfrak{r})\eta_2(\mathfrak{r}).$$

Now, define the function ω_1 by

$$\omega_1(\mathfrak{r}) = \gamma(\mathfrak{r})\left(\frac{h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^\alpha}{(\Phi(\mathfrak{r}))^\alpha} + \frac{1}{\eta_2^\alpha(\mathfrak{r})}\right), \quad \mathfrak{r} \geq \mathfrak{r}_1. \tag{38}$$

Then, we see that $\omega_1(\mathfrak{r}) > 0$ for $\mathfrak{r} \geq \mathfrak{r}_1$. Therefore, we have

$$\omega_1'(\mathfrak{r}) = \frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})}\omega_1(\mathfrak{r}) + \gamma(\mathfrak{r})\frac{\left(h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^\alpha\right)'}{(\Phi(\mathfrak{r}))^\alpha} - \alpha\gamma(\mathfrak{r})\frac{h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^\alpha\Phi'(\mathfrak{r})}{(\Phi(\mathfrak{r}))^{\alpha+1}} - \alpha\gamma(\mathfrak{r})\frac{\eta_2'(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})}.$$

It follows from (1) that

$$\omega_1'(\mathfrak{r}) = \frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})}\omega_1(\mathfrak{r}) - q(\mathfrak{r})\gamma(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\mathfrak{r})}{(\Phi(\mathfrak{r}))^\alpha}\frac{\Phi^\alpha(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} - \alpha\gamma(\mathfrak{r})\frac{h(\mathfrak{r})(\Phi'''(\mathfrak{r}))^\alpha\Phi'(\mathfrak{r})}{(\Phi(\mathfrak{r}))^{\alpha+1}} - \alpha\gamma(\mathfrak{r})\frac{\eta_2'(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})}.$$

From (37) and (38), we find

$$\begin{aligned}\omega_1'(\mathfrak{r}) \quad \leq \quad & \frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})}\omega_1(\mathfrak{r}) - q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\mathfrak{r})}{(\Phi(\mathfrak{r}))^\alpha}\frac{\Phi^\alpha(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} - \alpha\gamma(\mathfrak{r})\eta_1(\mathfrak{r})\left(\frac{\omega_1(\mathfrak{r})}{\gamma(\mathfrak{r})} - \frac{1}{\eta_2^\alpha(\mathfrak{r})}\right)^{1+1/\alpha} \\ & +\alpha\gamma(\mathfrak{r})\frac{\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})}.\end{aligned}$$

From Lemma 4, we obtain

$$-\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{(\Phi(\tau(\mathfrak{r})))^\alpha} \geq -\frac{\eta_2^{\alpha\beta_n}(\mathfrak{r})}{(\Phi(\mathfrak{r}))^\alpha},$$

hence,

$$\omega_1'(\mathfrak{r}) \leq \frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})}\omega_1(\mathfrak{r}) - \gamma(\mathfrak{r})q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} + \alpha\gamma(\mathfrak{r})\frac{\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})} - \alpha\gamma(\mathfrak{r})\eta_1(\mathfrak{r})\left(\frac{\omega_1(\mathfrak{r})}{\gamma(\mathfrak{r})} - \frac{1}{\eta_2^\alpha(\mathfrak{r})}\right)^{1+1/\alpha}.$$

By using the inequality (13) with $A = \omega_1(\mathfrak{r})/\gamma(\mathfrak{r})$ and $\jmath = 1/\eta_2^\alpha(\mathfrak{r})$, we obtain

$$\begin{aligned}\omega_1'(\mathfrak{r}) \quad \leq \quad & \frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})}\omega_1(\mathfrak{r}) - \gamma(\mathfrak{r})q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} + \alpha\gamma(\mathfrak{r})\frac{\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})} \\ & -\alpha\gamma(\mathfrak{r})\eta_1(\mathfrak{r})\left\{\left(\frac{\omega_1(\mathfrak{r})}{\gamma(\mathfrak{r})}\right)^{1+1/\alpha} - \frac{1}{\eta_2(\mathfrak{r})}\left((1+\alpha)\frac{\omega_1(\mathfrak{r})}{\gamma(\mathfrak{r})} - \frac{1}{\eta_2^\alpha(\mathfrak{r})}\right)\right\},\end{aligned}$$

hence,

$$\begin{aligned}\omega_1'(\mathfrak{r}) \quad \leq \quad & \left(\frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})} + \frac{(1+\alpha)\eta_1(\mathfrak{r})}{\eta_2(\mathfrak{r})}\right)\omega_1(\mathfrak{r}) - \gamma(\mathfrak{r})q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} - \frac{\alpha\eta_1(\mathfrak{r})}{\gamma^{1/\alpha}(\mathfrak{r})}\omega_1^{1+1/\alpha}(\mathfrak{r}) \\ & -\frac{\gamma(\mathfrak{r})\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})} + \frac{\alpha\gamma(\mathfrak{r})\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})}.\end{aligned}$$

