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On Some New Dynamic Inequalities Involving C-Monotonic Functions on Time Scales

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Abstract: In this paper, we establish some new dynamic inequalities involving C-monotonic functions with $C \geq 1$, on time scales. As a special case of our results when $C = 1$, we obtain the inequalities involving increasing or decreasing functions (where for $C = 1$, the 1-decreasing function is decreasing and the 1-increasing function is increasing). The main results are proved by applying the properties of C-monotonic functions and the chain rule formula on time scales. As a special case of our results, when $\mathbb{T} = \mathbb{R}$, we obtain refinements of some well-known continuous inequalities and when $\mathbb{T} = \mathbb{N}$, to the best of the authors' knowledge, the results are essentially new.

Keywords: C-monotonic functions; time scales; chain rule on time scales; inequalities

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1. Introduction

In 1995, Heinig and Maligranda [1] proved that if $-\infty \leq \varepsilon < \epsilon \leq \infty$, $\omega, \varpi \geq 0$, ω is decreasing on $[\varepsilon, \epsilon]$ and ϖ is increasing on $[\varepsilon, \epsilon]$ with $\varpi(\varepsilon) = 0$, then for any $\delta \in (0, 1]$,

$$\int_{\varepsilon}^{\epsilon} \omega(\vartheta) d\varpi(\vartheta) \leq \left(\int_{\varepsilon}^{\epsilon} \omega^{\delta}(\vartheta) d\left[\varpi^{\delta}(\vartheta)\right] \right)^{\frac{1}{\delta}}. \tag{1}$$

The inequality (1) is reversed when $1 \leq \delta < \infty$. In addition, the authors of [1] proved that if ω is increasing on $[\varepsilon, \epsilon]$ and ϖ is decreasing on $[\varepsilon, \epsilon]$ with $\varpi(\epsilon) = 0$, then for any $\delta \in (0, 1]$,

$$\int_{\varepsilon}^{\epsilon} \omega(\vartheta) d[-\varpi(\vartheta)] \leq \left(\int_{\varepsilon}^{\epsilon} \omega^{\delta}(\vartheta) d\left[-\varpi^{\delta}(\vartheta)\right] \right)^{\frac{1}{\delta}}. \tag{2}$$

We define that if $s \leq \theta$ implies $\omega(\theta) \leq C\omega(s)$ with $C \geq 1$, then ω is C-decreasing and if $s \leq \theta$ implies $\omega(s) \leq C\omega(\theta)$, $C \geq 1$, then ω is C-increasing. We observe that for $C = 1$, the 1-decreasing function is the normal decreasing function and the 1-increasing function is the normal increasing function.

By using the definition of C-monotonic functions, Pečarić et al. [2] generalized (1) and (2) for C-monotone functions with $C \geq 1$. they proved that if $0 < p < q < \infty$, ω is C-decreasing on $[\varepsilon, \epsilon]$ for $C \geq 1$ and ϖ is increasing and differentiable on $[\varepsilon, \epsilon]$, such that $\varpi(\varepsilon) = 0$, then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^{q}(\vartheta) d[\varpi^{q}(\vartheta)] \right)^{\frac{1}{q}} \leq C^{1-p/q} \left(\int_{\varepsilon}^{\epsilon} \omega^{p}(\vartheta) d[\varpi^{p}(\vartheta)] \right)^{\frac{1}{p}}. \tag{3}$$

In addition, they proved that if $0 < p < q < \infty$, ω is C-increasing on $[\varepsilon, \epsilon]$ for $C \geq 1$ and ϖ is increasing and differentiable on $[\varepsilon, \epsilon]$, such that $\varpi(\varepsilon) = 0$, then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^q(\vartheta) d[\varpi^q(\vartheta)] \right)^{\frac{1}{q}} \geq C^{p/q-1} \left(\int_{\varepsilon}^{\epsilon} \omega^p(\vartheta) d[\varpi^p(\vartheta)] \right)^{\frac{1}{p}}. \tag{4}$$

The authors of [2] proved that if $0 < p < q < \infty$, ω is C-increasing on $[\varepsilon, \epsilon]$ with $C \geq 1$ and ϖ is decreasing and differentiable on $[\varepsilon, \epsilon]$, such that $\varpi(\epsilon) = 0$, then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^q(\vartheta) d[-\varpi^q(\vartheta)] \right)^{\frac{1}{q}} \leq C^{1-p/q} \left(\int_{\varepsilon}^{\epsilon} \omega^p(\vartheta) d[-\varpi^p(\vartheta)] \right)^{\frac{1}{p}}, \tag{5}$$

and they also proved that if $0 < p < q < \infty$, ω is C-decreasing on $[\varepsilon, \epsilon]$ for $C \geq 1$ and ϖ is decreasing and differentiable on $[\varepsilon, \epsilon]$, such that $\varpi(\epsilon) = 0$, then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^q(\vartheta) d[-\varpi^q(\vartheta)] \right)^{\frac{1}{q}} \geq C^{p/q-1} \left(\int_{\varepsilon}^{\epsilon} \omega^p(\vartheta) d[-\varpi^p(\vartheta)] \right)^{\frac{1}{p}}. \tag{6}$$

