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Optical Solitons in Fiber Bragg Gratings with Dispersive Reflectivity Having Five Nonlinear Forms of Refractive Index

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Abstract: This paper implements the trial equation approach to retrieve cubic–quartic optical solitons in fiber Bragg gratings with the aid of the trial equation methodology. Five forms of nonlinear refractive index structures are considered. They are the Kerr law, the parabolic law, the polynomial law, the quadratic–cubic law, and the parabolic nonlocal law. Dark and singular soliton solutions are recovered along with Jacobi's elliptic functions with an appropriate modulus of ellipticity.

Keywords: Bragg gratings; solitons; cubic–quartic

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1. Introduction

The dynamics of optical solitons have revolutionized modern-day telecommunication engineering. Intercontinental long-distance communication technology has immeasurably advanced. These soliton molecules or pulses constitute the fundamentals of the engineering of such communications. Yet, there pressing issues still need to be addressed for smooth and sustainable soliton propagation across long distances through optical fibers [1–5].

Occasionally, the chromatic dispersion (CD) gradually becomes depleted during soliton transmission over intercontinental distances. This has led to the replacement of CD with the collective effect of third-order and fourth-order dispersions. Together, this is referred to as cubic–quartic (CQ) dispersion, which has led to the emergence of CQ solitons [6–10]. Another countermeasure to circumvent the effect of the depletion of CD is to introduce a grating structure along the internal walls of the fiber core, which produces dispersive reflectivity [11–15]. This technology was first introduced by Bragg; hence, it is referred to as a Bragg grating. The current paper combines both such measures to lead to the structure of the governing models [16–20]. Five forms of nonlinear refractive index structures are also studied in this paper.

The model is modified for each of these nonlinear forms. They are individually studied by the aid of the trial equation approach, and the soliton solutions are recovered. The fundamental solutions in terms of Jacobi's elliptic functions naturally emerge from the integration scheme. Subsequently, when the limiting approach to these elliptic functions are implemented, with respect to the modulus of ellipticity, the soliton solutions emerge. Thus, the integration scheme is a double-layered process to yield soliton solutions from the model. It must also be noted that only dark and singular soliton solutions emerge from this mathematical scheme.

2. Trial Equation Method

Step 1. Consider a model equation

$$G(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{1}$$

where u is the dependent variable, and x and t are the independent variables. Take the wave variable

$$\xi = h(x - mt), \quad u(x,t) = u(\xi), \tag{2}$$

where h and m are constants. Substituting Equation (2) into Equation (1) yields an ordinary differential equation

$$G(u, u', u'', \cdots) = 0. \tag{3}$$

Step 2. Take the trial equation [21–28]

$$u'' = \sum_{i=0}^{n} s_i u^i(\xi), \tag{4}$$

where n comes from the balance algorithm, and we derive

$$(u')^2 = F(u) = \sum_{i=0}^{n} \frac{2s_i}{i+1} u^{i+1} + s, \tag{5}$$

$$u'''' = (\sum_{i=2}^{n} i(i-1)s_i u^{i-2})(\sum_{i=0}^{n} \frac{2s_i}{i+1} u^{i+1} + s) + (\sum_{i=1}^{n} i s_i u^{i-1})(\sum_{i=0}^{n} s_i u^i). \tag{6}$$

Inserting Equation (4) into Equation (3) leaves us with a system of equations that enables us to obtain the values of s_i and s.

Step 3. Write Equation (5) as the standard integral form

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{F(u)}}. \tag{7}$$

Then, we can obtain the solutions to Equation (1) by solving the integral (7).

3. Application to Fiber Bragg Gratings

3.1. Kerr Law

In this case, the model is

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + (c_1|q|^2 + d_1|r|^2)q + i\alpha_1 q_x + \beta_1 r = 0, \tag{8}$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + (c_2|r|^2 + d_2|q|^2)r + i\alpha_2 r_x + \beta_2 q = 0, \tag{9}$$

where t and x account for time in dimensionless form and the non-dimensional distance, respectively, while $q(x,t)$ and $r(x,t)$ represent the wave profiles. a_j, b_j, c_j, α_j, d_j, and β_j $(j = 1,2)$ come from the third-order dispersion, fourth-order dispersion, self-phase modulation, intermodal dispersion, cross-phase modulation, and the detuning parameters, respectively. The first terms stand for the linear temporal evolution, and $i = \sqrt{-1}$.

We consider the wave profiles

$$q(x,t) = P_1(\xi)e^{i\phi(x,t)}, \tag{10}$$

$$r(x,t) = P_2(\xi)e^{i\phi(x,t)}, \tag{11}$$

along with

$$\xi = k(x - vt), \tag{12}$$

$$\phi(x,t) = -\kappa x + \omega t + \theta, \tag{13}$$

where $P_l(\xi)(l = 1,2)$ are the soliton amplitude components, and ξ signifies the wave variable. $v, \kappa, \omega, \theta$, and $\phi(x,t)$ stand for the soliton velocity, the soliton frequency, the soliton wave number, the phase constant, and the soliton phase component, respectively.

Inserting Equations (10) and (11) into Equations (8) and (9) provides us with the ancillary equations

$$3\kappa k^2 P_{\bar{l}}''(a_l - 2\kappa b_l) + P_{\bar{l}}(-\kappa^3 a_l + \kappa^4 b_l + \beta_l) + k^4 b_l P_{\bar{l}}'''' + d_l P_l P_{\bar{l}}^2 + c_l P_l^3 + P_l(\alpha_l \kappa - \omega) = 0, \tag{14}$$

$$k^3 P_{\bar{l}}'''(a_l - 4\kappa b_l) + \kappa^2 k P_{\bar{l}}'(4\kappa b_l - 3a_l) + kP_l'(\alpha_l - v) = 0, \tag{15}$$

where $\tilde{l} = 3 - l$. Setting

$$P_2 = \chi P_1, \tag{16}$$

Equations (14) and (15) evolve as

$$3\chi\kappa k^2(a_1 - 2b_1\kappa)P_1'' + P_1(-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega) + b_1\chi k^4 P_1'''' + P_1^3(c_1 + \chi^2 d_1) = 0, \tag{17}$$

$$\chi k^3(a_1 - 4b_1\kappa)P_1''' + (-3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3 - v)kP_1' = 0, \tag{18}$$

with the usage of the following restrictions

$$\begin{aligned}
&c_2\chi^3 + \chi d_2 = c_1 + \chi^2 d_1,\\
&a_2 - 2b_2\kappa = \chi(a_1 - 2b_1\kappa),\\
&b_2 = \chi b_1,\\
&\alpha_2\chi\kappa - a_2\kappa^3 + b_2\kappa^4 + \beta_2 - \chi\omega = -a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega.
\end{aligned} \tag{19}$$

Equation (18) provides the velocity

$$v = -3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3, \tag{20}$$

by virtue of the relations

$$\alpha_2\chi - 3a_2\kappa^2 + 4b_2\kappa^3 = -3a_1\chi^2\kappa^2 + \alpha_1\chi + 4b_1\chi^2\kappa^3, \tag{21}$$

$$a_l = 4b_l\kappa, \tag{22}$$

while Equation (17) reads as

$$A_1 P_1'' + A_3 P_1^3 + A_2 P_1 + k^2 P_1'''' = 0, \tag{23}$$

where

$$A_1 = \frac{3\kappa(a_1\chi - 2b_1\chi\kappa)}{b_1\chi}, A_2 = \frac{-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega}{k^2 b_1\chi}, A_3 = \frac{c_1 + \chi^2 d_1}{k^2 b_1\chi}. \tag{24}$$

With the usage of the balance algorithm in (23), Equation (5) becomes

$$(P_1')^2 = \frac{2s_2}{3}P_1^3 + s_1 P_1^2 + 2s_0 P_1 + s, \tag{25}$$

where

$$s_2 = (-\frac{3A_3}{10k^2})^{\frac{1}{2}}, s_1 = -\frac{A_1}{5k^2}, s_0 = (-\frac{A_2}{6k^2} + \frac{2A_1^2}{75k^4})(-\frac{3A_3}{10k^2})^{-\frac{1}{2}},$$
$$s = \frac{4A_1}{3A_3}(-\frac{A_2}{6k^2} + \frac{2A_1^2}{75k^4}). \tag{26}$$

