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New Subclasses of Bi-Univalent Functions with Respect to the Symmetric Points Defined by Bernoulli Polynomials

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Abstract: In this paper, we introduce and investigate new subclasses of bi-univalent functions with respect to the symmetric points in $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by Bernoulli polynomials. We obtain upper bounds for Taylor–Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete–Szegö inequalities $|a_3 - \mu a_2^2|$ for these new subclasses.

Keywords: Fekete–Szegö inequality; Bernoulli polynomial; analytic and bi-univalent functions; subordination; symmetric points

MSC: 30C45; 30C50

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1. Introduction

Let the class of analytic functions in $U = \{z \in \mathbb{C} : |z| < 1\}$, denoted by A, contain all the functions of the type

$$l(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U), \tag{1}$$

which satisfy the usual normalization condition $l(0) = l'(0) - 1 = 0$.

Let S be the subclass of A consisting of all functions $l \in A$, which are also univalent in U. The Koebe one quarter theorem [1] ensures that the image of U under every univalent function $l \in A$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function l has an inverse l^{-1} satisfying

$$l^{-1}(l(z)) = z, (z \in U) \text{ and } l\left(l^{-1}(\omega)\right) = \omega, \ (|\omega| < r_0(l), r_0(l) \geq \frac{1}{4}).$$

If l and l^{-1} are univalent in U, then $l \in A$ is said to be bi-univalent in U, and the class of bi-univalent functions defined in the unit disk U is denoted by Σ. Since $l \in \Sigma$ has the Maclaurin series given by (1), a computation shows that $m = l^{-1}$ has the expansion

$$m(\omega) = l^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 + \cdots. \tag{2}$$

The expression Σ is a non-empty class of functions, as it contains at least the functions

$$l_1(z) = -\frac{z}{1-z}, \quad l_2(z) = \frac{1}{2}\log\frac{1+z}{1-z},$$

with their corresponding inverses

$$l_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \quad l_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1}.$$

In addition, the Koebe function $l(z) = \frac{z}{(1-z)^2} \notin \Sigma$.

The study of analytical and bi-univalent functions is reintroduced in the publication of [2] and is then followed by work such as [3–8]. The initial coefficient constraints have been determined by several authors who have also presented new subclasses of bi-univalent functions (see [2–4,6,9–11]).

Consider α and β to be analytic functions in U. We say that α is subordinate to β, if a Schwarz function w exists that is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$ such that

$$\alpha(z) = \beta(w(z)), \quad (z \in U).$$

This subordination is denoted by $\alpha \prec \beta$ or $\alpha(z) \prec \beta(z)$, $(z \in U)$. Given that β is a univalent function in U, then

$$\alpha(z) \prec \beta(z) \Leftrightarrow \alpha(0) = \beta(0) \quad \text{and} \quad \alpha(U) \subset \beta(U).$$

Using Loewner's technique, the Fekete–Szegö problem for the coefficients of $l \in S$ in [6] is

$$\left| a_3 - \mu a_2^2 \right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \text{ for } 0 \leq \mu < 1.$$

The elementary inequality $\left|a_3 - a_2^2\right| \leq 1$ is obtained as $\mu \to 1$. The coefficient functional

$$F_\mu(l) = a_3 - \mu a_2^2$$

on the normalized analytic functions l in the open unit disk U also has a significant impact on geometric function theory. The Fekete–Szegö problem is known as the maximization problem for functional $\left|F_\mu(l)\right|$.

Researchers were concerned about several classes of univalent functions (see [12–15]) due to the Fekete–Szegö problem, proposed in 1933 ([16]); therefore, it stands to reason that similar inequalities were also discovered for bi-univalent functions, and fairly recent publications can be cited to back up the claim that the subject still yields intriguing findings [17–19].

Because of their importance in probability theory, mathematical statistics, mathematical physics, and engineering, orthogonal polynomials have been the subject of substantial research in recent years from a variety of angles. The classical orthogonal polynomials are the orthogonal polynomials that are most commonly used in applications (Hermite polynomials, Laguerre polynomials, Jacobi polynomials, and Bernoulli). We point out [17,18,20–24] as more recent examples of the relationship between geometric function theory and classical orthogonal polynomials.