In the last decades, some authors have been interested in finding some discrete results on $l^p(\mathbb{N})$ analogues to $L^p(\mathbb{R})$-bounds in different fields in analysis and, as a result, this subject becomes a topic of ongoing research. One reason for this upsurge of interest in the discrete case is also due to the fact that discrete operators may even behave differently from their continuous counterparts. In this paper, we obtain the discrete inequalities as special cases of the results with a general domain called the time scale $\mathbb{T}$. The time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. These results contain the classical continuous and discrete inequalities as special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ and can be extended to different inequalities on different time scales such as $\mathbb{T} = h\mathbb{N}$, $h > 0$, $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, etc. In recent years, the study of dynamic inequalities on time scales has received a lot of attention and become a major field in pure and applied mathematics. For more details about the dynamic inequalities on time scales, we refer the reader to the papers [3–16]. For example, Saker et al. [17] proved some dynamic inequalities for C-monotonic functions and proved that if ω is C-decreasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$ with $C \geq 1$ and ϖ is increasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$, such that $\varpi(\varepsilon) = 0$, then

$$\varphi\left(C \int_{\varepsilon}^{\epsilon} \omega(\vartheta) \varpi^{\Delta}(\vartheta) \Delta\vartheta \right) \leq C \int_{\varepsilon}^{\epsilon} \omega(\vartheta) \varpi^{\Delta}(\vartheta) \varphi'[\omega(\vartheta)\varpi(\vartheta)] \Delta\vartheta,$$

and if ω is C-increasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$ for $C \geq 1$ and ϖ is increasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$, such that $\varpi(\varepsilon) = 0$, then

$$\varphi\left(\frac{1}{C} \int_{\varepsilon}^{\epsilon} \omega(\vartheta) \varpi^{\Delta}(\vartheta) \Delta\vartheta \right) \geq \frac{1}{C} \int_{\varepsilon}^{\epsilon} \omega(\vartheta) \varpi^{\Delta}(\vartheta) \varphi'[\omega^{\sigma}(\vartheta)\varpi^{\sigma}(\vartheta)] \Delta\vartheta.$$

In addition, they proved that if ω is C-increasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$ with $C \geq 1$ and ϖ is decreasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$, such that $\varpi(\epsilon) = 0$, then

$$\varphi\left(C \int_{\varepsilon}^{\epsilon} \omega(\vartheta) [-\varpi(\vartheta)]^{\Delta} \Delta\vartheta \right) \leq C \int_{\varepsilon}^{\epsilon} \omega(\vartheta) [-\varpi(\vartheta)]^{\Delta} \varphi'[\omega^{\sigma}(\vartheta)\varpi^{\sigma}(\vartheta)] \Delta\vartheta,$$

and if ω is C-decreasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$ with $C \geq 1$ and ϖ is decreasing on $[\varepsilon, \epsilon] \cap \mathbb{T}$, such that $\varpi(\epsilon) = 0$, then

$$\varphi\left(\frac{1}{C} \int_{\varepsilon}^{\epsilon} \omega(\vartheta) [-\varpi(\vartheta)]^{\Delta} \Delta\vartheta \right) \geq \frac{1}{C} \int_{\varepsilon}^{\epsilon} \omega(\vartheta) [-\varpi(\vartheta)]^{\Delta} \varphi'[\omega(\vartheta)\varpi(\vartheta)] \Delta\vartheta.$$

Our aim in this paper is to generalize the inequalities (1)–(6) on time scales by establishing some new dynamic inequalities involving C-monotonic functions.

The paper is organized as follows. In Section 2, we present some preliminaries concerning the theory of time scales and the definitions of C-monotonic functions. In Section 3, we prove the main results using the chain rule on time scales and the properties of C-monotonic functions. Our results when $\mathbb{T}=\mathbb{R}$ give the inequalities (1)–(6) proved by Heinig, Maligranda, Pečarić, Perić and Persson, respectively. Our results for $\mathbb{T}=\mathbb{N}$ are essentially new.

2. Preliminaries and Basic Lemmas

In this section, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [18,19] which summarize and organize much of the time scale calculus. We define the time scale interval $[\varepsilon,\epsilon]_{\mathbb{T}}$ by $[\varepsilon,\epsilon]_{\mathbb{T}}:=[\varepsilon,\epsilon]\cap\mathbb{T}$. A function $\omega:\mathbb{T}\to\mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided that ω is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})=C_{rd}(\mathbb{T},\mathbb{R})$. The product and quotient rules for the derivative of the product $\omega\varpi$ and the quotient ω/ϖ (where $\varpi\varpi^{\sigma}\neq 0$, here $\varpi^{\sigma}=\varpi\circ\sigma$) of two differentiable functions ω and ϖ are given by

$$(\omega\varpi)^{\Delta}=\omega\varpi^{\Delta}+\omega^{\Delta}\varpi^{\sigma}=\omega^{\Delta}\varpi+\omega^{\sigma}\varpi^{\Delta},\text{ and }\left(\frac{\omega}{\varpi}\right)^{\Delta}=\frac{\omega^{\Delta}\varpi-\omega\varpi^{\Delta}}{\varpi\,\varpi^{\sigma}}.$$

Let $\omega:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose that $\varpi:\mathbb{T}\to\mathbb{R}$ is delta-differentiable. Then, $\omega\circ\varpi:\mathbb{T}\to\mathbb{R}$ is delta-differentiable and there exists ξ in the real interval $[\theta,\sigma(\theta)]$ with

$$(\omega\circ\varpi)^{\Delta}(\theta)=\omega'(\varpi(\xi))\varpi^{\Delta}(\theta).$$

In addition, the formula

$$(\omega\circ\varpi)^{\Delta}(\theta)=\left\{\int_0^1\omega'\Big(\varpi(\theta)+h\mu(\theta)\varpi^{\Delta}(\theta)\Big)dh\right\}\varpi^{\Delta}(\theta), \tag{7}$$

holds. A special case of (7) is

$$\left[u^{\lambda}(\theta)\right]^{\Delta}=\lambda\int_0^1[hu^{\sigma}+(1-h)u]^{\lambda-1}u^{\Delta}(\theta)dh.$$