With the help of the criterions

$$p = (\frac{2s_2}{3})^{\frac{1}{3}} P_1, \xi_1 = (\frac{2s_2}{3})^{\frac{1}{3}} \xi, \tag{27}$$

Equation (25) becomes

$$(p_{\xi_1})^2 = p^3 + z_2 p^2 + z_1 p + z_0, \tag{28}$$

where

$$z_2 = -\frac{A_1}{5k^2}(-\frac{2A_3}{15k^2})^{-\frac{1}{3}},$$
$$z_1 = (-\frac{A_2}{3k^2} + \frac{4A_1^2}{75k^4})(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}, \tag{29}$$
$$z_0 = \frac{4A_1}{3A_3}(-\frac{A_2}{6k^2} + \frac{2A_1^2}{75k^4}).$$

We rewrite Equation (28) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dp}{\sqrt{F(p)}}, \tag{30}$$

where

$$F(p) = p^3 + z_2 p^2 + z_1 p + z_0. \tag{31}$$

We consider the discriminant system

$$\Delta = -27(\frac{2z_2^3}{27} + z_0 - \frac{z_1 z_2}{3})^2 - 4(z_1 - \frac{z_2^2}{3})^3, D_1 = z_1 - \frac{z_2^2}{3}. \tag{32}$$

Case 1. $\Delta = 0, D_1 < 0$, and $F(p) = (p-\gamma_1)^2(p-\gamma_2)$. When $p > \gamma_2$, if $\gamma_1 > \gamma_2$, the dark and singular solitons are

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[(\gamma_1 - \gamma_2)\tanh^2(\frac{\sqrt{\gamma_1 - \gamma_2}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0)) + \gamma_2]\}e^{i(-\kappa x + \omega t + \theta)}, \tag{33}$$

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[(\gamma_1 - \gamma_2)\coth^2(\frac{\sqrt{\gamma_1 - \gamma_2}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0)) + \gamma_2]\}e^{i(-\kappa x + \omega t + \theta)}, \tag{34}$$

and if $\gamma_1 < \gamma_2$, the singular periodic solution is

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[(\gamma_2 - \gamma_1)\tan^2(\frac{\sqrt{\gamma_2 - \gamma_1}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0)) + \gamma_2]\}e^{i(-\kappa x + \omega t + \theta)}, \tag{35}$$

where γ_1 and γ_2 are real numbers.

Figure 1 depicts the plot of a dark soliton (33). The parameter values chosen are $\gamma_1 = 2$, $\gamma_2 = 1$, $k = 1$, $A_3 = -1$, $a_1 = 1$, $\kappa = 1$, $\alpha_1 = 1$, $b_1 = 1$, and $\chi = 2$. The dark soliton is featured as a localized intensity dip below a continuous wave background. Dark solitons are meaningful only if the background that supports the dark solitons is modulationally stable. Dark solitons are observed when the carrier wave is modulationally stable [29,30].

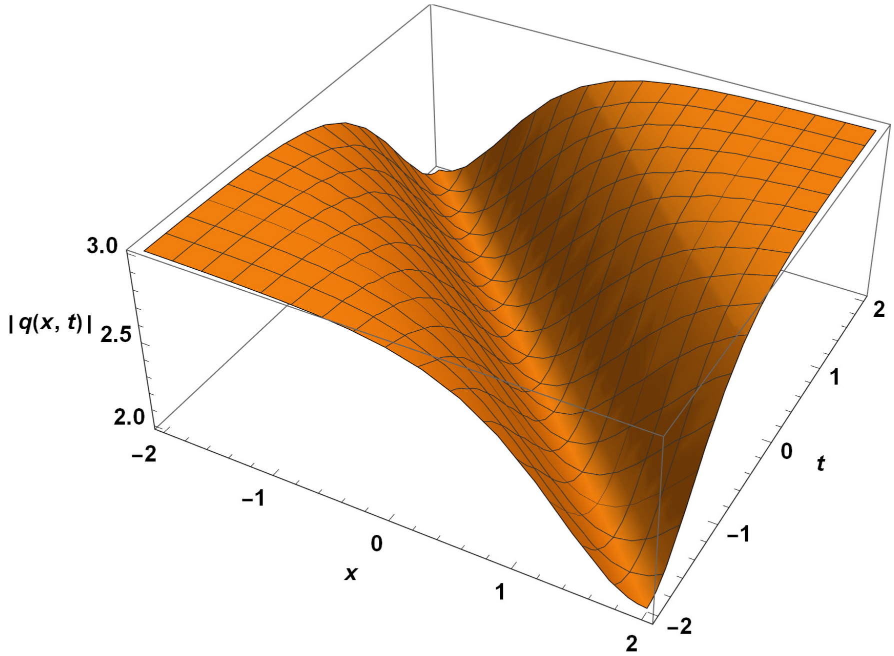

Figure 1. Profile of a dark soliton in fiber Bragg gratings.

Case 2. $\Delta = 0, D_1 = 0$, and $F(p) = (p-\gamma_1)^3$. The solution is

$$q = \{4((-\frac{2A_3}{15k^2})^{-\frac{1}{6}}\xi - \xi_0)^{-2} + \gamma_1\}e^{i(-\kappa x+\omega t+\theta)}, \tag{36}$$

where γ_1 is a real number.

Case 3. $\Delta > 0, D_1 < 0$, and $F(p) = (p-\gamma_1)(p-\gamma_2)(p-\gamma_3)$. When $\gamma_1 < \gamma_2 < \gamma_3$, if $\gamma_1 < p < \gamma_2$, the solution is

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[\gamma_1 + (\gamma_2-\gamma_1)sn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0), m)]\}e^{i(-\kappa x+\omega t+\theta)}, \tag{37}$$

and if $p > \gamma_3$, the solution is

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[\frac{\gamma_3 - \gamma_2 sn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0), m)}{cn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0), m)}]\}e^{i(-\kappa x+\omega t+\theta)}, \tag{38}$$

where γ_1, γ_2, and γ_3 are real numbers, and $m^2 = \frac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}$.

Case 4. $\Delta < 0$, and $F(p) = (p-\gamma_1)(p^2+lp+j), l^2-4j < 0$. The solution is

$$q = \{(-\frac{2A_3}{15k^2})^{-\frac{1}{6}}[\gamma_1 - \sqrt{\gamma_1^2+l\gamma_1+j} + \frac{2\sqrt{\gamma_1^2+l\gamma_1+j}}{1+cn((\gamma_1^2+l\gamma_1+j)^{\frac{1}{4}}((-\frac{2A_3}{15k^2})^{\frac{1}{6}}\xi - \xi_0), m)}]\}e^{i(-\kappa x+\omega t+\theta)}, \tag{39}$$

where γ_1, l, and j are real numbers, and $m^2 = \frac{1}{2}(1 - \frac{\gamma_1+\frac{l}{2}}{\sqrt{\gamma_1^2+l\gamma_1+j}})$.

3.2. Parabolic Law

In this case, the model evolves as

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + (c_1|q|^2 + d_1|r|^2)q + (\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4)q + i\alpha_1 q_x + \beta_1 r = 0, \tag{40}$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + (c_2|r|^2 + d_2|q|^2)r + (\xi_2|r|^4 + \eta_2|q|^2|r|^2 + \zeta_2|q|^4)r + i\alpha_2 r_x + \beta_2 q = 0. \tag{41}$$

Substituting Equations (10) and (11) into Equations (40) and (41) provides us with the equations

$$3k^2 P_{\tilde{l}}''(\kappa a_l P_{\tilde{l}}'' - 2\kappa^2 b_l) + P_2(-\kappa^3 a_l P_{\tilde{l}} + \kappa^4 b_l P_{\tilde{l}} + \beta_l) + k^4 b_l P_{\tilde{l}}''''$$
$$+ P_l^3(\eta_l P_{\tilde{l}}^2 + c_l) + P_l(d_l P_{\tilde{l}}^2 + \zeta_l P_{\tilde{l}}^4 + \kappa\alpha_l - \omega) + \xi_l P_l^5 = 0, \tag{42}$$

$$k^3 P_{\tilde{l}}'''(a_l - 4\kappa b_l) + P_{\tilde{l}}'\left(4\kappa^3 k b_l - 3\kappa^2 k a_l\right) + kP_l'(\alpha_l - v) = 0. \tag{43}$$