Fractional calculus, a classical branch of mathematical analysis whose foundations were laid by Liouville in an 1832 paper and is currently a very active research field [25], is one of many special functions that are studied. This branch of mathematics is known as the Bernoulli polynomials, named after Jacob Bernoulli (1654–1705). A novel approximation method based on orthonormal Bernoulli's polynomials has been developed to solve fractional order differential equations of the Lane–Emden type [26], whereas in [27–29], Bernoulli polynomials are utilized to numerically resolve Fredholm fractional integro-differential equations with right-sided Caputo derivatives.

The Bernoulli polynomials $B_n(x)$ are often defined (see, e.g., [30]) using the generating function:

$$F(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \; |t| < 2\pi, \tag{3}$$

where $B_n(x)$ are polynomials in x, for each nonnegative integer n.

The Bernoulli polynomials are easily computed by recursion since

$$\sum_{j=0}^{n-1}\binom{n}{j}B_j(x) = nx^{n-1}, n = 2,3,\cdots. \tag{4}$$

The initial few polynomials of Bernoulli are

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6},\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \cdots. \tag{5}$$

Sakaguchi [31] introduced the class S_s^* of functions starlike with respect to symmetric points, which consists of functions $l \in S$ satisfying the condition

$$Re\left\{\frac{zl'(z)}{l(z) - l(-z)}\right\} > 0, \quad (z \in U).$$

In addition, Wang et al. [32] introduced the class C_s of functions convex with respect to symmetric points, which consists of functions $l \in S$ satisfying the condition

$$Re\left\{\frac{[zl'(z)]'}{[l(z) - l(-z)]'}\right\} > 0, \quad (z \in U).$$

In this paper, we consider two subclasses of Σ: the class $S_s^{\Sigma}(x)$ of functions bi-starlike with respect to the symmetric points and the relative class $C_s^{\Sigma}(x)$ of functions bi-convex with respect to the symmetric points associated with Bernoulli polynomials. The definitions are as follows:

Definition 1. *$l \in S_s^{\Sigma}(x)$, if the next subordinations hold:*

$$\frac{2zl'(z)}{l(-z) - l(z)} \prec F(x,z), \tag{6}$$

and

$$\frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} \prec F(x,\omega), \tag{7}$$

where $z, \omega \in U$, $F(x,z)$ is given by (3), and $m = l^{-1}$ is given by (2).

Definition 2. *$l \in C_s^{\Sigma}(x)$, if the following subordinations hold:*

$$\frac{2[zl'(z)]'}{[l(z) - l(-z)]'} \prec F(x,z), \tag{8}$$

and

$$\frac{2[\omega m'(\omega)]'}{[m(\omega) - m(-\omega)]'} \prec F(x,\omega), \tag{9}$$

where $z, \omega \in U$, $F(x,z)$ is given by (3), and $m = l^{-1}$ is given by (2).

Lemma 1 ([33], p. 172)**.** *Suppose that $c(z) = \sum_{n=1}^{\infty} c_n z^n$, $|c(z)| < 1$, $z \in U$, is an analytic function in U. Then,*

$$|c_1| \le 1,\ |c_n| \le 1 - |c_1|^2,\ n = 2,3,\cdots.$$

2. Coefficients Estimates for the Class $S_s^{\Sigma}(x)$

We obtain upper bounds of $|a_2|$ and $|a_3|$ for the functions belonging to the class $S_s^{\Sigma}(x)$.

Theorem 1. *If $l \in S_s^{\Sigma}(x)$, then*

$$|a_2| \leq |B_1(x)|\sqrt{6|B_1(x)|}, \tag{10}$$

and

$$|a_3| \leq \frac{B_1(x)}{2} + \frac{[B_1(x)]^2}{4}. \tag{11}$$

Proof. Let $l \in S_s^{\Sigma}(x)$ and $m = l^{-1}$. From definition in (6) and (7), we have

$$\frac{2l'(z)z}{l(z) - l(-z)} = F(x, \varphi(z)), \tag{12}$$

and

$$\frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} = F(x, \chi(\omega)), \tag{13}$$

where φ and χ are analytic functions in U given by

$$\varphi(z) = r_1 z + r_2 z^2 + \cdots, \tag{14}$$

$$\chi(\omega) = s_1\omega + s_2\omega^2 + \cdots, \tag{15}$$

and $\varphi(0) = \chi(0) = 0$, and $|\varphi(z)| < 1$, $|\chi(\omega)| < 1$, $z, \omega \in U$.