Using the inequality (12) with $K = \gamma'(\mathfrak{r})/\gamma(\mathfrak{r}) + (1+\alpha)\eta_1(\mathfrak{r})/\eta_2(\mathfrak{r})$, $L = \alpha\eta_1(\mathfrak{r})/\gamma^{1/\alpha}(\mathfrak{r})$, and $v = \omega_1(\mathfrak{r})$, we obtain

$$\begin{aligned}
\omega_1'(\mathfrak{r}) \quad \leq \quad & -\gamma(\mathfrak{r})q(\mathfrak{r})\frac{\eta_2^{\alpha\beta_n}(\tau(\mathfrak{r}))}{\eta_2^{\alpha\beta_n}(\mathfrak{r})} + (\alpha-1)\frac{\gamma(\mathfrak{r})\eta_1(\mathfrak{r})}{\eta_2^{\alpha+1}(\mathfrak{r})} \\
& + \frac{\gamma(\mathfrak{r})}{(\alpha+1)^{(\alpha+1)}\eta_1^{\alpha}(\mathfrak{r})}\left(\frac{\gamma'(\mathfrak{r})}{\gamma(\mathfrak{r})} + \frac{(1+\alpha)\eta_1(\mathfrak{r})}{\eta_2(\mathfrak{r})}\right)^{\alpha+1}. \qquad (39)
\end{aligned}$$

Integrating (39) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$\int_{\mathfrak{r}_1}^{\mathfrak{r}}\left[\psi(\jmath) - \frac{\gamma(\jmath)}{(\alpha+1)^{(\alpha+1)}\eta_1^{\alpha}(\mathfrak{r})}\left(\frac{\gamma'(\jmath)}{\gamma(\jmath)} + \frac{(1+\alpha)\eta_1(\jmath)}{\eta_2(\jmath)}\right)^{\alpha+1}\right]\mathrm{d}\jmath \leq \omega_1(\mathfrak{r}_1),$$

we obtain a contradiction with (35) by taking the lim sup on both sides of this inequality. The proof is now complete. □

Theorem 8. *Suppose that $\Phi \in C((\mathfrak{r}_o,\infty),(0,\infty))$ is a solution of (1). If the differential equation*

$$\Phi'(\mathfrak{r}) + \frac{1}{\eta_2(\tau(\mathfrak{r}))}\left(\int_{\mathfrak{r}}^{\infty}\int_{\zeta}^{\infty}\frac{\eta_2(\tau(v))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v\mathrm{d}\zeta\right)\Phi(\tau(\mathfrak{r})) = 0. \qquad (40)$$

is oscillatory, then $_3^+ = \varnothing$.

Proof. Assume that $\Phi \in {}^+$ and satisfies case (3). From (1) and integrating from $\mathfrak{r}_1$ to $\mathfrak{r}$, we obtain

$$h(\mathfrak{r})\left(\Phi'''(\mathfrak{r})\right)^{\alpha} \leq -\Phi^{\alpha}(\tau(\mathfrak{r}))\int_{\mathfrak{r}_1}^{\mathfrak{r}}q(\jmath)\mathrm{d}\jmath. \qquad (41)$$

As in the proof of Lemma 3, we obtain that (19), (21), and (23) hold. Now, integrating (41) from $\mathfrak{r}$ to ∞ and using (23), we obtain

$$-\Phi''(\mathfrak{r}) \leq -\int_{\mathfrak{r}}^{\infty}\frac{\Phi(\tau(v))}{\eta_2(\tau(\mathfrak{r}))}\frac{\eta_2(\tau(\mathfrak{r}))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v.$$

From Lemma 3, note that $\Phi(\mathfrak{r})/\eta_2(\mathfrak{r})$ is nondecreasing and yields

$$-\Phi''(\mathfrak{r}) \leq -\frac{\Phi(\tau(\mathfrak{r}))}{\eta_2(\tau(\mathfrak{r}))}\int_{\mathfrak{r}}^{\infty}\frac{\eta_2(\tau(\mathfrak{r}))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v. \qquad (42)$$

Integrating (42) from $\mathfrak{r}$ to ∞, we find

$$\begin{aligned}
\Phi'(\mathfrak{r}) \quad & \leq \quad -\int_{\mathfrak{r}}^{\infty}\frac{\Phi(\tau(\zeta))}{\eta_2(\tau(\zeta))}\int_{\zeta}^{\infty}\frac{\eta_2(\tau(\mathfrak{r}))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v\mathrm{d}\zeta \\
& \leq \quad -\frac{\Phi(\tau(\mathfrak{r}))}{\eta_2(\tau(\mathfrak{r}))}\int_{\mathfrak{r}}^{\infty}\int_{\zeta}^{\infty}\frac{\eta_2(\tau(v))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v\mathrm{d}\zeta.
\end{aligned}$$