Taking

$$P_2 = \chi P_1, \tag{44}$$

Equations (42) and (43) turn into

$$\begin{aligned} &k^2 P_1''(3a_1\chi\kappa - 6b_1\chi\kappa^2) + P_1(-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega) \\ &+ b_1\chi k^4 P_1'''' + P_1^3(c_1 + \chi^2 d_1) + P_1^5(\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1) = 0, \end{aligned} \tag{45}$$

and

$$k^2 P_1'''(a_1\chi - 4b_1\chi\kappa) + P_1'(-3a_1\chi\kappa^2 + 4b_1\chi\kappa^3 + (\alpha_1 - v)) = 0, \tag{46}$$

along with the constraints

$$\begin{aligned} &c_1 + \chi^2 d_1 = \chi(c_2\chi^2 + d_2), \\ &-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega = \alpha_2\chi\kappa - a_2\kappa^3 + b_2\kappa^4 + \beta_2 - \chi\omega, \\ &\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1 = \chi P_1^5(\chi^4\xi_2 + \chi^2\eta_2 + \zeta_2), \\ &\chi(a_1\kappa - 2b_1\kappa^2) = a_2\kappa - 2b_2\kappa^2, \\ &b_1\chi = b_2. \end{aligned} \tag{47}$$

Equation (46) yields the velocity

$$v = -3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3, \tag{48}$$

with the aid of the restrictions

$$\alpha_2\chi - 3a_2\kappa^2 + 4b_2\kappa^3 = -3a_1\chi^2\kappa^2 + \alpha_1\chi + 4b_1\chi^2\kappa^3, \tag{49}$$

$$a_l = 4b_l\kappa, \tag{50}$$

while Equation (45) becomes

$$A_1 P_1'' + A_4 P_1^5 + A_3 P_1^3 + A_2 P_1 + k^2 P_1'''' = 0, \tag{51}$$

where

$$\begin{aligned} &A_1 = \frac{3a_1\chi\kappa - 6b_1\chi\kappa^2}{b_1\chi}, A_2 = \frac{-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega}{b_1\chi k^2}, \\ &A_3 = \frac{c_1 + \chi^2 d_1}{b_1\chi k^2}, A_4 = \frac{\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1}{b_1\chi k^2}. \end{aligned} \tag{52}$$

With the usage of the balance procedure in (51), Equation (5) becomes

$$(P_1')^2 = \frac{s_3}{2}P_1^4 + \frac{2s_2}{3}P_1^3 + s_1 P_1^2 + 2s_0 P_1 + s, \tag{53}$$

where

$$\begin{aligned}
s_3 &= (-\frac{A_4}{6k^2})^{\frac{1}{2}},\\
s_2 &= 0,\\
s_1 &= -\frac{A_3}{10k^2}(-\frac{A_4}{6k^2})^{-\frac{1}{2}} - \frac{A_1}{10k^2},\\
s_0 &= 0,\\
s &= (-\frac{A_3}{10}(-\frac{A_4}{6k^2})^{-\frac{1}{2}} + \frac{9A_1}{10})(-\frac{A_3}{10A_4k^2} + \frac{A_1}{60k^4}(-\frac{A_4}{6k^2})^{-\frac{1}{2}}) - \frac{A_2}{6k^2}(-\frac{A_4}{6k^2})^{-\frac{1}{2}}.
\end{aligned} \tag{54}$$

With the help of the relations

$$g = (2s_3)^{\frac{1}{3}} P_1^2, \xi_1 = (2s_3)^{\frac{1}{3}}\xi, \tag{55}$$

Equation (53) is

$$(g_{\xi_1})^2 = g^3 + \epsilon_2 g^2 + \epsilon_1 g, \tag{56}$$

where

$$\begin{aligned}
\epsilon_2 &= (-\frac{2A_3}{5k^2}(-\frac{A_4}{6k^2})^{-\frac{1}{2}} - \frac{2A_1}{5k^2})(-\frac{2A_4}{3k^2})^{-\frac{1}{3}},\\
\epsilon_1 &= ((-\frac{A_3}{10}(-\frac{A_4}{6k^2})^{-\frac{1}{2}} + \frac{9A_1}{10})(-\frac{2A_3}{5A_4k^2} + \frac{A_1}{15k^4}(-\frac{A_4}{6k^2})^{-\frac{1}{2}}) - \frac{2A_2}{3k^2}(-\frac{A_4}{6k^2})^{-\frac{1}{2}})(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}.
\end{aligned} \tag{57}$$

We rewrite Equation (56) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dg}{\sqrt{F(g)}}, \tag{58}$$

where

$$F(g) = g(g^2 + \epsilon_2 g + \epsilon_1). \tag{59}$$

We consider the discriminant system

$$\Delta = \epsilon_2^2 - 4\epsilon_1. \tag{60}$$

Case 1. $\Delta = 0$. For $g > 0$, if $\epsilon_2 < 0$, we have the dark and singular solitons

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(-\frac{\epsilon_2}{2}\tanh^2(\frac{1}{2}\sqrt{-\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \tag{61}$$

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(-\frac{\epsilon_2}{2}\coth^2(\frac{1}{2}\sqrt{-\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}; \tag{62}$$

if $\epsilon_2 > 0$, we have the singular periodic solution

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(\frac{\epsilon_2}{2}\tan^2(\frac{1}{2}\sqrt{\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}; \tag{63}$$

if $\epsilon_2 = 0$, we have the solution

$$q = \{\frac{4(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}}{((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)^2}\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}. \tag{64}$$

Figure 2 depicts the plot of a singular soliton (62). The parameter values chosen are $\epsilon_2 = -1$, $k = 1$, $A_4 = -1$, $a_1 = 1$, $\kappa = 1$, $\alpha_1 = 1$, $b_1 = 1$, and $\chi = 2$.

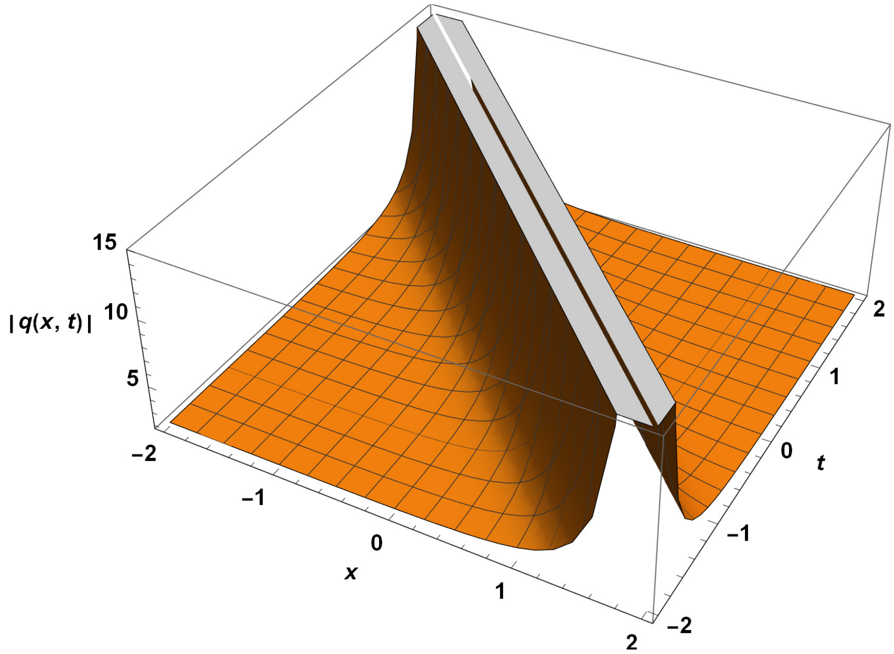

Figure 2. Profile of a singular soliton in fiber Bragg gratings.