In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(\theta)=\varpi(\theta)$, then the Cauchy (delta) integral of ϖ is defined by $\int_{\varepsilon}^{\theta}\varpi(\vartheta)\Delta\vartheta:=G(\theta)-G(\varepsilon)$. It can be shown (see [18]) that if $\varpi\in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(\theta):=\int_{\theta_0}^{\theta}\varpi(\vartheta)\Delta\vartheta$ exists, $\theta_0\in\mathbb{T}$ and satisfies $G^{\Delta}(\theta)=\varpi(\theta)$, $\theta\in\mathbb{T}$. The integration on discrete time scales is defined by $\int_{\varepsilon}^{\epsilon}\varpi(\theta)\Delta\theta=\sum_{\theta\in[\varepsilon,\epsilon)}\mu(\theta)\varpi(\theta)$. In case $\mathbb{T}=\mathbb{R}$, we have

$$\sigma(\theta)=\rho(\theta)=\theta,\ \mu(\theta)=0,\ \omega^{\Delta}=\omega',\text{ and }\int_{\varepsilon}^{\epsilon}\omega(\theta)\Delta\theta=\int_{\varepsilon}^{\epsilon}\omega(\theta)d\theta,$$

and in case $\mathbb{T}=\mathbb{Z}$, we have

$$\sigma(\theta)=\theta+1,\ \rho(\theta)=\theta-1,\ \mu(\theta)=1,\ \omega^{\Delta}=\Delta\omega,\text{ and }\int_{\varepsilon}^{\epsilon}\omega(\theta)\Delta\theta=\sum_{\theta=\varepsilon}^{\epsilon-1}\omega(\theta).$$

The integration by parts formula on time scale is given by

$$\int_{\varepsilon}^{\epsilon}u^{\Delta}(\theta)v^{\sigma}(\theta)\,\Delta\theta=u(\theta)v(\theta)|_{\varepsilon}^{\epsilon}-\int_{\varepsilon}^{\epsilon}u(\theta)v^{\Delta}(\theta)\Delta\theta.$$

In addition, we have for $\omega \in C_{rd}$ and $\theta \in \mathbb{T}$ that

$$\int_{\theta}^{\sigma(\theta)} \omega(\tau)\Delta\tau = \mu(\theta)\omega(\theta).$$

Definition 1. *Assume that $\mathbb{T}$ is a time scale and $\omega : \mathbb{T} \to \mathbb{R}$. If $s \leq \theta$ implies $\omega(\theta) \leq \omega(s)$, then ω is decreasing and if $s \leq \theta$ implies $\omega(s) \leq \omega(\theta)$, then ω is increasing.*

We can generalize the definition of the increasing and decreasing function to be C-increasing and C-decreasing, respectively, which is given in the following.

Definition 2 ([17])**.** *Assume that $\mathbb{T}$ is a time scale, $\omega : \mathbb{T} \to \mathbb{R}$ and $C \geq 1$. If $s \leq \theta$ implies $\omega(\theta) \leq C\omega(s)$, then ω is C-decreasing. If $s \leq \theta$ implies $\omega(s) \leq C\omega(\theta)$, then ω is C-increasing.*

As a special case, when $C = 1$, we observe that the 1-decreasing function is decreasing and the 1-increasing function is increasing.

Lemma 1. *Let $0 < q < \infty$. If ω is C-decreasing for $C \geq 1$, then ω^q is C^q-decreasing and if ϖ is C-increasing, then ϖ^q is C^q-increasing.*

Proof. Since ω is C-decreasing, we have for $s \leq \theta$ that $\omega(\theta) \leq C\omega(s)$, and then, we obtain (where $q > 0$) that

$$\omega^q(\theta) \leq C^q\omega^q(s).$$

Thus, ω^q is C^q-decreasing.

Since ϖ is C-increasing, we have for $s \leq \theta$ that $\varpi(s) \leq C\varpi(\theta)$, and then, we see (where $q > 0$) that

$$\varpi^q(s) \leq C^q\varpi^q(\theta),$$

which indicates that ϖ^q is C^q-increasing. The proof is completed. □

3. Main Results

Throughout the paper, we assume that the functions (without mentioning) are rd-continuous nonnegative and Δ-differentiable functions, locally Δ-integrable on $[\varepsilon, \infty)_{\mathbb{T}}$, and the considered integrals are assumed to exist.

In this section, we state and prove our main results.

Theorem 1. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$, $q > 0$ and $0 < \delta < 1$. Furthermore, assume that if χ is C^q-decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, $C \geq 1$ and λ is increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, such that $\lambda(\varepsilon) = 0$. If*

$$\lambda^{\delta-1}(\sigma(\theta))\lambda^{1-\delta}(\theta) \geq 1, \tag{8}$$

then

$$\left(\int_{\varepsilon}^{\epsilon} \chi(\vartheta)[\lambda(\vartheta)]^{\Delta}\Delta\vartheta\right) \leq C^{q(1-\delta)}\left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}. \tag{9}$$

Proof. Since λ is an increasing function with $\lambda(\varepsilon) = 0$ and χ is C^q-decreasing function, we have for $\vartheta \leq \theta$ that $\chi(\theta) \leq C^q\chi(\vartheta)$, and then,

$$\begin{aligned} \int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \geq{} & \left(\tfrac{1}{C}\right)^{q\delta} \int_{\varepsilon}^{\theta} \chi^{\delta}(\theta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ ={} & \left(\tfrac{1}{C}\right)^{q\delta} \chi^{\delta}(\theta) \int_{\varepsilon}^{\theta}\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ ={} & \left(\tfrac{1}{C}\right)^{q\delta} \chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta) - \lambda^{\delta}(\varepsilon)\right] \\ ={} & \left(\tfrac{1}{C}\right)^{q\delta} \chi^{\delta}(\theta)\lambda^{\delta}(\theta), \end{aligned} \tag{10}$$

and then,

$$\chi^{\delta}(\theta)\lambda^{\delta}(\theta) \leq C^{q\delta} \int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta. \tag{11}$$

Consider the function

$$\Lambda(\theta) = C^{q(1-\delta)}\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}} - \int_{\varepsilon}^{\theta} \chi(\vartheta)[\lambda(\vartheta)]^{\Delta}\Delta\vartheta. \tag{12}$$

By applying chain rule formula on the term

$$\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}},$$

we have for $\xi \in [\theta, \sigma(\theta)]$ that

$$\begin{aligned} &\left[\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}\right]^{\Delta} \\ &= \frac{1}{\delta}\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta}. \end{aligned} \tag{13}$$

