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Solving Time-Fractional Partial Differential Equation Using Chebyshev Cardinal Functions

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Abstract: We propose a numerical scheme based on the Galerkin method for solving the time-fractional partial differential equations. To this end, after introducing the Chebyshev cardinal functions (CCFs), using the relation between fractional integral and derivative, we represent the Caputo fractional derivative based on these bases and obtain an operational matrix. Applying the Galerkin method and using the operational matrix for the Caputo fractional derivative, the desired equation reduces to a system of linear algebraic equations. By solving this system, the unknown solution is obtained. The convergence analysis for this method is investigated, and some numerical simulations show the accuracy and ability of the technique.

Keywords: Galerkin method; Chebyshev cardinal functions; time-fractional partial differential equations

MSC: 65M60; 41A50; 35R11

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1. Introduction

In this paper, we propose and analyze an efficient algorithm for solving the time-fractional partial differential equation (TFPDE)

$$\frac{\partial^\beta w(x,t)}{\partial t^\beta} + \eta \frac{\partial w(x,t)}{\partial x} + \sigma \frac{\partial^2 w(x,t)}{\partial x^2} = g(x,t), \qquad t \geq 0,\ x \in [0,1], \beta \in [0,1], \tag{1}$$

with Dirichlet boundary and initial conditions

$$\begin{aligned} &w(x,0) = w_0(x), \quad x \in [0,1], \\ &w(0,t) = f_1(t),\ w(1,t) = f_2(t), \quad t \geq 0, \end{aligned}$$

where η and σ are constant, and $\beta \in \mathbb{R}^+$. Moreover, $\frac{\partial^\beta w(x,t)}{\partial t^\beta}$ is a Caputo derivative with respect to t and in the following, we introduce it.

Fractional calculus was developed in the 17th century and Gottfried Wilhelm Leibniz and Niels Henrik Abel played a unique role in introducing the fractional derivative [1]. Almost all concepts about fractional calculus were introduced by Abel. In recent years, fractional calculus and its application have had a brilliant role in physics, mathematics, and engineering. This branch of mathematics has attracted many enthusiasts, and many papers in this field have been published in various journals. Fractional differential equations are found in the modeling of many physical phenomena such as mathematical biology, fluid mechanics, electrochemistry, viscoelasticity, and many other fields [2–5]. Some differential equations of the fractional type with particular forms can be solved by the Fourier transform method or the Laplace transform method. However, an analytical solution to many generalized fractional differential equations is difficult to obtain. Thus, numerical

techniques play a vital role in this regard. Here are some of the papers that solve fractional differential equations numerically using methods such as the discretized piecewise polynomial collocation method [6], the Kuratowski MNC technique [7], Simpson's method [8], the multiwavelet Galerkin method [9], the B-spline collocation method [10], the Adomian decomposition [11], the least-squares finite element method [12], an operational method [13], and Gegenbauer's wavelets method [14].

Several papers can be found in the literature for solving the fractional partial differential equation. Alikhanov [15] studied a finite difference method to solve the time-fractional diffusion equation. Du et al. [16] discussed a high-order scheme for approximating the Caputo fractional derivative and then solved the fractional diffusion wave equation. Langlands and Henry [17] analyzed the stability and accuracy of an implicit method for the fractional diffusion equation. Lakestani et al. [18] introduced an operational matrix for representing the fractional derivative in the Caputo sense and then used it to solve the fractional partial differential equation. The asymptotic homotopy method was utilized to find the approximate solution of fourth-order TFPDEs [19]. Uddin et al. [20] used a numerical scheme for solving TFPDEs based on radial basis functions. In [21], the authors used a neural network method to solve the advection–diffusion equation. There exist also some valuable references that considered the Caputo fractional derivative in their models [22–24].

Cardinal functions are nonzero at just one point. Therefore, any function can be approximated easily without applying integrals. This is the main property of cardinal functions. The general framework and how to build these types of functions can be found in reference [25]. Due to the high approximation power and characteristics of these types of functions, they have been widely used in solving equations. Afarideh et al. [26] used these bases for solving fractional Sturm–Liouville problem. Lakestani et al. [10], applied them to solve a partial differential equation with an unknown time-dependent coefficient. In [27], the authors used these bases to solve the second-order one-dimensional telegraph equation. Tchier et al. [28] employed a pseudospectral method based on the CCFs to solve the partial integro-differential equations.