Case 2. $\Delta > 0$, and $\epsilon_1 = 0$. For $g > -\epsilon_2$, if $\epsilon_2 > 0$, we have the dark and singular solitons

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(\frac{\epsilon_2}{2}\tanh^2(\frac{1}{2}\sqrt{\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)) - \epsilon_2)\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}, \tag{65}$$

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(\frac{\epsilon_2}{2}\coth^2(\frac{1}{2}\sqrt{\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)) - \epsilon_2)\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}; \tag{66}$$

if $\epsilon_2 < 0$, we have the singular periodic solution

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(-\frac{\epsilon_2}{2}\tan^2(\frac{1}{2}\sqrt{-\frac{\epsilon_2}{2}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0)) - \epsilon_2)\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}. \tag{67}$$

Case 3. $\Delta > 0$, and $\epsilon_1 \neq 0$. Suppose that $\gamma_1 < \gamma_2 < \gamma_3$, one of them is zero, and the other two are roots of $g^2 + \epsilon_2 g + \epsilon_1$. For $\gamma_1 < g < \gamma_2$, we have

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}[\gamma_1 + (\gamma_2 - \gamma_1)sn^2(\frac{\sqrt{\gamma_3 - \gamma_1}}{2}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0), m)]\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}, \tag{68}$$

and for $g > \gamma_3$, we have

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}[\frac{\gamma_3 - \gamma_2 sn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0), m)}{cn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0), m)}]\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}, \tag{69}$$

where $m^2 = \frac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}$.

Case 4. $\Delta < 0$. For $g > 0$, we have

$$q = \{(-\frac{2A_4}{3k^2})^{-\frac{1}{6}}(\frac{2\sqrt{\epsilon_1}}{1 + cn(\epsilon_1^{\frac{1}{4}}((-\frac{2A_4}{3k^2})^{\frac{1}{6}}\xi - \xi_0), m)} - \sqrt{\epsilon_1})\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}, \tag{70}$$

where $m^2 = \frac{1-\frac{\epsilon_2}{2\sqrt{\epsilon_1}}}{2}$.

3.3. Polynomial Law

In this case, the model reads as

$$\begin{aligned} & iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + (c_1|q|^2 + d_1|r|^2)q + (\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4)q \\ & + (l_1|q|^6 + m_1|q|^4|r|^2 + n_1|q|^2|r|^4 + p_1|r|^6)q + i\alpha_1 q_x + \beta_1 r = 0, \end{aligned} \tag{71}$$

$$\begin{aligned} & ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + (c_2|r|^2 + d_2|q|^2)r + (\xi_2|r|^4 + \eta_2|r|^2|q|^2 + \zeta_2|q|^4)r \\ & + (l_2|r|^6 + m_2|r|^4|q|^2 + n_2|r|^2|q|^4 + p_2|q|^6)r + i\alpha_2 r_x + \beta_2 q = 0. \end{aligned} \tag{72}$$

Putting Equations (10) and (11) into Equations (71) and (72) provides us with the ancillary equations

$$\begin{aligned} & 3\kappa k^2 a_l P_{\bar{l}}'' - \kappa^3 a_l P_{\bar{l}} + k^4 b_l P_{\bar{l}}'''' - 6\kappa^2 k^2 b_l P_{\bar{l}}'' + \kappa^4 b_l P_{\bar{l}} + P_l^3(n_l P_{\bar{l}}^4 + \eta_l P_{\bar{l}}^2 + c_l) \\ & + P_l(d_l P_{\bar{l}}^2 + p_l P_{\bar{l}}^6 + \zeta_l P_{\bar{l}}^4 + \kappa\alpha_l - \omega) + P_l^5(m_l P_{\bar{l}}^2 + \xi_l) + \beta_l P_{\bar{l}} + l_l P_l^7 = 0, \end{aligned} \tag{73}$$

$$k^3 P_{\bar{l}}'''(a_l - 4\kappa b_l) + kP_{\bar{l}}'(4\kappa^3 b_l - 3\kappa^2 a_l) + kP_l'(\alpha_l - v) = 0. \tag{74}$$

Setting

$$P_2 = \chi P_1, \tag{75}$$

Equations (73) and (74) becomes

$$\begin{aligned} & \chi k^2 P_1''(3a_1\kappa - 6b_1\kappa^2) + P_1(-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega) + b_1\chi k^4 P_1'''' \\ & + P_1^3(c_1 + \chi^2 d_1) + P_1^7(l_1 + \chi^2 m_1 + \chi^4 n_1 + \chi^6 p_1) + P_1^5(\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1) = 0, \end{aligned} \tag{76}$$

$$k^2 P_1''''(a_1\chi - 4b_1\chi\kappa) + P_1'(-3a_1\chi\kappa^2 + 4b_1\chi\kappa^3 + (\alpha_1 - v)) = 0, \tag{77}$$

together with the conditions

$$\begin{aligned} & c_1 + \chi^2 d_1 = \chi(c_2\chi^2 + d_2), \\ & l_1 + \chi^2 m_1 + \chi^4 n_1 + \chi^6 p_1 = \chi(\chi^6 l_2 + \chi^4 m_2 + \chi^2 n_2 + p_2), \\ & -a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega = \alpha_2\chi\kappa - a_2\kappa^3 + b_2\kappa^4 + \beta_2 - \chi\omega, \\ & \chi^4\zeta_1 + \chi^2\eta_1 + \xi_1 = \chi(\chi^4\xi_2 + \chi^2\eta_2 + \zeta_2), \\ & \chi(a_1\kappa - 2b_1\kappa^2) = a_2\kappa - 2b_2\kappa^2, \\ & b_1\chi = b_2. \end{aligned} \tag{78}$$

Equation (77) gives way to the velocity

$$v = -3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3, \tag{79}$$

with the aid of the constraints

$$\alpha_2\chi - 3a_2\kappa^2 + 4b_2\kappa^3 = -3a_1\chi^2\kappa^2 + \alpha_1\chi + 4b_1\chi^2\kappa^3, \tag{80}$$

$$a_l = 4b_l\kappa, \tag{81}$$

while Equation (76) becomes

$$A_1 P_1'' + A_5 P_1^7 + A_4 P_1^5 + A_3 P_1^3 + A_2 P_1 + k^2 P_1'''' = 0, \tag{82}$$

where

$$A_1=\frac{\chi(3a_1\kappa-6b_1\kappa^2)}{b_1\chi}, A_2=\frac{-a_1\chi\kappa^3+\alpha_1\kappa+b_1\chi\kappa^4+\beta_1\chi-\omega}{b_1\chi k^2}, A_3=\frac{c_1+\chi^2 d_1}{b_1\chi k^2},$$
$$A_4=\frac{\chi^4\zeta_1+\chi^2\eta_1+\xi_1}{b_1\chi k^2}, A_5=\frac{l_1+\chi^2 m_1+\chi^4 n_1+\chi^6 p_1}{b_1\chi k^2}. \tag{83}$$

With the help of the balance technique in (82), Equation (5) simplifies to

$$(P_1')^2=\frac{2s_4}{5}P_1^5+\frac{s_3}{2}P_1^4+\frac{2s_2}{3}P_1^3+s_1P_1^2+2s_0P_1+s, \tag{84}$$

where

$$s_4=(-\frac{5A_5}{44k^2})^{\frac{1}{2}}, s_3=0, s_2=-\frac{5A_4}{74k^2}(-\frac{5A_5}{44k^2})^{-\frac{1}{2}}, s_1=-\frac{A_1}{17k^2},$$
$$s_0=-\frac{A_3}{28k^2}(-\frac{5A_5}{44k^2})^{-\frac{1}{2}}-\frac{125A_4^2}{229992k^4}(-\frac{5A_5}{44k^2})^{-\frac{3}{2}}, s=-\frac{14652A_1A_2}{209457A_5}. \tag{85}$$

By the aid of the criterions

$$f=(\frac{2s_4}{5})^{\frac{1}{5}}P_1, \xi_1=(\frac{2s_4}{5})^{\frac{1}{5}}\xi, \tag{86}$$

Equation (84) becomes

$$(f_{\xi_1})^2=f^5+y_3f^3+y_2f^2+y_1f+y_0, \tag{87}$$

where

$$y_3=-(\frac{2}{5})^{-\frac{3}{5}}\frac{15A_4}{222k^2}(-\frac{5A_5}{44k^2})^{-\frac{4}{5}}, y_2=-\frac{A_1}{17k^2}(-\frac{A_5}{55k^2})^{-\frac{1}{5}},$$
$$y_1=-(\frac{2}{5})^{-\frac{1}{5}}(\frac{A_3}{14k^2}(-\frac{5A_5}{44k^2})^{-\frac{3}{5}}+\frac{125A_4^2}{114996k^4}(-\frac{5A_5}{44k^2})^{-\frac{8}{5}}), y_0=h. \tag{88}$$