As a result, it is clear that Φ is a positive solution to the first-order delay differential inequality

$$\Phi'(\mathfrak{r}) + \frac{1}{\eta_2(\tau(\mathfrak{r}))}\left(\int_{\mathfrak{r}}^{\infty}\int_{\zeta}^{\infty}\frac{\eta_2(\tau(v))}{h^{1/\alpha}(v)}\left(\int_{\mathfrak{r}_1}^{v}q(\jmath)\mathrm{d}\jmath\right)^{1/\alpha}\mathrm{d}v\mathrm{d}\zeta\right)\Phi(\tau(\mathfrak{r})) \leq 0.$$

According to [20], Equation (40) also has a solution that is positive, creating a contradiction. The proof is now complete. □

Corollary 1. *Assume that $\Phi \in C((\mathfrak{r}_o, \infty), (0, \infty))$ is a solution of (1). If*

$$\liminf_{\mathfrak{r}\to\infty} \int_{\tau(\mathfrak{r})}^{\mathfrak{r}} \frac{1}{\eta_2(\tau(\xi))} \left(\int_{\xi}^{\infty} \int_{\zeta}^{\infty} \frac{\eta_2(\tau(v))}{h^{1/\alpha}(v)} \left(\int_{\mathfrak{r}_1}^{v} q(\jmath) \mathrm{d}\jmath \right)^{1/\alpha} \mathrm{d}v \mathrm{d}\zeta \right) \mathrm{d}\xi > \frac{1}{\mathrm{e}}, \tag{43}$$

then $_{3}^{+} = \varnothing$.

Proof. We remark that (43) ensures the oscillation of (40) using [20]. The proof is now complete. □

4. Application in Oscillation Theory

The criteria for oscillation depend on finding conditions that exclude each case of the derivatives of the solution separately. In many cases, we note that the most influential condition in the test of oscillation of the equation is the condition of excluding decreasing solutions. Therefore, improving the conditions for excluding decreasing solutions necessarily affects the improvement of oscillation criteria. In this section, we will set the criteria for testing oscillation for (1) to combine conditions known in the literature that exclude cases (1) and (2) of the derivatives of the solution with the new conditions in the previous section that exclude the existence of solutions that fulfill case (3).

In the next theorems, the proof of the case where (1) or (2) holds is the same as that of [16] (Theorem 2.1, Theorem 2.2). Moreover, either conditions (29) or (35), or (43), excludes case (3) .

Theorem 9. *Assume that (29) holds. If (8) and (10) hold for some $\lambda_1 \in (0,1)$, then (1) oscillates.*

Theorem 10. *Assume that (35) holds. If (8) and (10) hold for some $\lambda_1 \in (0,1)$, then (1) oscillates.*

Theorem 11. *Assume that (43) holds. If (8) and (10) hold for some $\lambda_1 \in (0,1)$, then (1) oscillates.*

Example 1. *We consider*

$$\left(\mathrm{e}^{\alpha\mathfrak{r}} \left(\Phi'''(\mathfrak{r}) \right)^{\alpha} \right)' + q_o \mathrm{e}^{\alpha\mathfrak{r}} \Phi^{\alpha} \left(\mathfrak{r} - \arcsin\left(\sqrt{10}/10 \right) \right) = 0, \tag{44}$$

where $h(\mathfrak{r}) = \mathrm{e}^{\alpha\mathfrak{r}}$, $q(\mathfrak{r}) = q_o \mathrm{e}^{\alpha\mathfrak{r}}$, $\tau(\mathfrak{r}) = \mathfrak{r} - \arcsin\left(\sqrt{10}/10\right)$ *and* $\eta(\mathfrak{r}) = \mathrm{e}^{-\mathfrak{r}}$. *Note that*

$$\begin{aligned} H(\mathfrak{r}) &= \int_{\mathfrak{r}}^{\infty} (\jmath - \mathfrak{r}) \mathrm{e}^{-\jmath} \mathrm{d}\jmath \\ &= \mathrm{e}^{-\mathfrak{r}}. \end{aligned}$$

If we choose $\gamma(\mathfrak{r}) = \mathrm{e}^{-\alpha\mathfrak{r}}$, *then we see that*

$$\eta_1(\mathfrak{r}) = \mathrm{e}^{-\mathfrak{r}},\ \eta_2(\mathfrak{r}) = \mathrm{e}^{-\mathfrak{r}} \text{ and } \eta_2(\tau(\mathfrak{r})) = \mathrm{e}^{-\left(\mathfrak{r} - \arcsin\left(\sqrt{10}/10\right)\right)}.$$