The outline of this article is as follows: In Section 2, we describe and introduce the Chebyshev cardinal functions and their properties. Section 3 is dedicated to constructing the Galerkin method for solving TFPDEs based on the CCFs, and the convergence analysis is investigated for the proposed method. Numerical results are given in Section 4 to demonstrate the ability and efficiency of the method.

2. Chebyshev Cardinal Functions

Assume that $\mathcal{Y}$ is the set of the roots of the Chebyshev polynomials of the first kind, i.e.,

$$\mathcal{Y} := \{y_i : T_{m+1}(y_i) = 0, i \in \mathcal{M}\}, \quad \mathcal{M} := \{1, 2, \ldots, m+1\},$$

in which T_{m+1} denotes the Chebyshev polynomials of the first kind on $[-1, 1]$ and the roots of Chebyshev polynomial T_{m+1} can be determined by

$$y_i := \cos\left(\frac{(2i-1)\pi}{2m+2}\right), \quad \forall i \in \mathcal{M}. \tag{2}$$

To define the shifted Chebyshev polynomials for any arbitrary interval $[a, b]$, by putting $y = \left(\frac{2(x-a)}{b-a} - 1\right)$, we can introduce them as follows:

$$T^*_{m+1}(x) := T_{m+1}\left(\frac{2(x-a)}{b-a} - 1\right), \quad x \in [a, b]. \tag{3}$$

It can be easily verified that the shifted Chebyshev function has the roots $x_i = \frac{(y_i+1)(b-a)}{2} + a$.

One of the significant cases among the cardinal functions for orthogonal polynomials is the Chebyshev cardinal functions (CCFs) [26]. Given $i \in \mathcal{M}$, let $T^*_{m+1,x}(x_i)$ interpret

the differentiation with respect to x at the value $x = x_i$. Now, we determine the CCFs via [10,26]

$$\psi_i(x) = \frac{T^*_{m+1}(x)}{T^*_{m+1,x}(x_i)(x - x_i)}, \quad i \in \mathcal{M}. \tag{4}$$

With a simple review, it can be found that the CCFs satisfy the well-known cardinality condition, i.e.,

$$\psi_i(x_j) = \delta_{ij}, \tag{5}$$

where δ_{ij} denotes the Kronecker delta, which is 1 if the variables are equal, and 0 otherwise. This is an important property and it helps us to approximate any function $f(x) \in \mathcal{H}^\nu([a,b])$ ($\mathcal{H}^\nu([a,b])$ is defined below). Motivated by (5), the function $f(x) \in \mathcal{H}^\nu([a,b])$ can be expressed as a linear combination of CCFs as

$$f(x) \approx \sum_{i=1}^{m+1} f(x_i)\psi_i(x). \tag{6}$$

Given $\nu \in \mathbb{N}$, the Sobolev space $H^\nu([a,b])$ consists of functions $f \in C^\nu([a,b])$ such that $\mathcal{D}^\beta f \in L^2([a,b])$, where $\mathbb{N} \ni \beta \leq \nu$, and $\mathcal{D}$ is the derivative operator. If $f \in H^\nu([a,b])$, we determine its norm

$$\|f\|^2_{H^\nu([a,b])} = \sum_{i=0}^{\nu} \|f^{(i)}(t)\|^2_{L^2([a,b])}, \tag{7}$$

and a seminorm as follows

$$|f|^2_{H^{\nu,m}([a,b])} = \sum_{i=\min\{\nu,m\}}^{m} \|f^{(i)}(t)\|^2_{L^2([a,b])}. \tag{8}$$

Lemma 1 ([29]). *Assume that $\mathcal{X}^* := \{x_i : T^*_{m+1}(x_i) = 0, i \in \mathcal{M}\}$ is the set of shifted Gauss–Chebyshev points, and f_m is an approximation of $f(t) \in \mathcal{H}^\nu([a,b])$, i.e.,*

$$f_m(t) = \sum_{i=1}^{m+1} f(x_i)\psi_i(t).$$

Then, one can bound the error obtained from this expansion via

$$\|f - f_m\|_{L^2([a,b])} \leq \mathcal{C} m^{-\nu} |f|_{H^{\nu,m}([a,b])}, \tag{9}$$

in which $\mathcal{C}$ is a constant and independent of m. Moreover, for $\nu \geq 1$ and $1 \leq l \leq \nu$, we can verify that

$$\|f - f_m\|_{H^l([a,b])} \leq \mathcal{C} m^{2l-1/2-\nu} |f|_{H^{\nu,m}([a,b])}. \tag{10}$$