We rewrite Equation (87) as

$$\pm(\xi_1-\xi_0)=\int\frac{df}{\sqrt{F(f)}}, \tag{89}$$

where

$$F(f)=f^5+y_3f^3+y_2f^2+y_1f+y_0. \tag{90}$$

We consider the discriminant system

$$\begin{aligned}
D_2=&-y_3,\\
D_3=&40y_1y_3-12y_3^3-45y_2^2,\\
D_4=&12y_3^4y_1-4y_3^3y_2^2+117y_3y_1y_2^2-88y_1^2y_3^2-40y_2y_0y_3^2-27y_2^4-300y_2y_1y_0+160y_1^3,\\
D_5=&-1600y_2y_0y_1^3-3750y_3y_2y_0^3+2000y_3y_0^2y_1^2-4y_3^3y_2^2y_1^2+16y_3^3y_2^3y_0-900y_1y_0^2y_3^3+825y_3^2y_2^2y_0^2\\
&+144y_3y_2^2y_1^3+2250y_1y_2^2y_0^2+16y_3^4y_1^3+108y_3^5y_0^2-128y_1^4y_3^2-27y_1^2y_2^4+108y_0y_2^5+256y_1^5\\
&+3125y_0^4-72y_1y_0y_2y_3^4+560y_0y_2y_1^2y_3^2-630y_3y_1y_0y_2^3,\\
E_2=&160y_1^2y_3^3+900y_2^2y_1^2-48y_1y_3^5+60y_1y_3^2y_2^2+1500y_3y_2y_1y_0+16y_2^2y_3^4-1100y_2y_0y_3^3+625y_0^2y_3^2-3375y_0y_2^3,\\
F_2=&3y_2^2-8y_1y_3.
\end{aligned} \tag{91}$$

Case 1. $D_5 = 0$, $D_4 = 0$, $D_3 > 0$, $E_2 \neq 0$, and $F(f) = (f-\delta_1)^2(f-\delta_2)^2(f-\delta_3)$. When $f > \delta_3$, if $\delta_3 > \delta_1$, $\delta_3 > \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(\sqrt{\delta_3-\delta_1}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}}{\sqrt{\delta_3-\delta_1}} \\ &-\sqrt{\delta_3-\delta_2}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}}{\sqrt{\delta_3-\delta_2}}),\end{aligned} \tag{92}$$

if $\delta_3 > \delta_1$, $\delta_3 < \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(\sqrt{\delta_3-\delta_1}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}}{\sqrt{\delta_3-\delta_1}} \\ &-\frac{1}{\sqrt{\delta_2-\delta_3}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}-\sqrt{\delta_2-\delta_3}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}+\sqrt{\delta_2-\delta_3}}|);\end{aligned} \tag{93}$$

if $\delta_3 < \delta_1$, $\delta_3 > \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(-\sqrt{\delta_3-\delta_2}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}}{\sqrt{\delta_3-\delta_2}} \\ &+\frac{1}{2\sqrt{\delta_1-\delta_3}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}-\sqrt{\delta_1-\delta_3}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}+\sqrt{\delta_1-\delta_3}}|);\end{aligned} \tag{94}$$

if $\delta_3 < \delta_1$, $\delta_3 < \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(\frac{1}{2\sqrt{\delta_1-\delta_3}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}-\sqrt{\delta_1-\delta_3}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}+\sqrt{\delta_1-\delta_3}}| \\ &-\frac{1}{2\sqrt{\delta_2-\delta_3}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}-\sqrt{\delta_2-\delta_3}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_3}+\sqrt{\delta_2-\delta_3}}|),\end{aligned} \tag{95}$$

where δ_1, δ_2, and δ_3 are real numbers.

Case 2. $D_5 = 0$, $D_4 = 0$, $D_3 = 0$, $D_2 \neq 0$, $F_2 \neq 0$, and $F(f) = (f-\delta_1)^3(f-\delta_2)^2$. When $f > \delta_1$, if $\delta_1 > \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(-\frac{1}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}} \\ &-\sqrt{\delta_1-\delta_2}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}}{\sqrt{\delta_1-\delta_2}});\end{aligned} \tag{96}$$

if $\delta_1 < \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) =& \frac{2}{\delta_1-\delta_2}(-\frac{1}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}} \\ &-\frac{1}{2\sqrt{\delta_2-\delta_1}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}-\sqrt{\delta_2-\delta_1}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}+\sqrt{\delta_2-\delta_1}}|),\end{aligned} \tag{97}$$

where δ_1 and δ_2 are real numbers.

Case 3. $D_5 = 0$, $D_4 = 0$, $D_3 = 0$, $D_2 \neq 0$, $F_2 = 0$, and we have $F(f) = (f-\delta_1)^4(f-\delta_2)$. When $f > \delta_1$, if $\delta_1 < \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) = &\frac{2}{\delta_1-\delta_2}(-\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_2}}{2((-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1)} \\ &-\frac{1}{2\sqrt{\delta_1-\delta_2}}\arctan\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_2}}{\sqrt{\delta_2-\delta_1}});\end{aligned} \tag{98}$$

if $\delta_1 > \delta_2$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) = &\frac{2}{\delta_1-\delta_2}(-\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_2}}{2((-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1)} \\ &-\frac{1}{4\sqrt{\delta_1-\delta_2}}\ln|\frac{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_2}-\sqrt{\delta_1-\delta_2}}{\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_2}+\sqrt{\delta_1-\delta_2}}|),\end{aligned} \tag{99}$$

where δ_1 and δ_2 are real numbers.

Case 4. $D_5 = 0$, $D_4 = 0$, $D_3 = 0$, $D_2 = 0$, and $F(f) = (f-\delta_1)^5$. When $f > \delta_1$, we obtain

$$P_1 = \{(-\frac{A_5}{55k^2})^{-\frac{1}{10}}[(\mp\frac{3}{2}((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0))^{-\frac{3}{2}}+\delta_1]\}^{\frac{1}{2}}, \tag{100}$$

where δ_1 is a real number.

Case 5. $D_5 = 0$, $D_4 = 0$, $D_3 < 0$, $E_2 \neq 0$, and $F(f) = (f-\delta_1)(f^2+mf+n)^2$. When $f > \delta_1$, we obtain

$$\begin{aligned}\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) = &-\frac{2}{\sigma\sqrt{4n-m^2}}(\cos\psi\arctan\frac{2\sigma\sin\psi\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}}{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1-\sigma^2} \\ &+\frac{\sin\psi}{2}\ln|\frac{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1-\sigma^2-2\sigma\cos\psi\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}}{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1-\sigma^2+2\sigma\cos\psi\sqrt{(-\frac{A_5}{55k^2})^{\frac{1}{10}}P_1-\delta_1}}|),\end{aligned} \tag{101}$$

where δ_1 is a real number, $m^2-4n<0$, $\sigma = (\delta_1^2+m\delta_1+n)^{\frac{1}{4}}$, and $\psi = \frac{1}{2}\arctan\frac{\sqrt{4n-m^2}}{-2\delta_1-m}$.

Case 6. $D_5 = 0$, $D_4 > 0$, and $F(f) = (f-\delta_1)^2(f-\delta_2)(f-\delta_3)(f-\delta_4)$. When $\delta_2 > \delta_3 > \delta_4$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi - \xi_0) = -\frac{2}{(\delta_1-\delta_3)\sqrt{\delta_3-\delta_4}}\{F(\psi,k)-\frac{\delta_2-\delta_3}{\delta_2-\delta_1}\Pi(\psi,\frac{\delta_2-\delta_3}{\delta_2-\delta_1},k)\}, \tag{102}$$

where δ_1, δ_2, δ_3, and δ_4 are real numbers, and $\delta_2 > \delta_2 > \delta_3$, $F(\psi,k) = \int_0^\psi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$, and $\Pi(\psi,n,k) = \int_0^\psi \frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}$.