As a result of Lemma 1,

$$|r_k| \leq 1 \text{ and } |s_k| \leq 1,\ k \in \mathbb{N}. \tag{16}$$

If we replace (14) and (15) in (12) and (13), respectively, we obtain

$$\frac{2zl'(z)}{l(z) - l(-z)} = B_0(x) + B_1(x)\varphi(z) + \frac{B_2(x)}{2!}\varphi^2(z) + \cdots, \tag{17}$$

and

$$\frac{2\omega m'(\omega)}{m(\omega) - m(-\omega)} = B_0(x) + B_1(x)\chi(\omega) + \frac{B_2(x)}{2!}\chi^2(\omega) + \cdots. \tag{18}$$

In view of (1) and (2), from (17) and (18), we obtain

$$1 + 2a_2 z + 2a_3 z^2 + \cdots = 1 + B_1(x) r_1 z + \left[B_1(x) r_2 + \frac{B_2(x)}{2!} r_1^2\right] z^2 + \cdots$$

and

$$1 - 2a_2\omega + (4a_2^2 - 2a_3)\omega^2 + \cdots = 1 + B_1(x) s_1\omega + \left[B_1(x) s_2 + \frac{B_2(x)}{2!} s_1^2\right]\omega^2 + \cdots,$$

which yields the following relations:

$$2a_2 = B_1(x) r_1, \tag{19}$$

$$2a_3 = B_1(x) r_2 + \frac{B_2(x)}{2!} r_1^2, \tag{20}$$

and

$$-2a_2 = B_1(x) s_1, \tag{21}$$

$$4a_2^2 - 2a_3 = B_1(x) s_2 + \frac{B_2(x)}{2!} s_1^2. \tag{22}$$

From (19) and (21), it follows that

$$r_1 = -s_1, \tag{23}$$

and

$$8a_2^2 = [B_1(x)]^2\left(r_1^2 + s_1^2\right)$$

$$a_2^2 = \frac{[B_1(x)]^2(r_1^2 + s_1^2)}{8}. \tag{24}$$

Adding (20) and (22), using (24), we obtain

$$a_2^2 = \frac{[B_1(x)]^3(r_2 + s_2)}{4([B_1(x)]^2 - B_2(x))}. \tag{25}$$

Using relation (5), from (16) for r_2 and s_2, we get (10).

Using (23) and (24), by subtracting (22) from relation (20), we get

$$\begin{aligned} a_3 &= \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2!}(r_1^2 - s_1^2)}{4} + a_2^2 \\ &= \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2!}(r_1^2 - s_1^2)}{4} + \frac{[B_1(x)]^2(r_1^2 + s_1^2)}{8}. \end{aligned} \tag{26}$$

Once again applying (23) and using (5), for the coefficients r_1, s_1, r_2, s_2, we deduce (11). □

3. The Fekete–Szegö Problem for the Function Class $S_s^\Sigma(x)$

We obtain the Fekete–Szegö inequality for the class $S_s^\Sigma(x)$ due to the result of Zaprawa; see [19].

Theorem 2. *If l given by (1) is in the class $S_s^\Sigma(x)$ where $\mu \in \mathbb{R}$, then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1(x)}{2}, & \text{if} \quad |h(\mu)| \leq \frac{1}{4}, \\ 2B_1(x)|h(\mu)|, & \text{if} \quad |h(\mu)| \geq \frac{1}{4}, \end{cases}$$

where

$$h(\mu) = 3(1-\mu)[B_1(x)]^2.$$

Proof. If $l \in S_s^\Sigma(x)$ is given by (1), from (25) and (26), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{B_1(x)(r_2 - s_2)}{4} + (1-\mu)a_2^2 \\ &= \frac{B_1(x)(r_2 - s_2)}{4} + \frac{(1-\mu)[B_1(x)]^3(r_2 + s_2)}{4([B_1(x)]^2 - B_2(x))} \\ &= B_1(x)\left[\frac{r_2}{4} - \frac{s_2}{4} + \frac{(1-\mu)[B_1(x)]^2 r_2}{4([B_1(x)]^2 - B_2(x))} + \frac{(1-\mu)[B_1(x)]^2 s_2}{4([B_1(x)]^2 - B_2(x))}\right] \\ &= B_1(x)\left[\left(h(\mu) + \frac{1}{4}\right)r_2 + \left(h(\mu) - \frac{1}{4}\right)s_2\right], \end{aligned}$$