2.1. Operational Matrix of Derivation

In this section, our objective is to find an $(m+1) \times (m+1)$ dimensional matrix D, so that

$$\mathcal{D}(\Psi)(x) = D\Psi(x), \tag{11}$$

where $\Psi(x)$ is a vector function of dimension $m+1$ consisting of CCFs. To find the elements of matrix D, one can employ the property (5) and Equation (6), i.e.,

$$D_{i,j} = \mathcal{D}(\psi_i)(x_j). \tag{12}$$

To find the elements of the matrix, we make some changes to the CCFs. Assuming $\omega = 2^{2m+1}/((b-a)^{m+1} T^*_{m+1,x}(x_i))$ and using (4), we can obtain

$$\psi_i(x) = \omega \prod_{\rho=1,\rho\neq i}^{m+1} (x - x_\rho). \tag{13}$$

In the sequel, one can obtain the following relation via taking the derivative from both sides of (13), viz.,

$$\begin{aligned}\mathcal{D}(\psi_i)(x) &= \omega \prod_{\substack{\rho=1\\ \rho\neq i}}^{m+1} \mathcal{D}(x-x_\rho) = \omega \sum_{\substack{k=1\\ k\neq i}}^{m+1} \prod_{\substack{\rho=1\\ \rho\neq i,\rho}}^{m+1} (x-x_\rho)\\ &= \sum_{\substack{k=1\\ k\neq i}}^{m+1} \frac{T^*_{m+1}(x)}{(x-x_i)(x-x_k)T^*_{m+1,x}(x_i)}\\ &= \sum_{\substack{k=1\\ k\neq i}}^{m+1} \frac{k}{(x-x_k)}\psi_i(x). \end{aligned} \tag{14}$$

Note here that depending on the selection of nodes, two cases can be considered as follows:

1. If $i = j$, due to the fact that $\psi_i(x_i) = 1$, we have

$$\mathcal{D}(\psi_i)(x_j) = \sum_{\substack{k=1\\ k\neq i}}^{m+1} \frac{1}{(x_j - x_k)}. \tag{15}$$

2. If $i \neq j$, according to Equation (14), we get

$$\mathcal{D}(\psi_i)(x_j) = \eta \prod_{\substack{\rho=1\\ \rho\neq i,j}}^{m+1} (x_j - x_\rho). \tag{16}$$

2.2. Operational Matrix of Fractional Integration

Definition 1. *Let $\beta \in \mathbb{R}^+$. We specify the fractional integral operator $\mathcal{I}_a^\beta$ via*

$$\mathcal{I}_a^\beta(f)(x) := \frac{1}{\Gamma(\beta)} \int_a^x (x-\zeta)^{\beta-1} f(\zeta) d\zeta, \quad x \in [a,b], \quad f \in L_1[a,b], \tag{17}$$

in which $\Gamma(\beta)$ determines the Gamma function.

By taking the fractional integral of the vector function $\Psi(x)$ and expanding the results based on these bases (namely $\Psi(x)$), one can find an N-dimensional square matrix such that

$$\mathcal{I}_0^\beta(\Psi(x)) \approx I^\beta \Psi(x), \quad x \in (0,1), \tag{18}$$

where I^β is called the operational matrix of fractional integration. Finding the elements of matrix I^β is our objective. Motivated by Equation (6), one can obtain the elements of I^β via

$$I^\beta_{i,j} = \mathcal{I}_a^\beta \psi_i(x_j). \tag{19}$$

By performing simple calculations, it follows from [30] that

$$\prod_{\substack{\rho=1\\ \rho\neq i}}^{m+1} (x-x_\rho) = \sum_{\rho=0}^{m} \gamma_{i,\rho} x^{m-\rho}, \tag{20}$$

where $\gamma_{i,k}$ is given by

$$\gamma_{i,0} = 1, \gamma_{i,\rho} = \frac{1}{\rho} \sum_{k=0}^{\rho} \theta_{i,k} \gamma_{i,\rho-k}, \ i = 1,\ldots,m+1, \rho = 1,\ldots,m,$$

with

$$\theta_{i,\rho} = \sum_{\substack{j=1 \\ j\neq i}}^{m+1} x_j^{\rho},\ i = 1,\ldots,m+1, \rho = 1,\ldots,m.$$