Case 7. $D_5 = 0$, $D_4 = 0$, $D_3 < 0$, $E_2 = 0$, and $F(f) = (f-\delta_1)^3((f-\delta_2)^2+\delta_3^2)$. When $f > \delta_1$, if $\delta_1 \neq \delta_2+\delta_3$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0)=\frac{\tan\theta+\cot\theta}{2(\delta_3\tan\theta-\delta_2-\delta_1)\sqrt{\frac{\delta_3}{\sin^3 2\theta}}}F(\psi,k)-\frac{\delta_3\tan\theta+\delta_3\cot\theta}{\delta_3\cot\theta+\delta_2+\delta_1} \times\{\frac{\tan\theta+\delta_2+\delta_1}{(\delta_3\cot\theta+\delta_2-\delta_1)\sin\psi}\sqrt{1-k^2\sin^2\psi}+F(\psi,k)-E(\psi,k)\}; \quad (103)$$

if $\delta_1=\delta_2+\delta_3$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0)=\sqrt{\frac{\sin^3 2\theta}{4\delta_3^3}}(\frac{1}{k}\arcsin(k\sin\psi)-F(\psi,k)), \quad (104)$$

where δ_1, δ_2, and δ_3 are real numbers, $\tan 2\theta=\frac{\delta_3}{\delta_1-\delta_2}$, $k=\sin\theta(0<\theta<\frac{\pi}{2})$, and $E(\psi,k)=\int_0^{\psi}\sqrt{1-k^2\sin^2\varphi}d\varphi$.

Case 8. $D_5=0$, $D_4<0$, and $F(f)=(f-\delta_1)^2(f-\delta_2)((f-\delta_3)^2+\delta_4^2)$. When $f>\delta_2$, if $\delta_1\neq\delta_3-\delta_4\tan\theta$, $\delta_1\neq\delta_3+\delta_4\cot\theta$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0)=\frac{\tan\theta+\cot\theta}{2(\delta_4\tan\theta-\delta_3-\delta_1)\sqrt{\frac{\delta_4}{\sin^3 2\theta}}}F(\psi,k)-\frac{\delta_4\tan\theta+\delta_4\cot\theta}{\delta_4\cot\theta+\delta_3+\delta_1} \times\{\frac{\tan\theta+\delta_3+\delta_1}{(\delta_4\cot\theta+\delta_3-\delta_1)\sin\psi}\sqrt{1-k^2\sin^2\psi}+F(\psi,k)-E(\psi,k)\}; \quad (105)$$

if $\delta_1=\delta_3-\delta_4\tan\theta$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0)=\sqrt{\frac{\sin^3 2\theta}{4\delta_4^3}}(\frac{1}{k}\arcsin(k\sin\psi)-F(\psi,k)); \quad (106)$$

if $\delta_1=\delta_3+\delta_4\cot\theta$, we obtain

$$\pm((-\frac{A_5}{55k^2})^{\frac{1}{10}}\xi-\xi_0)=\sqrt{\frac{\sin^3 2\theta}{4\delta_4^3}}(F(\psi,k)-\frac{1}{\sqrt{1-k^2}}\ln\frac{\sqrt{1-k^2\sin^2\psi}+\sqrt{1-k^2}\sin\psi}{\cos\psi}), \quad (107)$$

where δ_1, δ_2, δ_3, and δ_4 are real numbers, $\tan 2\theta=\frac{\delta_4}{\delta_2-\delta_3}$, and $k=\sin\theta(0<\theta<\frac{\pi}{2})$.

When $D_5=0$, $D_4=0$, $D_3>0$, and $E_2=0$; $D_5>0$, $D_4>0$, $D_3>0$, and $D_2>0$; $D_5<0$; and $D_5>0$ and $D_4\leq 0$ or $D_3\leq 0$ or $D_2\leq 0$, the solutions of these four cases can be structured by Legendre elliptic functions or hyperelliptic integral or hyperelliptic functions.

3.4. Quadratic–Cubic Law

In this case, the model is

$$iq_t+ia_1r_{xxx}+b_1r_{xxxx}+c_1q\sqrt{|q|^2+|r|^2+qr^*+q^*r}+(d_1|q|^2+f_1|r|^2)q+p_1r^2q^*+i\alpha q_x+\beta_1 r=0, \quad (108)$$

$$ir_t+ia_2q_{xxx}+b_2q_{xxxx}+c_2q\sqrt{|r|^2+|q|^2+qr^*+qr^*}+(d_2|r|^2+f_2|q|^2)r+p_2q^2r^*+i\alpha r_x+\beta_2 q=0. \quad (109)$$

Putting Equations (10) and (11) into Equations (108) and (109) provides us with the auxiliary equations

$$3k^2P_{\bar{l}}''(\kappa a_l-2\kappa^2 b_l)+P_{\bar{l}}(-\kappa^3 a_l+\kappa^4 b_l+\beta_l)+k^4 b_l P_{\bar{l}}'''' + P_l(c_l(P_{\bar{l}}+P_l)+f_l P_{\bar{l}}^2+p_l P_{\bar{l}}^2+\alpha\kappa-\omega)+d_l P_l^3=0, \quad (110)$$

$$k^3P_{\bar{l}}'''(a_l-4\kappa b_l)+kP_{\bar{l}}'(4\kappa^3 b_l-3\kappa^2 k a_l)+k(\alpha-v)P_l'=0. \quad (111)$$

Taking

$$P_2 = \chi P_1, \tag{112}$$

Equations (110) and (111) become

$$\begin{gathered} 3\chi\kappa k^2 P_1''(a_1 - 2b_1\kappa) + P_1(-a_1\chi\kappa^3 + \alpha\kappa \\ +b_1\chi\kappa^4 + \beta_1\chi - \omega) + b_1\chi k^4 P_1'''' + (\chi+1)c_1P_1^2 + P_1^3(d_1 + \chi^2 f_1 + \chi^2 p_1) = 0, \end{gathered} \tag{113}$$

$$\chi k^2 P_1'''(a_1 - 4b_1\kappa) + P_1'(-3a_1\chi\kappa^2 + 4b_1\chi\kappa^3 + (\alpha - v)) = 0, \tag{114}$$

by virtue of the restrictions

$$\begin{aligned} &c_1 = \chi c_2, \\ &d_1 + \chi^2 f_1 + \chi^2 p_1 = \chi^3 d_2 + \chi f_2 + \chi p_2, \\ &-a_1\chi\kappa^3 + \alpha\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega = \alpha\chi\kappa - a_2\kappa^3 + b_2\kappa^4 + \beta_2 - \chi\omega, \\ &\chi(a_1 - 2b_1\kappa) = a_2 - 2b_2\kappa, \\ &b_1\chi = b_2. \end{aligned} \tag{115}$$

Equation (114) exposes the velocity

$$v = -3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3, \tag{116}$$

along with the constraints

$$\alpha_2\chi - 3a_2\kappa^2 + 4b_2\kappa^3 = -3a_1\chi^2\kappa^2 + \alpha_1\chi + 4b_1\chi^2\kappa^3, \tag{117}$$

$$a_l = 4b_l\kappa, \tag{118}$$

while Equation (113) becomes

$$A_1P_1'' + A_3P_1^3 + A_4P_1^2 + A_2P_1 + k^2P_1'''' = 0, \tag{119}$$

where

$$\begin{aligned} &A_1 = \frac{3\chi\kappa k^2(a_1 - 2b_1\kappa)}{b_1\chi k^2}, A_2 = \frac{-a_1\chi\kappa^3 + \alpha\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega}{b_1\chi k^2}, \\ &A_3 = \frac{d_1 + \chi^2 f_1 + \chi^2 p_1}{b_1\chi k^2}, A_4 = \frac{(\chi+1)c_1}{b_1\chi k^2}. \end{aligned} \tag{120}$$