It is easy to verify that

$$\ell_o = \frac{\mathrm{e}^{\sqrt{q_o/\alpha}\arcsin\left(\sqrt{10}/10\right)}}{\sqrt{1 - \sqrt{\sqrt{q_o/\alpha}}}},\ \mu_* = \frac{\mathrm{e}^{-\left(\mathfrak{r} - \arcsin\left(\sqrt{10}/10\right)\right)}}{\mathrm{e}^{-\mathfrak{r}}} = \mathrm{e}^{\arcsin\left(\sqrt{10}/10\right)},$$

$n = 0$, $\beta_* = q_o/\alpha$, *and* $\beta_o = \sqrt{q_o/\alpha}$.

By using Theorem 9, we find conditions (8) and (10) are satisfied and the condition (29) holds if

$$q_o > \frac{(\alpha)^{\alpha+1}}{(\alpha+1)^{\alpha+1} \mathrm{e}^{\alpha\sqrt{q_o/\alpha}\arcsin\left(\sqrt{10}/10\right)}}. \tag{45}$$

Therefore, Equation (44) is oscillatory if (45) holds. Additionally, by using Theorem 10, we find that condition (35) is satisfied if

$$q_o > \frac{1}{e^{\alpha\sqrt{q_o/\alpha}\arcsin(\sqrt{10}/10)}}\left(\frac{1}{(\alpha+1)^{\alpha+1}} - (1-\alpha)\right). \tag{46}$$

Therefore, Equation (44) is oscillatory if (46) holds. Now, by using Theorem 2 and Theorem 5, Equation (44) is oscillatory if

$$q_o > \frac{(\alpha)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{47}$$

Figure 1 illustrates the efficiency of conditions (45)–(47) in studying the oscillation of solutions of (44).

Remark 2. *To the best of our knowledge, the known related sharp criterion for (44) based on Example 1 gives*

$$q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{48}$$

Note firstly that our criteria (45) and (46) essentially take into account the influence of the delay argument $\tau(\mathfrak{r})$, which has been neglected in all previous results of fourth-order equations.

Secondly, in the case where $\alpha = 1$, we get the results in Table 1. Therefore, we note that conditions (45) and (46) support the most efficient and sharp criterion for oscillation of Equation (44).

Table 1. Comparison of the different oscillation criteria of (44) with $\alpha = 1$.

Condition	(45)	(46)	(48)
Criterion	$q_0 > 0.215$	$q_0 > 0.215$	$q_0 > 0.250$

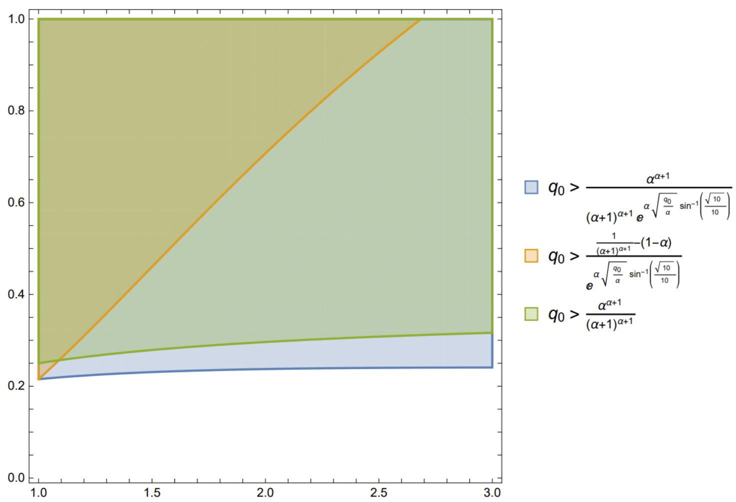

Figure 1. Regions for which conditions (45)–(47) are satisfied.

5. Conclusions

The study of oscillations for delay differential equations always begins with the classification of positive solutions based on the sign of their derivatives. The oscillation criteria depend on the conditions that exclude each case of the positive solutions. In many cases, the exclusion of decreasing solutions is the condition that has the most effect on the test for the oscillation of the equation. Therefore, improving the criteria for oscillation must obviously have an effect on improving the conditions for excluding decreasing solutions. In this work, we study the asymptotic properties of solutions to the fourth-order delay differential equation with the non-canonical operator. We have created new properties that help us have more effective terms in the oscillation of the Equation (1). We use the comparison theorem and more than one compensation for Riccatti to obtain criteria that guarantee the exclusion of decreasing solutions. After that, by combining well-known results with the results of Section 3, we set new criteria for the oscillation of the studied equation. Finally, we gave an example to illustrate the novelty and importance of our results. An open question is whether the neutral delay equation can be studied with the same technique used in this research.

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