Using this rearrangement, the CCFs can be modified to

$$\psi_i(x) = \eta \sum_{\rho=0}^{m} \gamma_{i,\rho} x^{m-\rho}. \tag{21}$$

Subtracting (21) from (18), it can be shown that

$$\begin{aligned}\mathcal{I}_a^{\beta}\psi_i(x) &= \omega\mathcal{I}_a^{\beta}(\sum_{\rho=0}^{m}\gamma_{i,\rho}x^{m-\rho}) \\ &= \omega\sum_{\rho=0}^{m}\gamma_{i,\rho}\mathcal{I}_a^{\beta}(x^{m-\rho}) \\ &= \omega\sum_{\rho=0}^{m}\gamma_{i,\rho}\frac{\Gamma(m-\rho+1)}{\Gamma(m-\rho+\beta+1)}x^{m-\rho+\beta}.\end{aligned}$$

With this modification, we can now find the elements of matrix I^{β} as follows

$$I_{i,j}^{\beta} = \omega\sum_{\rho=0}^{m}\gamma_{i,\rho}\frac{\Gamma(m-\rho+1)}{\Gamma(m-\rho+\beta+1)}x_j^{m-\rho+\beta}. \tag{22}$$

Since the fractional integral operator is invertible [26,31], then it follows that the matrix I^{β} is invertible.

2.3. Operational Matrix of Fractional Derivation

Definition 2. *Let $\beta \in \mathbb{R}^+$. We specify the fractional derivative operator $\mathcal{D}_a^{\beta}$ of the Riemann–Liouville (RL) type via*

$${}^{R}\mathcal{D}_a^{\beta}(f)(x) := \mathcal{D}^n\mathcal{I}_a^{n-\beta}(f)(x) = \frac{1}{\Gamma(n-\beta)}\mathcal{D}^n\int_a^x (x-\zeta)^{n-\beta-1}u(\zeta)d\zeta,$$

where $[\beta]+1 := n \in \mathbb{N}$ and $\mathcal{D}^n := \frac{d^n}{dx^n}$.

Definition 3. *The Caputo fractional derivative is determined by [26,32].*

$${}^{c}\mathcal{D}_a^{\beta}(f)(x) := \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{f^{(n)}(\zeta)d\zeta}{(x-\zeta)^{\beta-n+1}} =: \mathcal{I}_a^{n-\beta}\mathcal{D}^n(f)(x), \tag{23}$$

in which $\beta \in \mathbb{R}^+$ and $[\beta]+1 := n \in \mathbb{N}$.

Lemma 2 (cf Corollary 2.3 (a), [32])**.** *It can be proved that the Caputo fractional derivative operator ${}^{c}\mathcal{D}_a^{\beta}$ is bounded via*

$$\|{}^{c}\mathcal{D}_a^{\beta}(f)\|_C \leq \frac{1}{\Gamma(n-\beta)(n-\beta+1)}\|f\|_{C^n}, \tag{24}$$

where $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}_0$ and $n = -[-\beta]$.

There exists an $(m+1)$-dimensional square matrix D^{β} such that

$${}^{c}\mathcal{D}_a^{\beta}(\Psi(x)) \approx D^{\beta}\Psi(x). \tag{25}$$

Using the operational matrices for derivation and fractional integration, it follows from (23) that

$$ {}^{c}\mathcal{D}_{a}^{\beta}(\Psi(x)) = \mathcal{I}_{a}^{n-\beta}\mathcal{D}^{n}(\Psi(x)) \approx D^{n}(I^{\beta})^{n-\beta}\Psi(x). \tag{26}$$

3. Galerkin Method

In this section, we introduce an effective method based on Galerkin to solve TFPDEs (1). Assume that the unknown solution $w(x,t)$ can be expressed as a linear combination of CCFs as follows

$$ w(x,t) \approx \sum_{i=1}^{m+1}\sum_{j=1}^{m+1} w(x_i,x_j)\psi_i(x)\psi_j(t). \tag{27}$$

Using the vector function $\Psi(x)$, this equation can be written as

$$ w(x,t) \approx \Psi^{T}(x)W\Psi(t), \tag{28}$$

where W is a square matrix of dimension $m+1$ and includes unknown coefficients. Using the operational matrix for derivation and fractional derivation, a similar approximation (28) can be found for all terms in Equation (1) as follows.

$$\begin{aligned} \frac{\partial^{\beta} w(x,t)}{\partial t^{\beta}} \approx& \Psi^{T}(x)D^{\beta}W\Psi(t), \\ \frac{\partial w(x,t)}{\partial x} \approx& \Psi^{T}(x)WD\Psi(t), \\ \frac{\partial^{2} w(x,t)}{\partial x^{2}} \approx& \Psi^{T}(x)WD^{2}\Psi(t). \end{aligned} \tag{29}$$