Again, by applying the chain rule formula on the term $\lambda^{\delta}(\theta)$, we obtain

$$\left[\lambda^{\delta}(\theta)\right]^{\Delta} = \delta\lambda^{\delta-1}(\xi)\lambda^{\Delta}(\theta), \tag{14}$$

where $\xi \in [\theta, \sigma(\theta)]$. From (12), we observe that

$$\Lambda^{\Delta}(\theta) = C^{q(1-\delta)}\left[\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}\right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta). \tag{15}$$

Substituting (13) and (14) into (15), we get

$$\begin{aligned} \Lambda^{\Delta}(\theta) =\;& \frac{1}{\delta}C^{q(1-\delta)}\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta) \\ =\;& C^{q(1-\delta)}\lambda^{\delta-1}(\xi)\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\lambda^{\Delta}(\theta) - \chi(\theta)\lambda^{\Delta}(\theta) \\ =\;& \lambda^{\Delta}(\theta)\left[C^{q(1-\delta)}\lambda^{\delta-1}(\xi)\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) - \chi(\theta)\right]. \end{aligned} \tag{16}$$

Since $\xi \in [\theta, \sigma(\theta)]$, $0 < \delta < 1$ and λ is an increasing function, then

$$\begin{aligned} &\lambda^{\delta-1}(\xi)\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1} \\ &\geq \lambda^{\delta-1}(\sigma(\theta))\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}. \end{aligned} \tag{17}$$

Substituting (17) into (16), we observe that

$$\Lambda^{\Delta}(\theta) \geq \lambda^{\Delta}(\theta)\left[C^{q(1-\delta)}\lambda^{\delta-1}(\sigma(\theta))\left(\int_{\varepsilon}^{\theta} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) - \chi(\theta)\right]. \tag{18}$$

Substituting (11) into (18), we get

$$\begin{aligned}\Lambda^{\Delta}(\theta) \geq\; & \lambda^{\Delta}(\theta)\left[\lambda^{\delta-1}(\sigma(\theta))\left[\chi^{\delta}(\theta)\lambda^{\delta}(\theta)\right]^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)-\chi(\theta)\right] \\ =\; & \lambda^{\Delta}(\theta)\left[\lambda^{\delta-1}(\sigma(\theta))\lambda^{1-\delta}(\theta)\chi(\theta)-\chi(\theta)\right].\end{aligned} \tag{19}$$

By using (8) and λ is an increasing function, we have from (19) that

$$\Lambda^{\Delta}(\theta) \geq 0,$$

and then, the function Λ is increasing on $[\varepsilon,\epsilon]_{\lesssim}$.

Since Λ is an increasing function, we have for $\epsilon > \varepsilon$ that $\Lambda(\epsilon) \geq \Lambda(\varepsilon)$ and then (note that $\Lambda(\varepsilon) = 0$),

$$\int_{\varepsilon}^{\epsilon}\chi(\vartheta)[\lambda(\vartheta)]^{\Delta}\Delta\vartheta \leq C^{q(1-\delta)}\left(\int_{\varepsilon}^{\epsilon}\chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}},$$

which is the desired inequality (9). The proof is completed. □

Corollary 1. *When $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$, $C = 1$, we observe that (8) holds already with equality and we get the inequality (1) proved by Heinig and Maligranda* [1].

As a special case of Theorem 1, when $0 < p < q < \infty$ such that $0 < \delta = p/q < 1$, $\chi(\vartheta) = \omega^{q}(\vartheta)$ and $\lambda(\vartheta) = \varpi^{q}(\vartheta)$, we have the following corollary.

Corollary 2. *Assume that $0 < p < q < \infty$, ω is C-decreasing on $[\varepsilon,\epsilon]_{\mathbb{T}}$, $C \geq 1$ and ϖ is increasing on $[\varepsilon,\epsilon]_{\mathbb{T}}$, such that $\varpi(\varepsilon) = 0$. If*

$$[\varpi^{\sigma}(\theta)]^{p-q}\varpi^{q-p}(\theta) \geq 1, \tag{20}$$

then

$$\left(\int_{\varepsilon}^{\epsilon}\omega^{q}(\vartheta)[\varpi^{q}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{q}} \leq C^{1-\frac{p}{q}}\left(\int_{\varepsilon}^{\epsilon}\omega^{p}(\vartheta)[\varpi^{p}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{p}}. \tag{21}$$

Corollary 3. *In Corollary 2, when $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$, we observe that (20) holds with equality and then we obtain the inequality (3) proved by Pečarić et al.* [2].

Corollary 4. *In Corollary 2, when $\mathbb{T} = \mathbb{N}$, $\sigma(n) = n+1$, we have that if $\varepsilon, \epsilon \in \mathbb{N}$, $0 < p < q < \infty$, ω is C-decreasing sequence for $C \geq 1$ and ϖ is increasing with $\varpi(\varepsilon) = 0$ such that*

$$[\varpi(n+1)]^{p-q}\varpi^{q-p}(n) \geq 1,$$

then

$$\left(\sum_{n=\varepsilon}^{\epsilon}\omega^{q}(n)\Delta[\varpi^{q}(n)]\right)^{\frac{1}{q}} \leq C^{1-\frac{p}{q}}\left(\sum_{n=\varepsilon}^{\epsilon}\omega^{p}(n)\Delta[\varpi^{p}(n)]\right)^{\frac{1}{p}}.$$

Theorem 2. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$, $q > 0$ and $0 < \delta < 1$. Furthermore assume that χ is C^{q}-increasing on $[\varepsilon,\epsilon]_{\mathbb{T}}$ with $C \geq 1$ and λ is increasing on $[\varepsilon,\epsilon]_{\mathbb{T}}$, such that $\lambda(\varepsilon) = 0$. If*

$$\lambda^{\delta-1}(\theta)[\lambda^{\sigma}(\theta)]^{1-\delta}[\chi^{\sigma}(\theta)]^{1-\delta}\chi^{\delta}(\theta) \leq \chi(\theta), \tag{22}$$

then

$$\left(\int_{\varepsilon}^{\epsilon}\chi(\vartheta)[\lambda(\vartheta)]^{\Delta}\Delta\vartheta\right) \geq C^{q(\delta-1)}\left(\int_{\varepsilon}^{\epsilon}\chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}. \tag{23}$$