where

$$h(\mu) = \frac{(1-\mu)[B_1(x)]^2}{4([B_1(x)]^2 - B_2(x))}$$

Now, by using (5)

$$a_3 - \mu a_2^2 = \left(x - \frac{1}{2}\right)\left[\left(h(\mu) + \frac{1}{4}\right)r_2 + \left(h(\mu) - \frac{1}{4}\right)s_2\right],$$

where

$$h(\mu) = 3(1-\mu)\left(x - \frac{1}{2}\right)^2.$$

Therefore, given (5) and (16), we conclude that the necessary inequality holds. □

4. Coefficients Estimates for the Class $C_s^{\Sigma}(x)$

We will obtain upper bounds of $|a_2|$ and $|a_3|$ for the functions belonging to a class $C_S^{\Sigma}(x)$.

Theorem 3. *If $l \in C_s^{\Sigma}(x)$, then*

$$|a_2| \leq \frac{|B_1(x)|\sqrt{|B_1(x)|}}{\sqrt{\left|6[B_1(x)]^2 - 8B_2(x)\right|}}, \tag{27}$$

and

$$|a_3| \leq \frac{B_1(x)}{6} + \frac{[B_1(x)]^2}{16}. \tag{28}$$

Proof. Let $l \in C_s^{\Sigma}(x)$ and $m = l^{-1}$. From (8) and (9), we get

$$\frac{2[zl'(z)]'}{[l(z) - l(-z)]'} = F(x, \varphi(z)), \tag{29}$$

and

$$\frac{2[\omega m'(\omega)]'}{[m(\omega) - m(-\omega)]'} = F(x, \chi(\omega)) \tag{30}$$

where φ and χ are analytic functions in U given by

$$\varphi(z) = r_1 z + r_2 z^2 + \cdots, \tag{31}$$

$$\chi(\omega) = s_1\omega + s_2\omega^2 + \cdots, \tag{32}$$

where $\varphi(0) = \chi(0) = 0$, and $|\varphi(z)| < 1, |\chi(\omega)| < 1, z, \omega \in U$.

As a result of Lemma 1,

$$|r_k| \leq 1 \text{ and } |s_k| \leq 1,\ k \in \mathbb{N}. \tag{33}$$

If we replace (31) and (32) in (29) and (30), respectively, we obtain

$$\frac{2[zl'(z)]'}{[l(z) - l(-z)]'} = B_0(x) + B_1(x)\varphi(z) + \frac{B_2(x)}{2!}\varphi^2(z) + \cdots, \tag{34}$$

and

$$\frac{2[\omega m'(\omega)]'}{[m(\omega) - m(-\omega)]'} = B_0(x) + B_1(x)\chi(\omega) + \frac{B_2(x)}{2!}\chi^2(\omega) + \cdots. \tag{35}$$

In view of (1) and (2), from (34) and (35), we obtain

$$1 + 4a_2 z + 6a_3 z^2 + \cdots = 1 + B_1(x)r_1 z + \left[B_1(x)r_2 + \frac{B_2(x)}{2!}{r_1}^2\right]z^2 + \cdots$$

and

$$1 - 4a_2\omega + \left(12a_2{}^2 - 6a_3\right)\omega^2 + \cdots = 1 + B_1(x)s_1\omega + \left[B_1(x)s_2 + \frac{B_2(x)}{2!}s_1{}^2\right]\omega^2 + \cdots,$$

which yields the following relations:

$$4a_2 = B_1(x)r_1, \tag{36}$$

$$6a_3 = B_1(x)r_2 + \frac{B_2(x)}{2!}r_1^2, \tag{37}$$

and

$$-4a_2 = B_1(x)s_1, \tag{38}$$

$$12a_2^2 - 6a_3 = B_1(x)s_2 + \frac{B_2(x)}{2!}s_1^2. \tag{39}$$

From (36) and (38), it follows that

$$r_1 = -s_1, \tag{40}$$

and

$$32a_2^2 = [B_1(x)]^2\left(r_1^2 + s_1^2\right)$$

$$a_2^2 = \frac{[B_1(x)]^2(r_1^2 + s_1^2)}{32}. \tag{41}$$

Adding (37) and (39), using (41), we obtain

$$a_2^2 = \frac{[B_1(x)]^3(r_2 + s_2)}{4(3[B_1(x)]^2 - 4B_2(x))}. \tag{42}$$

Using relation (5), from (33) for r_2 and s_2, we get (27). Using (40) and (41), by subtracting (39) from relation (37), we get

$$\begin{aligned} a_3 &= \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2!}(r_1^2 - s_1^2)}{12} + a_2^2 \\ &= \frac{B_1(x)(r_2 - s_2) + \frac{B_2(x)}{2!}(r_1^2 - s_1^2)}{12} + \frac{[B_1(x)]^2(r_1^2 + s_1^2)}{32}. \end{aligned} \tag{43}$$

Once again applying (40) and using (5), for the coefficients r_1, s_1, r_2, s_2, we deduce (28). □

5. The Fekete–Szegö Problem for the Function Class $C_s^\Sigma(x)$

We obtain the Fekete–Szegö inequality for the class $C_s^\Sigma(x)$ due to the result of Zaprawa; see [19].

Theorem 4. *If l given by (1) is in the class $C_s^\Sigma(x)$ where $\mu \in \mathbb{R}$, then, we have*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1(x)}{6}, & \text{if } \ |h(\mu)| \le \frac{1}{12}, \\ 2B_1(x)|h(\mu)|, & \text{if } \ |h(\mu)| \ge \frac{1}{12}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)[B_1(x)]^2}{4(3[B_1(x)]^2 - 4B_2(x))}.$$

Proof. If $l \in C_s^{\Sigma}(x)$ is given by (1), from (42) and (43), we have

$$\begin{aligned}
a_3 - \mu a_2^2 &= \frac{B_1(x)(r_2 - s_2)}{12} + (1-\mu)a_2^2 \\
&= \frac{B_1(x)(r_2 - s_2)}{12} + \frac{(1-\mu)[B_1(x)]^3(r_2+s_2)}{4(3B_1(x)^2 - 4B_2(x))} \\
&= B_1(x)\left[\frac{r_2 - s_2}{12} + \frac{(1-\mu)[B_1(x)]^2 r_2}{4(3[B_1(x)]^2 - 4B_2(x))} + \frac{(1-\mu)[B_1(x)]^2 s_2}{4(3[B_1(x)]^2 - 4B_2(x))}\right] \\
&= B_1(x)\left[\left(h(\mu) + \frac{1}{12}\right)r_2 + \left(h(\mu) - \frac{1}{12}\right)s_2\right],
\end{aligned}$$

where

$$h(\mu) = \frac{(1-\mu)[B_1(x)]^2}{4(3[B_1(x)]^2 - 4B_2(x))}.$$

Now, by using (5)

$$a_3 - \mu a_2^2 = \left(x - \frac{1}{2}\right)\left[\left(h(\mu) + \frac{1}{12}\right)r_2 + \left(h(\mu) - \frac{1}{12}\right)s_2\right],$$

where

$$h(\mu) = \frac{(1-\mu)\left[x - \frac{1}{2}\right]^2}{4(3\left(x - \frac{1}{2}\right)^2 - 4(x^2 - x + \frac{1}{6}))}.$$

Therefore, given (5) and (33), we conclude that the required inequality holds. □

6. Conclusions

We introduce and investigate new subclasses of bi-univalent functions in U associated with Bernoulli polynomials and satisfying subordination conditions. Moreover, we obtain upper bounds for the initial Taylor–Maclaurin coeffcients $|a_2|$, $|a_3|$ and Fekete–Szegö problem $\left|a_3 - \mu a_2^2\right|$ for functions in these subclasses.

The approach employed here has also been extended to generate new bi-univalent function subfamilies using the other special functions. The researchers may carry out the linked outcomes in practice.

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