Substituting (28) and (29) into Equation (1), one can obtain the residual function via

$$ r(x,t) := \Psi^{T}(x)\Big(D^{\beta}W + \eta WD + \sigma WD^{2} - G\Big)\Psi(t), \tag{30}$$

where G is a square matrix with a dimension equal to W, and it is obtained as follows:

$$ g(x,t) \approx \sum_{i=1}^{m+1}\sum_{j=1}^{m+1} g(x_i,x_j)\psi_i(x)\psi_j(t) = \Psi^{T}(x)G\Psi(t) := g_m(x,t). \tag{31}$$

Our objective is to force $r(x,t)$ to be approximately zero. To this end, we use the Galerkin method to obtain the following linear system of algebraic equations, i.e.,

$$ \Lambda(W) := D^{\beta}W + \eta WD + \sigma WD^{2} - G = 0. \tag{32}$$

Equation (32) consists of some dependent equations,

$$ \Lambda(W)_{i,j} = 0, i = 2,3,\ldots n+1, \quad j = 2,3,\ldots,m. \tag{33}$$

The boundary conditions are used to find the other $3m+1$ independent linear equations.

$$\begin{aligned} \Lambda(W)_{1,j} &= W\Psi(0) - W_0, \quad i = 1,\ldots,m+1, \\ \Lambda(W)_{i,1} &= \Psi^{T}(0)W - F_1^{T}, \quad j = 2,\ldots,m+1, \\ \Lambda(W)_{i,m+1} &= \Psi^{T}(1)W - F_2^{T}, \quad j = 2,\ldots,m+1, \end{aligned}$$

where

$$ w_0(t) \approx W_0^{T}\Psi(t), \quad f_1(0,t) \approx F_1^{T}\Psi(x), \quad f_2(1,t) \approx F_2^{T}\Psi(x). $$

Because the desired Equation (1) is linear, using vectorization, we have the linear system of $(m+1)^2$ equations

$$\mathcal{A}\mathcal{W} = \mathcal{G}, \tag{34}$$

in which $\mathcal{W}$ and $\mathcal{G}$ are obtained from the vectorization of W and G, respectively. The solution of this system gives the approximate solution of the desired equation.

Convergence Analysis

Suppose that the function $p(x,y)$ interpolates the function values

$$g_{i,j} = g(x_i, y_j), \qquad i,j = 1,\ldots,m,$$

where x_i and y_j are distinct points. It follows from [33] that the reminder formula for a sufficiently smooth function $g(x,y)$ is equal to

$$\begin{aligned}|g(x,y) - p(x,y)| = {} & \frac{\partial^m}{\partial x^m} g(\xi, y) \frac{\Pi_{i=1}^m (x - x_i)}{m!} + \frac{\partial^m}{\partial y^m} g(x, \tau) \frac{\Pi_{j=1}^m (y - y_j)}{m!} \\ & - \frac{\partial^{2m}}{\partial x^m y^m} g(\xi', \tau') \frac{\Pi_{i=1}^m (x - x_i) \Pi_{j=1}^m (y - y_j)}{m! m!},\end{aligned}$$

in which $\xi, \tau, \xi', \tau' \in [0,1]$. By choosing the Chebyshev polynomial zeros as the interpolation points, one can find the Chebyshev interpolation polynomial for the desired function $g(x,y)$. Notice that the leading coefficient of the Chebyshev polynomial is 2^{m-1}, and this gives rise to

$$\begin{aligned}|g(x,y) - p(x,y)| \leq {} & \left(\frac{1}{2}\right)^m \frac{1}{2^{m-1} m!} \sup_{\xi \in [0,1)} |\frac{\partial^m}{\partial x^m} g(\xi, y)| + \left(\frac{1}{2}\right)^m \frac{1}{2^{m-1} m!} \sup_{\tau \in [0,1)} |\frac{\partial^m}{\partial y^m} g(x, \tau)| \\ & + \left(\frac{1}{2}\right)^{2m} \frac{1}{4^{m-1} (m!)^2} \sup_{\xi', \tau' \in [0,1)} |\frac{\partial^{2m}}{\partial x^m \partial y^m} g(\xi', \tau')|.\end{aligned} \tag{35}$$