Proof. Since λ is an increasing function with $\lambda(\varepsilon) = 0$ and χ is a C^q-increasing function, we have for $\vartheta \leq \theta$ that $\chi(\vartheta) \leq C^q \chi(\theta)$, and thus,

$$\begin{aligned} \int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta \leq \;& C^{q\delta} \int_\varepsilon^\theta \chi^\delta(\theta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta \\ =\;& C^{q\delta}\chi^\delta(\theta) \int_\varepsilon^\theta \left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta \\ =\;& C^{q\delta}\chi^\delta(\theta)\left[\lambda^\delta(\theta) - \lambda^\delta(\varepsilon)\right] \\ =\;& C^{q\delta}\chi^\delta(\theta)\lambda^\delta(\theta), \end{aligned} \tag{24}$$

and then,

$$\chi^\delta(\theta)\lambda^\delta(\theta) \geq \frac{1}{C^{q\delta}} \int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta. \tag{25}$$

Consider the function

$$\Lambda(\theta) = C^{q(\delta-1)}\left(\int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}} - \int_\varepsilon^\theta \chi(\vartheta)[\lambda(\vartheta)]^\Delta \Delta\vartheta. \tag{26}$$

By applying the chain rule formula on the term

$$\left(\int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}},$$

we have for $\xi \in [\theta, \sigma(\theta)]$ that

$$\begin{aligned} &\left[\left(\int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}}\right]^\Delta \\ &= \frac{1}{\delta}\left(\int_\varepsilon^\xi \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}-1} \chi^\delta(\theta)\left[\lambda^\delta(\theta)\right]^\Delta. \end{aligned} \tag{27}$$

Again, by applying the chain rule formula on the terms $\lambda^\delta(\theta)$, we obtain

$$\left[\lambda^\delta(\theta)\right]^\Delta = \delta\lambda^{\delta-1}(\xi)\lambda^\Delta(\theta), \tag{28}$$

where $\xi \in [\theta, \sigma(\theta)]$. From (26), we observe that

$$\Lambda^\Delta(\theta) = C^{q(\delta-1)}\left[\left(\int_\varepsilon^\theta \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}}\right]^\Delta - \chi(\theta)\lambda^\Delta(\theta). \tag{29}$$

Substituting (27) and (28) into (29), we get

$$\begin{aligned} \Lambda^\Delta(\theta) =\;& \frac{1}{\delta}C^{q(\delta-1)}\left(\int_\varepsilon^\xi \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}-1} \chi^\delta(\theta)\left[\lambda^\delta(\theta)\right]^\Delta - \chi(\theta)\lambda^\Delta(\theta) \\ =\;& C^{q(\delta-1)}\lambda^{\delta-1}(\xi)\left(\int_\varepsilon^\xi \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}-1} \chi^\delta(\theta)\lambda^\Delta(\theta) - \chi(\theta)\lambda^\Delta(\theta) \\ =\;& \lambda^\Delta(\theta)\left[C^{q(\delta-1)}\lambda^{\delta-1}(\xi)\left(\int_\varepsilon^\xi \chi^\delta(\vartheta)\left[\lambda^\delta(\vartheta)\right]^\Delta \Delta\vartheta\right)^{\frac{1}{\delta}-1} \chi^\delta(\theta) - \chi(\theta)\right]. \end{aligned} \tag{30}$$

Since $\xi \in [\theta, \sigma(\theta)]$, $0 < \delta < 1$ and λ is an increasing function, then

$$\begin{aligned} &\lambda^{\delta-1}(\xi)\left(\int_{\varepsilon}^{\xi} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1} \\ &\leq \lambda^{\delta-1}(\theta)\left(\int_{\varepsilon}^{\sigma(\theta)} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}. \end{aligned} \tag{31}$$

Substituting (31) into (30), we see that

$$\Lambda^{\Delta}(\theta) \leq \lambda^{\Delta}(\theta)\left[C^{q(\delta-1)}\lambda^{\delta-1}(\theta)\left(\int_{\varepsilon}^{\sigma(\theta)} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) - \chi(\theta)\right]. \tag{32}$$

Substituting (25) into (32), we get

$$\begin{aligned} \Lambda^{\Delta}(\theta) \leq\; & \lambda^{\Delta}(\theta)\left[\lambda^{\delta-1}(\theta)\left[\chi^{\delta}(\sigma(\theta))\lambda^{\delta}(\sigma(\theta))\right]^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) - \chi(\theta)\right] \\ =\; & \lambda^{\Delta}(\theta)\left[\lambda^{\delta-1}(\theta)[\lambda^{\sigma}(\theta)]^{1-\delta}[\chi^{\sigma}(\theta)]^{1-\delta}\chi^{\delta}(\theta) - \chi(\theta)\right]. \end{aligned} \tag{33}$$

By using (22) and λ is an increasing function, the inequality (33) becomes

$$\Lambda^{\Delta}(\theta) \leq 0,$$

and then, the function Λ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$.

Since Λ is a decreasing function, then we have for $\epsilon > \varepsilon$ that $\Lambda(\epsilon) \leq \Lambda(\varepsilon)$ and then (note that $\Lambda(\varepsilon) = 0$),

$$C^{q(\delta-1)}\left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta)\left[\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}} \leq \int_{\varepsilon}^{\epsilon} \chi(\vartheta)[\lambda(\vartheta)]^{\Delta}\Delta\vartheta,$$

which is the desired inequality (23). The proof is completed. □

As a special case of Theorem 2, when $0 < p < q < \infty$ such that $0 < \delta = p/q < 1$, $\chi(\vartheta) = \omega^{q}(\vartheta)$ and $\lambda(\vartheta) = \varpi^{q}(\vartheta)$, we get the following corollary.