By virtue of the balance procedure in (119), Equation (5) becomes

$$(P_1')^2 = \frac{2s_2}{3}P_1^3 + s_1P_1^2 + 2s_0P_1 + s, \tag{121}$$

where

$$\begin{aligned} &s_2 = (-\frac{3A_3}{10k^2})^{\frac{1}{2}}, s_1 = -\frac{A_1}{5k^2} - \frac{A_4}{5k^2}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}, \\ &s_0 = (\frac{2A_1^2}{75k^4} - \frac{A_2}{6k^2} + \frac{A_4^2}{45k^2A_3} + \frac{A_1A_4}{50k^4}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}})(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}, \\ &s = (\frac{4A_1}{3A_3} - \frac{A_4}{3A_3}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}})(\frac{2A_1^2}{75k^4} - \frac{A_2}{6k^2} + \frac{A_4^2}{45k^2A_3} + \frac{A_1A_4}{50k^4}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}). \end{aligned} \tag{122}$$

We rewrite Equation (121) as

$$\pm(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0) = \int \frac{dP_1}{\sqrt{F(P_1)}}, \tag{123}$$

where

$$F(P_1) = P_1^3 + j_2 P_1^2 + j_1 P_1 + j_0, \tag{124}$$

and

$$\begin{aligned} j_2 &= -\frac{3A_1}{10k^2}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}} + \frac{A_4}{A_3}, \\ j_1 &= -\frac{10k^2}{A_3}(\frac{2A_1^2}{75k^4} - \frac{A_2}{6k^2} + \frac{A_4^2}{45k^2A_3} + \frac{A_1A_4}{50k^4}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}), \\ j_0 &= (\frac{2A_1}{A_3}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}} + \frac{10k^2A_4}{6A_3^2})(\frac{2A_1^2}{75k^4} - \frac{A_2}{6k^2} + \frac{A_4^2}{45k^2A_3} + \frac{A_1A_4}{50k^4}(-\frac{3A_3}{10k^2})^{-\frac{1}{2}}). \end{aligned} \tag{125}$$

We consider the discriminant system

$$\Delta = -27(\frac{2j_2^3}{27} + j_0 - \frac{j_1j_2}{3})^2 - 4(j_1 - \frac{j_2^2}{3})^3, D_1 = j_1 - \frac{j_2^2}{3}. \tag{126}$$

Case 1. $\Delta = 0, D_1 < 0$, and $F(P_1) = (P_1 - \iota_1)^2(P_1 - \iota_2)$. When $P_1 > \iota_2$, if $\iota_1 > \iota_2$, the dark and singular solitons are

$$q = [(\iota_1 - \iota_2)\tanh^2(\frac{\sqrt{\iota_1 - \iota_2}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0) + \iota_2]e^{i(-\kappa x + \omega t + \theta)}, \tag{127}$$

$$q = [(\iota_1 - \iota_2)\coth^2(\frac{\sqrt{\iota_1 - \iota_2}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0) + \iota_2]e^{i(-\kappa x + \omega t + \theta)}, \tag{128}$$

and if $\iota_1 < \iota_2$, the singular periodic solution is

$$q = [(\iota_2 - \iota_1)\tan^2(\frac{\sqrt{\iota_2 - \iota_1}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0) + \iota_2]e^{i(-\kappa x + \omega t + \theta)}, \tag{129}$$

where ι_1 and ι_2 are real numbers.

Case 2. $\Delta = 0, D_1 = 0$, and $F(P_1) = (P_1 - \iota_1)^3$. The solution is

$$q = [4(-\frac{2A_3}{15k^2})^{-\frac{1}{2}}(\xi_1 - \xi_0)^{-2} + \iota_1]e^{i(-\kappa x + \omega t + \theta)}, \tag{130}$$

where ι_1 is a real number.

Case 3. $\Delta > 0, D_1 < 0$, and $F(P_1) = (P_1 - \iota_1)(P_1 - \iota_2)(P_1 - \iota_3)$. When $\iota_1 < \iota_2 < \iota_3$, if $\iota_1 < P_1 < \iota_2$, the solution is

$$q = [\iota_1 + (\iota_2 - \iota_1)sn^2(\frac{\sqrt{\iota_3 - \iota_1}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0), m)]]e^{i(-\kappa x + \omega t + \theta)}, \tag{131}$$

and if $P_1 > \iota_3$, the solution is

$$q = [\frac{\iota_3 - \iota_2 sn^2(\frac{\sqrt{\iota_3 - \iota_1}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0), m)}{cn^2(\frac{\sqrt{\iota_3 - \iota_1}}{2}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0), m)}]e^{i(-\kappa x + \omega t + \theta)}, \tag{132}$$

where ι_1, ι_2, and ι_3 are real numbers, and $m^2 = \frac{\iota_2 - \iota_1}{\iota_3 - \iota_1}$.

Case 4. $\Delta < 0$, and $F(P_1) = (P_1 - \iota_1)(P_1^2 + lP_1 + j), l^2 - 4j < 0$. The solution is

$$q = [\iota_1 - \sqrt{\iota_1^2 + l\iota_1 + j} + \frac{2\sqrt{\iota_1^2 + l\iota_1 + j}}{1 + cn((\iota_1^2 + l\iota_1 + j)^{\frac{1}{4}}(-\frac{2A_3}{15k^2})^{\frac{1}{4}}(\xi_1 - \xi_0), m)}]e^{i(-\kappa x + \omega t + \theta)}, \tag{133}$$

where ι_1, l, and j are real numbers, and $m^2 = \frac{1}{2}(1 - \frac{\iota_1 + \frac{l}{2}}{\sqrt{\iota_1^2 + l\iota_1 + j}})$.

3.5. Parabolic-Nonlocal Law

In this case, the model is

$$iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + (c_1|q|^2 + d_1|r|^2)_{xx} q + (\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4)q + i\alpha_1 q_x + \beta_1 r = 0, \quad (134)$$

$$ir_t + ia_2 q_{xxx} + b_2 q_{xxxx} + (c_2|r|^2 + d_2|q|^2)_{xx} r + (\xi_2|r|^4 + \eta_2|r|^2|q|^2 + \zeta_2|q|^4)r + i\alpha_2 r_x + \beta_2 q = 0. \quad (135)$$

Inserting Equations (10) and (11) into Equations (134) and (135) provides us with the ancillary equations

$$\begin{aligned} &P_{\bar{l}}(-\kappa^3 a_l + \kappa^4 b_l + \beta_l) + \eta_l P_l^3 P_{\bar{l}}^2 + \xi_l P_l^5 + 2k^2 c_l P_{\bar{l}}^2 P_l'' + 3\kappa k^2 a_l P_{\bar{l}}'' - 6\kappa^2 k^2 b_l P_{\bar{l}}'' \\ &+ k^4 b_l P_{\bar{l}}^{(4)} + P_l(2k^2 d_l P_{\bar{l}}'^2 + 2k^2 d_l P_{\bar{l}} P_{\bar{l}}'' + \zeta_l P_{\bar{l}}^4 + \alpha_1 \kappa + 2k^2 c_l P_l'^2 - \omega) = 0, \end{aligned} \quad (136)$$

$$k^3 P_2'''(a_1 - 4b_1\kappa) + kP_2'(4b_1\kappa^3 P_2' - 3a_1\kappa^2) + kP_1'(\alpha_1 - v) = 0, \quad (137)$$

Setting

$$P_2 = \chi P_1, \quad (138)$$

Equations (136) and (137) become

$$\begin{aligned} &P_1^5(\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1) + P_1(-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi + 2c_1k^2P_1'^2 + 2\chi^2 d_1 k^2 P_1'^2 - \omega) \\ &+ \chi k^2 P_1''(3a_1\kappa - 6b_1\kappa^2) + b_1\chi k^4 P_1'''' + 2k^2 P_1^2 P_1''(c_1 + \chi^2 d_1) = 0, \end{aligned} \quad (139)$$

and

$$k(a_1(\chi k^2 P_1''' - 3\chi\kappa^2 P_1') + 4b_1\kappa(\chi\kappa^2 P_1' - \chi k^2 P_1''') + P_1'(\alpha_1 - v)) = 0, \quad (140)$$

by virtue of the constraints

$$\begin{aligned} &\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1 = \chi(\chi^4\xi_2 + \chi^2\eta_2 + \zeta_2), \\ &- a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega = \alpha_2\chi\kappa - a_2\kappa^3 + b_2\kappa^4 + \beta_2 - \chi\omega, \\ &c_1 + \chi^2 d_1 = \chi(c_2\chi^2 + d_2), \\ &\chi(3a_1\kappa - 6b_1\kappa^2) = 3a_2\kappa - 6b_2\kappa^2, \\ &b_1\chi = b_2. \end{aligned} \quad (141)$$