Assume that w_m is the approximate solution of Equation (1). Thus, the global error $e = w - w_m$ satisfies

$$\frac{\partial^\beta e(x,t)}{\partial t^\beta} + \eta \frac{\partial e(x,t)}{\partial x} + \sigma \frac{\partial^2 e(x,t)}{\partial x^2} = g(x,t) - g_m(x,t), \tag{36}$$

and for this equation, the residual function is as follows

$$R(x,t) = \frac{\partial^\beta e(x,t)}{\partial t^\beta} + \eta \frac{\partial e(x,t)}{\partial x} + \sigma \frac{\partial^2 e(x,t)}{\partial x^2} - g(x,t) + g_m(x,t). \tag{37}$$

We have to show that residual function R tends to zero. To this end, we express all terms on the right-hand side of (37) as a linear combination of CCFs and obtain

$$R(x,t) = \Psi^T(x) \left(D^\beta E + \eta E D + \sigma E D^2\right) \Psi(t) - g(x,t) + g_m(x,t), \tag{38}$$

where

$$e(x,t) \approx \Psi^T(x) E \Psi(t).$$

Taking the L^2-norm from both sides of (38), we have

$$\begin{aligned}\|R(x,t)\| & \leq \|\Psi^T(x) \left(D^\beta E + \eta E D + \sigma E D^2\right) \Psi(t)\| + \|g(x,t) - g_m(x,t)\| \\ & \leq \|\Psi^T(x)\| \|D^\beta E + \eta E D + \sigma E D^2\| \|\Psi(t)\| + \|g(x,t) - g_m(x,t)\|\end{aligned} \tag{39}$$

Because the CCFs are polynomial, we conclude $\|\Psi^T(x)\| = M < \infty$. Thus, it follows from (39) that

$$\begin{aligned}\|R(x,t)\| &\leq M^2\|D^\beta E + \eta ED + \sigma ED^2\| + \|g(x,t) - g_m(x,t)\| \\ &\leq M^2\varepsilon\|E\| + \|g(x,t) + g_m(x,t)\|,\end{aligned} \tag{40}$$

where $\|D^\beta + \eta D + \sigma D^2\| \leq \varepsilon$.

Setting

$$M_{max} = \max\Bigg\{\sup_{\xi\in[0,1)}|\frac{\partial^m}{\partial x^m}g(\xi,y)|, \sup_{\tau\in[0,1)}|\frac{\partial^m}{\partial y^m}g(x,\tau)|, \sup_{\xi',\tau'\in[0,1)}|\frac{\partial^{2m}}{\partial x^m\partial y^m}g(\xi',\tau')|$$
$$\sup_{\xi\in[0,1)}|\frac{\partial^m}{\partial x^m}e(\xi,y)|, \sup_{\tau\in[0,1)}|\frac{\partial^m}{\partial y^m}e(x,\tau)|, \sup_{\xi',\tau'\in[0,1)}|\frac{\partial^{2m}}{\partial x^m\partial y^m}e(\xi',\tau')\Bigg\},$$

and using Equation (35), we conclude from (40) that

$$\|R(x,t)\| \leq \mathcal{C}\left(\frac{1}{2}\right)^m \frac{1}{2^{m-1}m!}\left(2 + \left(\frac{1}{2}\right)^m \frac{1}{2^{m-1}m!}\right), \tag{41}$$

where $\mathcal{C} = M^2 M_{max}\varepsilon$.

Finally, the norm of the residual tends to 0 as m tends to ∞. This proves the convergence of the method.

4. Numerical Experiments

Example 1. *For the first example, consider the time-fractional partial differential equation*

$$\frac{\partial^\beta w(x,t)}{\partial t^\beta} + \frac{\partial w(x,t)}{\partial x} - \frac{\partial^2 uw(x,t)}{\partial x^2} = 2\,\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\,x - 2, \qquad t \geq 0,\ x \in [0,1], \beta \in [0,1],$$

subject to initial and Dirichlet boundary conditions

$$\begin{aligned} &w(x,0) = x^2, \quad x \in [0,1], \\ &w(0,t) = t^2,\ w(1,t) = t^2 + 1, \quad t \geq 0. \end{aligned}$$

The exact solution for this example is provided in [20] and is equal to $w(x,t) = x^2 + t^2$.

Table 1 demonstrates a comparison between our technique and RBFs method [20], taking $m = 3$ *and* $\beta = 0.5$. *It can be seen that the proposed method provides better results than [20]. Figure 1 demonstrates the approximate solution on the left side and corresponding absolute errors on the right.*

Table 1. Comparison between our technique and RBFs method [20] for Example 1.