Corollary 5. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$ and $0 < p < q < \infty$. Furthermore, if ω is C-increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$ for $C \geq 1$ and ϖ is increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, with $\varpi(\varepsilon) = 0$, such that*

$$\varpi^{p-q}(\theta)[\varpi^{\sigma}(\theta)]^{q-p}[\omega^{\sigma}(\theta)]^{q-p}\omega^{p}(\theta) \leq \omega^{q}(\theta), \tag{34}$$

then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^{q}(\vartheta)[\varpi^{q}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{q}} \geq C^{p/q-1}\left(\int_{\varepsilon}^{\epsilon} \omega^{p}(\vartheta)[\varpi^{p}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{p}}.$$

Corollary 6. *As a special case of Corollary 5, when $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$, we have that (34) holds already with equality and we get the inequality (4) proved by Pečarić et al.* [2].

Corollary 7. *In Corollary 5, when $\mathbb{T} = \mathbb{N}$, $\sigma(n) = n+1$, we have that if $\varepsilon, \epsilon \in \mathbb{N}$, $0 < p < q < \infty$, ω is a C-increasing sequence for $C \geq 1$ and ϖ is increasing with $\varpi(\varepsilon) = 0$, such that*

$$\varpi^{p-q}(n)\varpi^{q-p}(n+1)\omega^{q-p}(n+1)\omega^{p}(n) \leq \omega^{q}(n),$$

then

$$\left(\sum_{n=\varepsilon}^{\epsilon} \omega^{q}(n)\Delta\varpi^{q}(n)\right)^{\frac{1}{q}} \geq C^{p/q-1}\left(\sum_{n=\varepsilon}^{\epsilon}\int_{\varepsilon}^{\epsilon} \omega^{p}(n)\Delta\varpi^{p}(n)\right)^{\frac{1}{p}}.$$

Theorem 3. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$, $q > 0$ and $0 < \delta < 1$. Furthermore, if χ is C^q-increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, $C \geq 1$ and λ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, with $\lambda(\epsilon) = 0$ such that*

$$\chi(\theta) \leq \lambda^{\delta-1}(\theta)[\lambda^{\sigma}(\theta)]^{1-\delta}\chi^{\delta}(\theta)[\chi^{\sigma}(\theta)]^{1-\delta}, \tag{35}$$

then

$$\int_{\varepsilon}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta}\Delta\vartheta \leq C^{q(1-\delta)}\left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}. \tag{36}$$

Proof. Since λ is a decreasing function with $\lambda(\epsilon) = 0$ and χ is a C^q-increasing function, we have for $\vartheta \geq \theta$ that $\chi(\theta) \leq C^q \chi(\vartheta)$, and thus,

$$\begin{aligned} \int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \geq \quad & \frac{1}{C^{q\delta}} \int_{\theta}^{\epsilon} \chi^{\delta}(\theta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ = \quad & \frac{1}{C^{q\delta}}\chi^{\delta}(\theta) \int_{\theta}^{\epsilon}\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ = \quad & \frac{1}{C^{q\delta}}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta) - \lambda^{\delta}(\epsilon)\right] \\ = \quad & \frac{1}{C^{q\delta}}\chi^{\delta}(\theta)\lambda^{\delta}(\theta). \end{aligned} \tag{37}$$

Consider the function

$$\Lambda(\theta) = C^{q(1-\delta)}\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}} - \int_{\theta}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta}\Delta\vartheta. \tag{38}$$

By applying the chain rule formula on the term

$$\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}},$$

we have for $\xi \in [\theta, \sigma(\theta)]$ that

$$\begin{aligned} &\left[\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}\right]^{\Delta} \\ &= \frac{1}{\delta}\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta}. \end{aligned} \tag{39}$$

Again by applying the chain rule formula on the terms $\lambda^{\delta}(\theta)$, we obtain

$$\left[\lambda^{\delta}(\theta)\right]^{\Delta} = \delta\lambda^{\delta-1}(\xi)\lambda^{\Delta}(\theta), \tag{40}$$

where $\xi \in [\theta, \sigma(\theta)]$. From (38), we observe that

$$\Lambda^{\Delta}(\theta) = C^{q(1-\delta)}\left[\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}}\right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta). \tag{41}$$

Substituting (39) and (40) into (41), we get

$$\begin{aligned} \Lambda^{\Delta}(\theta) = \quad & \frac{1}{\delta}C^{q(1-\delta)}\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta) \\ = \quad & C^{q(1-\delta)}\lambda^{\delta-1}(\xi)\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta)\lambda^{\Delta}(\theta) - \chi(\theta)\lambda^{\Delta}(\theta) \\ = \quad & \left[-\lambda^{\Delta}(\theta)\right]\left[-C^{q(1-\delta)}\lambda^{\delta-1}(\xi)\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) + \chi(\theta)\right]. \end{aligned} \tag{42}$$

Since $\xi \in [\theta, \sigma(\theta)]$, $0 < \delta < 1$ and λ is a decreasing function, then

$$\begin{aligned} &\lambda^{\delta-1}(\xi)\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1} \\ &\geq \lambda^{\delta-1}(\theta)\left(\int_{\sigma(\theta)}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}. \end{aligned} \tag{43}$$

Substituting (43) into (42), we see that

$$\Lambda^{\Delta}(\theta) \leq \left[-\lambda^{\Delta}(\theta)\right]\left[-C^{q(1-\delta)}\lambda^{\delta-1}(\theta)\left(\int_{\sigma(\theta)}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) + \chi(\theta)\right]. \tag{44}$$

Substituting (37) into (44), we get

$$\Lambda^{\Delta}(\theta) \leq \left[-\lambda^{\Delta}(\theta)\right]\left[-\lambda^{\delta-1}(\theta)[\lambda^{\sigma}(\theta)]^{1-\delta}\chi^{\delta}(\theta)[\chi^{\sigma}(\theta)]^{1-\delta} + \chi(\theta)\right]. \tag{45}$$

By using (35) and λ is a decreasing function, we have from (45) that

$$\Lambda^{\Delta}(\theta) \leq 0,$$

and then, the function Λ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$.