Equation (140) gives way to the velocity

$$v = -3a_1\chi\kappa^2 + \alpha_1 + 4b_1\chi\kappa^3, \quad (142)$$

with the usage of the conditions

$$\alpha_2\chi - 3a_2\kappa^2 + 4b_2\kappa^3 = -3a_1\chi^2\kappa^2 + \alpha_1\chi + 4b_1\chi^2\kappa^3, \quad (143)$$

$$a_l = 4b_l\kappa, \quad (144)$$

while Equation (139) becomes

$$A_2(P_1 P_1'^2 + P_1^2 P_1'') + A_1 P_1'' + A_4 P_1^5 + A_3 P_1 + k^2 P_1'''' = 0, \quad (145)$$

where

$$\begin{aligned} &A_1 = \frac{3\chi\kappa k^2(a_1 - 2b_1\kappa)}{b_1\chi k^2}, A_2 = \frac{2k^2(c_1 + \chi^2 d_1)}{b_1\chi k^2}, \\ &A_3 = \frac{-a_1\chi\kappa^3 + \alpha_1\kappa + b_1\chi\kappa^4 + \beta_1\chi - \omega}{b_1\chi k^2}, A_4 = \frac{\chi^4\zeta_1 + \chi^2\eta_1 + \xi_1}{b_1\chi k^2}. \end{aligned} \quad (146)$$

By virtue of the balance procedure in (145), Equation (5) becomes

$$(P_1')^2 = \frac{s_3}{2}P_1^4 + \frac{2s_2}{3}P_1^3 + s_1P_1^2 + 2s_0P_1 + s, \tag{147}$$

where

$$\begin{aligned}
s_3 =&\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{24k^2},\\
s_2 =&0,\\
s_1 =&-\frac{(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})A_1}{2k^2(5(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 24A_2)},\\
s_0 =&0,\\
s =&(9(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})^2A_1^2 + 48(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})A_1^2A_2\\
&- 100k^2(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})^2A_3 - 960k^2t(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})A_2A_3 - 2304k^2A_2^2A_3)/\\
&(k^2((-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 4A_2)(5(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 24A_2))^2).
\end{aligned} \tag{148}$$

With the usage of the relations

$$U = (2s_3)^{\frac{1}{3}}P_1^2, \xi_1 = (2s_3)^{\frac{1}{3}}\xi, \tag{149}$$

Equation (147) simplifies to

$$(U_{\xi_1})^2 = U^3 + \varsigma_2U^2 + \varsigma_1U, \tag{150}$$

where

$$\begin{aligned}
\varsigma_2 =& (\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{-\frac{2}{3}}(-\frac{2A_1(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})}{k^2(5(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 24A_2)}),\\
\varsigma_1 =& (36(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})^2A_1^2 + 192(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})A_1^2A_2\\
&- 400k^2(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})^2A_3 - 3840k^2t(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4})A_2A_3 - 9216k^2A_2^2A_3)/\\
&(k^2((-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 4A_2)(5(-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}) + 24A_2))^2)(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{\frac{1}{3}}.
\end{aligned} \tag{151}$$

We rewrite Equation (150) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dU}{\sqrt{F(U)}}, \tag{152}$$

where

$$F(U) = U(U^2 + \varsigma_2U + \varsigma_1). \tag{153}$$

We consider the discriminant system

$$\Delta = \varsigma_2^2 - 4\varsigma_1. \tag{154}$$

Case 1. $\Delta = 0$. For $U > 0$, if $\varsigma_2 < 0$, we have the dark and singular solitons

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{-\frac{1}{3}}(-\frac{\varsigma_2}{2}\tanh^2(\frac{1}{2}\sqrt{-\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}}e^{i(-\kappa x + \omega t + \theta)}, \tag{155}$$

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{-\frac{1}{3}}(-\frac{\varsigma_2}{2}\coth^2(\frac{1}{2}\sqrt{-\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}}e^{i(-\kappa x + \omega t + \theta)}; \tag{156}$$

if $\varsigma_2 > 0$, we have the singular periodic solution

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}(\frac{\varsigma_2}{2}\tan^2(\frac{1}{2}\sqrt{\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}; \quad (157)$$

if $\varsigma_2 = 0$, we have

$$q = \{\frac{4(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}}{((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)^2}\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}. \quad (158)$$

Case 2. $\Delta > 0$, and$\varsigma_1 = 0$. For $U > -\varsigma_2$, if $\varsigma_2 > 0$, we have the dark and singular solitons

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}(\frac{\varsigma_2}{2}\tanh^2(\frac{1}{2}\sqrt{\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)) - \varsigma_2)\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (159)$$

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}(\frac{\varsigma_2}{2}\coth^2(\frac{1}{2}\sqrt{\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)) - \varsigma_2)\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}; \quad (160)$$

if $\varsigma_2 < 0$, we have the singular periodic solution

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}(-\frac{\varsigma_2}{2}\tan^2(\frac{1}{2}\sqrt{-\frac{\varsigma_2}{2}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0)) - \varsigma_2)\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}. \quad (161)$$

Case 3. $\Delta > 0$, and$\varsigma_1 \neq 0$. Suppose that $\gamma_1 < \gamma_2 < \gamma_3$, one of them is zero, and the other two are roots of $U^2 + \varsigma_2 U + \varsigma_1$. For $\gamma_1 < U < \gamma_2$, we have

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}[\gamma_1 + (\gamma_2 - \gamma_1)sn^2(\frac{\sqrt{\gamma_3 - \gamma_1}}{2}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0), m)]\}^{\frac{1}{2}} \times e^{i(-\kappa x+\omega t+\theta)}, \quad (162)$$

and for $U > \gamma_3$, we have

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}[\frac{\gamma_3 - \gamma_2 sn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0), m)}{cn^2(\frac{\sqrt{\gamma_3-\gamma_1}}{2}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0), m)}]\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (163)$$

where $m^2 = \frac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}$.

Case 4. $\Delta < 0$, for $U > 0$, we have

$$q = \{(\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{-\frac{1}{3}}(\frac{2\sqrt{\varsigma_1}}{1 + cn(\varsigma_1^{\frac{1}{4}}((\frac{-3A_2 \pm \sqrt{3}\sqrt{3A_2^2 - 32k^2 A_4}}{12k^2})^{\frac{1}{3}}\xi - \xi_0), m)} - \sqrt{\varsigma_1})\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (164)$$

where $m^2 = \frac{1 - \frac{\varsigma_2}{2\sqrt{\varsigma_1}}}{2}$. We have obtained the exact solutions $q(x,t)$ in this section, so according to the relation $r = \chi q$, the solutions $r(x,t)$ can be obtained. We omit them.

4. Conclusions

The current paper retrieved CQ dark and singular soliton solutions in fiber Bragg gratings. Such solitons are a new kind of soliton, known as the Bragg soliton or the gap soliton, which can form in nonlinear media whose refractive index varies weakly in a periodic fashion along its length [29,30]. The study was conducted with five forms of self-phase modulation. The results are very useful when dark soliton dynamics are to be handled in detail. The integration scheme that yielded such dark and singular solitons was the trial equation approach. Visibly, this approach had a shortcoming. This approach failed to reveal the much-needed bright soliton solutions that are needed in fiber optic communication dynamics.

In future work, the model will be addressed from a numerical perspective with the usage of the Adomian Decomposition approach, the Laplace–Adomian decomposition method, the finite element method, the finite difference method, and other relevant approaches.

Apart from the regime of a deterministic system, it is also very important to explore the world of stochasticity where the random noise could be a detrimental factor. In this case, later, the model will be revisited with an additive stochastic perturbation term. This would be Gaussian noise; therefore, the corresponding dynamical system of the soliton parameters will be formulated in presence of noise. The corresponding Langevin equation will subsequently be revealed. Upon solving this Langevin equation, the mean free velocity will be computed. The results should align with the previously published results [31].

Multiplicative white noise will also be considered later as a derivative of the Weiner process. This will lead to the analysis of the model with the aid of Ito Calculus, which will lead to the stochastic effect only in the phase components of the solitons. The remaining physical factors of the solitons will remain unaffected. The results should align along the lines of the previously reported results [32].

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