		$t = 0.1$	$t = 0.5$	$t = 1$
RBFs method (m = 51)	L^∞	6.09×10^{-2}	2.96×10^{-2}	2.11×10^{-2}
	L^2	2.61×10^{-1}	1.28×10^{-1}	9.12×10^{-2}
CCFs method (m = 3)	L^∞	2.00×10^{-16}	1.40×10^{-16}	3.30×10^{-15}
	L^2	3.15×10^{-17}	6.90×10^{-17}	1.40×10^{-15}

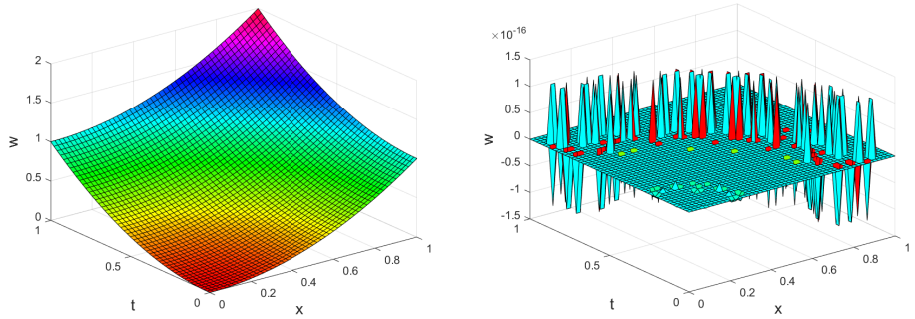


Figure 1. Approximate solution (**left**) and corresponding absolute errors (**right**), with $m = 3$ and $\beta = 0.4$, for Example 1.

Example 2. *The second example is devoted to using the CCFs–Galerkin method for solving the following TFPDE:*

$$\frac{\partial^\beta w(x,t)}{\partial t^\beta} + \frac{\partial w(x,t)}{\partial x} = \frac{t^{1-\alpha}\sin(x)}{\Gamma(2-\alpha)} + t\cos(x), \qquad t \geq 0,\ x \in [0,1], \beta \in [0,1],$$

subject to Dirichlet boundary and initial conditions

$$\begin{aligned} &w(x,0) = 0, \quad x \in [0,1], \\ &w(0,t) = 0,\ w(1,t) = t\sin(1), \quad t \geq 0. \end{aligned}$$

The solution of this equation is given by $w(x,t) = t\sin(x)$ *[20,34].*

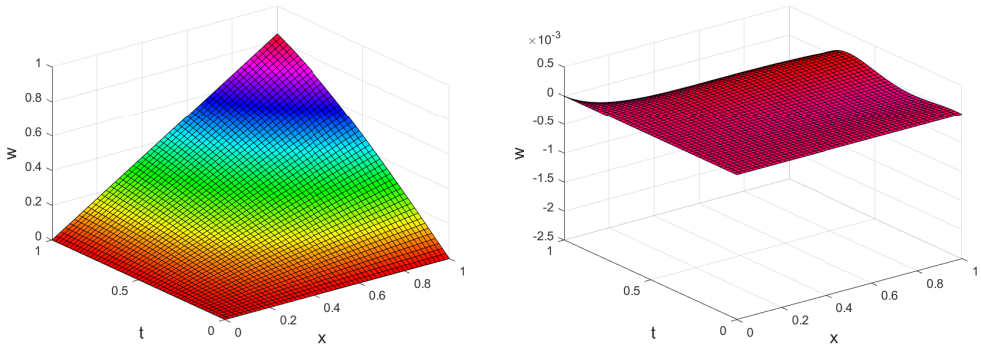
For the computation, we took $\beta = 0.6$*. We compare the proposed method in the previous section with the RBFs method [20] in Table 2. Our method gives better results than the RBFs method. The approximate solution and the absolute errors for different choices of parameter m are plotted in Figure 2. Figure 3 demonstrates the effect of parameter m on the* L^2*-error. The numerical results in Table 3 shows the approximate solution at different times with different m.*

Table 2. Comparison between the proposed method and RBFs method [20] for Example 2.