Since Λ is a decreasing function, we have for $\epsilon > \varepsilon$ that $\Lambda(\epsilon) \leq \Lambda(\varepsilon)$ and then (note that $\Lambda(\epsilon) = 0$),

$$\int_{\varepsilon}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta}\Delta\vartheta \leq C^{q(1-\delta)}\left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}},$$

which is the desired inequality (36). The proof is completed. □

Corollary 8. *As a special case of Theorem 3, when $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$ and $C = 1$, we observe that (35) holds with equality, and then, we get the inequality (2) proved by Heinig and Maligranda* [1].

As a special case of Theorem 3, when $0 < p < q < \infty$ such that $0 < \delta = p/q < 1$, $\chi(\vartheta) = \omega^{q}(\vartheta)$ and $\lambda(\vartheta) = \varpi^{q}(\vartheta)$, we get the following corollary.

Corollary 9. *If $0 < p < q < \infty$, ω is C-increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, $C \geq 1$ and ϖ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$ with $\varpi(\epsilon) = 0$ such that*

$$\omega^{q}(\theta) \leq \varpi^{p-q}(\theta)[\varpi^{\sigma}(\theta)]^{q-p}\omega^{p}(\theta)[\omega^{\sigma}(\theta)]^{q-p}, \tag{46}$$

then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^{q}(\vartheta)[-\varpi^{q}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{q}} \leq C^{1-p/q}\left(\int_{\varepsilon}^{\epsilon} \omega^{p}(\vartheta)[-\varpi^{p}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{p}}.$$

Corollary 10. *As a special case of Corollary 9, when $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$, we observe that (46) holds already with equality and we get the inequality (5) proved by Pečarić et al.* [2].

Corollary 11. *In Corollary 9, when $\mathbb{T} = \mathbb{N}$, $\sigma(n) = n + 1$, we have that if $\varepsilon, \epsilon \in \mathbb{N}$, $0 < p < q < \infty$, ω is C-increasing sequence for $C \geq 1$ and ϖ is decreasing with $\varpi(\epsilon) = 0$, such that*

$$\omega^{q}(n) \leq \varpi^{p-q}(n)\varpi^{q-p}(n+1)\omega^{p}(n)\omega^{q-p}(n+1),$$

then

$$\left(\sum_{n=\varepsilon}^{\epsilon} \omega^{q}(n)\Delta[-\varpi^{q}(n)]\right)^{\frac{1}{q}} \leq C^{1-p/q}\left(\sum_{n=\varepsilon}^{\epsilon} \omega^{p}(n)\Delta[-\varpi^{p}(n)]\right)^{\frac{1}{p}}.$$

Theorem 4. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$, $q > 0$ and $0 < \delta < 1$. Furthermore, if χ is C^q-decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, $C \geq 1$ and λ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$ with $\lambda(\epsilon) = 0$ such that*

$$\lambda^{1-\delta}(\theta)[\lambda^{\sigma}(\theta)]^{\delta-1} \leq 1, \tag{47}$$

then

$$\int_{\varepsilon}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta} \Delta\vartheta \geq C^{q(\delta-1)} \left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta) \left[-\lambda^{\delta}(\vartheta)\right]^{\Delta} \Delta\vartheta \right)^{\frac{1}{\delta}}. \tag{48}$$

Proof. Since λ is a decreasing function with $\lambda(\epsilon) = 0$ and χ is a C^q-decreasing function, we have for $\vartheta \geq \theta$ that $\chi(\vartheta) \leq C^q \chi(\theta)$, and then,

$$\begin{aligned} \textstyle\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \leq & \; C^{q\delta} \textstyle\int_{\theta}^{\epsilon} \chi^{\delta}(\theta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ = & \; C^{q\delta}\chi^{\delta}(\theta) \textstyle\int_{\theta}^{\epsilon} \left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \\ = & \; C^{q\delta}\chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta) - \lambda^{\delta}(\epsilon)\right] \\ = & \; C^{q\delta}\chi^{\delta}(\theta)\lambda^{\delta}(\theta). \end{aligned} \tag{49}$$

Consider the function

$$\Lambda(\theta) = C^{q(\delta-1)} \left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta) \left[-\lambda^{\delta}(\vartheta)\right]^{\Delta} \Delta\vartheta \right)^{\frac{1}{\delta}} - \int_{\theta}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta} \Delta\vartheta. \tag{50}$$

By applying the chain rule formula on the term

$$\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta) \left[-\lambda^{\delta}(\vartheta)\right]^{\Delta} \Delta\vartheta \right)^{\frac{1}{\delta}},$$

we have for $\xi \in [\theta, \sigma(\theta)]$ that

$$\begin{aligned} & \left[\left(\textstyle\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \right)^{\frac{1}{\delta}} \right]^{\Delta} \\ & = \tfrac{1}{\delta} \left(\textstyle\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \right)^{\frac{1}{\delta}-1} \chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta}. \end{aligned} \tag{51}$$

Again, by applying the chain rule formula on the terms $\lambda^{\delta}(\theta)$, we obtain

$$\left[\lambda^{\delta}(\theta)\right]^{\Delta} = \delta\lambda^{\delta-1}(\xi)\lambda^{\Delta}(\theta), \tag{52}$$

where $\xi \in [\theta, \sigma(\theta)]$. From (50), we observe that

$$\Lambda^{\Delta}(\theta) = C^{q(\delta-1)} \left[\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta) \left[-\lambda^{\delta}(\vartheta)\right]^{\Delta} \Delta\vartheta \right)^{\frac{1}{\delta}} \right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta). \tag{53}$$

Substituting (51) and (52) into (53), we get

$$\begin{aligned} \Lambda^{\Delta}(\theta) = & \; \tfrac{1}{\delta} C^{q(\delta-1)} \left(\textstyle\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \right)^{\frac{1}{\delta}-1} \chi^{\delta}(\theta)\left[\lambda^{\delta}(\theta)\right]^{\Delta} - \chi(\theta)\lambda^{\Delta}(\theta) \\ = & \; C^{q(\delta-1)}\lambda^{\delta-1}(\xi) \left(\textstyle\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \right)^{\frac{1}{\delta}-1} \chi^{\delta}(\theta)\lambda^{\Delta}(\theta) - \chi(\theta)\lambda^{\Delta}(\theta) \\ = & \; \left[-\lambda^{\Delta}(\theta)\right] \left[-C^{q(\delta-1)}\lambda^{\delta-1}(\xi) \left(\textstyle\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta \right)^{\frac{1}{\delta}-1} \chi^{\delta}(\theta) + \chi(\theta) \right]. \end{aligned} \tag{54}$$

Since $\xi \in [\theta, \sigma(\theta)]$, $0 < \delta < 1$ and λ is a decreasing function, we obtain

$$\begin{aligned} &\lambda^{\delta-1}(\xi)\left(\int_{\xi}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1} \\ &\leq \lambda^{\delta-1}(\sigma(\theta))\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}. \end{aligned} \tag{55}$$

Substituting (55) into (54), we observe that

$$\Lambda^{\Delta}(\theta) \geq \left[-\lambda^{\Delta}(\theta)\right]\left[-C^{q(\delta-1)}\lambda^{\delta-1}(\sigma(\theta))\left(\int_{\theta}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}-1}\chi^{\delta}(\theta) + \chi(\theta)\right]. \tag{56}$$

Substituting (49) into (56), we get

$$\Lambda^{\Delta}(\theta) \geq \left[-\lambda^{\Delta}(\theta)\right]\left[-\lambda^{\delta-1}(\sigma(\theta))\lambda^{1-\delta}(\theta)\chi(\theta) + \chi(\theta)\right]. \tag{57}$$

By using (47) and λ is a decreasing function, we have from (57) that

$$\Lambda^{\Delta}(\theta) \geq 0,$$

and then, the function Λ is increasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$.

Since Λ is an increasing function, we have for $\epsilon > \varepsilon$ that $\Lambda(\epsilon) \geq \Lambda(\varepsilon)$ and then (note that $\Lambda(\varepsilon) = 0$),

$$\int_{\varepsilon}^{\epsilon} \chi(\vartheta)[-\lambda(\vartheta)]^{\Delta}\Delta\vartheta \geq C^{q(\delta-1)}\left(\int_{\varepsilon}^{\epsilon} \chi^{\delta}(\vartheta)\left[-\lambda^{\delta}(\vartheta)\right]^{\Delta}\Delta\vartheta\right)^{\frac{1}{\delta}},$$

which is the desired inequality (48). The proof is completed. $\square$

As a special case of Theorem 4, when $0 < p < q < \infty$ such that $0 < \delta = p/q < 1$, $\chi(\vartheta) = \omega^{q}(\vartheta)$ and $\lambda(\vartheta) = \varpi^{q}(\vartheta)$, we get the following corollary.

Corollary 12. *Assume that $\mathbb{T}$ is a time scale with $\varepsilon, \epsilon \in \mathbb{T}$ and $0 < p < q < \infty$. Furthermore, if ω is C-decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$, $C \geq 1$ and ϖ is decreasing on $[\varepsilon, \epsilon]_{\mathbb{T}}$ with $\varpi(\epsilon) = 0$ such that*

$$\varpi^{q-p}(\theta)[\varpi^{\sigma}(\theta)]^{p-q} \leq 1, \tag{58}$$

then

$$\left(\int_{\varepsilon}^{\epsilon} \omega^{q}(\vartheta)[-\varpi^{q}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{q}} \geq C^{p/q-1}\left(\int_{\varepsilon}^{\epsilon} \omega^{p}(\vartheta)[-\varpi^{p}(\vartheta)]^{\Delta}\Delta\vartheta\right)^{\frac{1}{p}}.$$

Corollary 13. *As a special case of Corollary 12, when $\mathbb{T} = \mathbb{R}$, $\sigma(\theta) = \theta$, we have that (58) holds already with equality and we also get the inequality (6) proved by Pečarić et al.* [2].

Corollary 14. *In Corollary 12, when $\mathbb{T} = \mathbb{N}$, $\sigma(n) = n + 1$, we have that if $\varepsilon, \epsilon \in \mathbb{N}$, $0 < p < q < \infty$, ω is a C-decreasing sequence for $C \geq 1$ and ϖ is decreasing with $\varpi(\epsilon) = 0$, such that*

$$\varpi^{q-p}(n)\varpi^{p-q}(n+1) \leq 1,$$

then

$$\left(\sum_{n=\varepsilon}^{\epsilon} \omega^{q}(n)\Delta[-\varpi^{q}(n)]\right)^{\frac{1}{q}} \geq C^{p/q-1}\left(\sum_{n=\varepsilon}^{\epsilon} \omega^{p}(n)\Delta[-\varpi^{p}(n)]\right)^{\frac{1}{p}}.$$

4. Conclusions and Future Work

In this paper, we establish some new dynamic inequalities involving C-monotonic functions with $C \geq 1$, on time scales. It is known that if $C = 1$, then the 1-decreasing function is decreasing and the 1-increasing function increasing. Thus, our results are special cases when $C = 1$ and give the inequalities involving increasing or decreasing functions. These results can be proved by applying the properties of C-monotonic functions and the chain rule formula on time scales. In the future, we hope to study the dynamic inequalities involving C-monotonic functions via conformable delta fractional calculus on time scales.

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