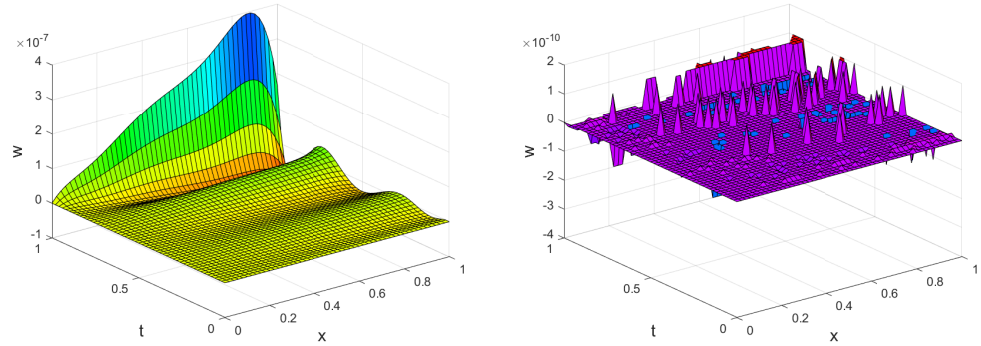
		$t = 0.1$	$t = 0.5$	$t = 1$
RBFs method (m = 121)	L^∞	1.045×10^{-7}	7.000×10^{-7}	1.657×10^{-6}
	L^2	8.765×10^{-8}	1.368×10^{-6}	4.769×10^{-6}
CCFs method (m = 9)	L^∞	4.951×10^{-12}	2.465×10^{-10}	9.131×10^{-11}
	L^2	5.577×10^{-12}	2.552×10^{-11}	3.058×10^{-11}

Table 3. The approximate solutions with different values for m at different times for Example 2.

m	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
3	3.036×10^{-03}	5.510×10^{-03}	7.448×10^{-03}	8.891×10^{-03}	9.903×10^{-03}
4	9.548×10^{-05}	1.659×10^{-04}	2.374×10^{-04}	3.345×10^{-04}	4.849×10^{-04}
5	9.056×10^{-06}	1.572×10^{-05}	2.453×10^{-05}	3.103×10^{-05}	2.495×10^{-05}
6	1.856×10^{-07}	3.609×10^{-07}	5.298×10^{-07}	6.550×10^{-07}	1.239×10^{-06}
7	1.220×10^{-08}	2.481×10^{-08}	3.382×10^{-08}	4.833×10^{-08}	3.392×10^{-08}
8	2.072×10^{-10}	4.010×10^{-10}	5.863×10^{-10}	7.608×10^{-10}	1.607×10^{-09}
9	1.107×10^{-11}	2.018×10^{-11}	3.115×10^{-11}	3.915×10^{-11}	3.058×10^{-11}



(a) Approximate solution **(b)** $m = 4$



(c) $m = 7$ **(d)** $m = 9$

Figure 2. The approximate solution (top-left) and absolute errors by choosing different values of m for Example 2.

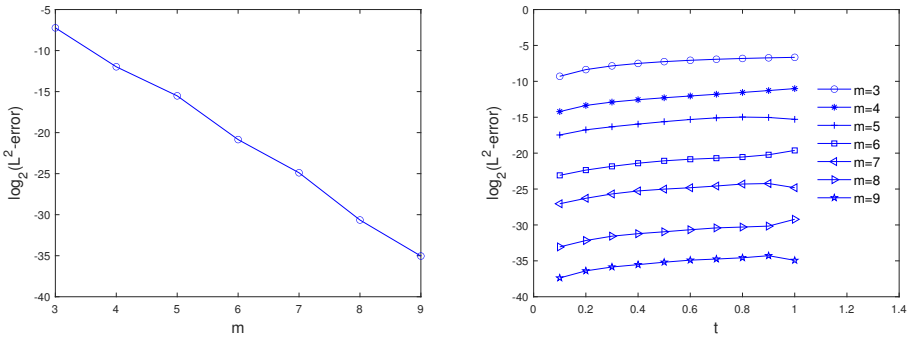


Figure 3. Effect of the parameter m on L^2–errors, with $\beta = 0.6$, for Example 2.

5. Conclusions

To solve time-fractional partial differential equations, we applied the Galerkin method based on Chebyshev cardinal functions. For this purpose, we found the operational matrix of fractional derivation for the CCFs in the Caputo sense via the operational matrices for the derivation operator and fractional integral operator. Using this operational matrix and applying the Galerkin method, the desired equation was reduced to a system of algebraic equations. By solving this system, we could find the unknown solution. We proved that the proposed method was convergent. Numerical results illustrated that our proposed method gave better results than other methods and in some cases, it was very accurate.

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Abbreviations

The following abbreviations are used in this manuscript:

TFPDEs	Time-fractional partial differential equations
CCFs	Chebyshev cardinal functions
RBFs	Radial basis functions

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