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Application of the Averaging Method to the Optimal Control Problem of Non-Linear Differential Inclusions on the Finite Interval

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Abstract: In this paper, we use the averaging method to find an approximate solution for the optimal control of non-linear differential inclusions with fast-oscillating coefficients on a finite time interval.

Keywords: non-linear differential inclusion; optimal control; averaging method; approximate solution

MSC: 49J21; 34A45

Citation: Zhuk, T.; Kasimova, N.; Ryzhov, A. Application of the Averaging Method to the Optimal Control Problem of Non-Linear Differential Inclusions on the Finite Interval. *Axioms* **2022**, *11*, 653. https://doi.org/10.3390/axioms11110653

Academic Editor: Francisco Javier Fernández Fernández, Alberto Cabada and Miguel Ernesto Vazquez-Mendez

Received: 25 October 2022
Accepted: 15 November 2022
Published: 17 November 2022

1. Introduction

It is known that the averaging method is one of the most effective tools for solving various optimal control problems for differential equations [1–4] as well as for differential inclusions with fast-oscillating coefficients [5–7]. Of the many published papers in which similar problems are considered (e.g., minimax, robust, and adaptive control), we mention [8–10]. The Krasnoselski–Krein theorem [2] and its multi-valued analogue [11] play an essential role for the investigation of the above-mentioned problems. When dealing with multi-valued mappings, one faces specific problems; nevertheless, the application of the well-developed averaging method for the optimal control problems is possible in this case.

In the present paper, we consider the optimal control problem of a non-linear system of differential inclusions with fast-oscillating parameters. First, we prove the existence of solutions for the initial perturbed optimal control problem and the corresponding problem with averaged coefficients. Then, we prove that the optimal control of the problem with averaging coefficients can be considered as "approximately" optimal for the initial perturbed one.

2. Statement of the Problem

Let us consider an optimal control problem as follows.

$$\begin{cases} \dot{x}(t) \in X(\frac{t}{\varepsilon}, x(t), u(t)), t \in (0,T), \\ x(0) = x_0, u(\cdot) \in U, \\ J[x,u] = \int\limits_0^T L(t, x(t), u(t))dt + \Phi(x(T)) \to \inf. \end{cases} \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, $x : [0,T] \to \mathbb{R}$ is an unknown phase variable, $u : [0,T] \to \mathbb{R}^m$ is an unknown control function, $X : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathrm{conv}(\mathbb{R}^n)$ is a multi-valued function, and $U \subset L^2(0,T)$ is a fixed set.

Assume that uniformly with respect to x for every $u \in \mathbb{R}^m$ that we have the following:

$$\operatorname{dist}_H\left(\frac{1}{s}\int_0^s X(\tau, x, u)d\tau, Y(x, u)\right) \to 0, s \to \infty, \tag{2}$$

where the limits for multi-valued functions are considered in the sense of [12,13], dist_H is the Hausdorff metric, $Y : \mathbb{R}^n \times \mathbb{R}^m \to \operatorname{conv}(\mathbb{R}^n)$, and the integral of multi-valued function is considered in the sense of Aumann [14]. We consider the following problem with an averaged right-hand side.

$$\begin{cases} \dot{y}(t) \in Y(y(t), u(t)), \\ y(0) = x_0, u(\cdot) \in U, \\ J[y, u] = \int_0^T L(t, y(t), u(t))dt + \Phi(y(T)) \to \inf. \end{cases} \tag{3}$$

Under the natural assumptions on X, L, Φ, and U, we will show that problems (1) and (3) have solutions $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ and $\{\overline{y}, \overline{u}\}$, respectively:

$$\overline{J}_{\varepsilon_n} \to \overline{J}, \varepsilon_n \to 0,$$

where $\overline{J}_{\varepsilon_n} := J[\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}]$, $\overline{J} := J[\overline{y}, \overline{u}]$, and up to a subsequence.

$$\overline{u}_{\varepsilon_n} \to \overline{u} \text{ in } L^2(0, T),$$

$$\overline{x}_{\varepsilon_n} \to \overline{y} \text{ in } \mathbb{C}([0, T]).$$

In what follows, we consider the problem of finding an approximate solution of (1) by transitions to the problem with averaged coefficients. We note that such transitions can essentially simplify the problem.

3. Assumptions and Notations

Let $Q = \{t \geq 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m\}$ and assume that the following assumptions hold.

Assumption 1. *Mapping $t, x, u \mapsto X(t, x, u)$ is continuous in Hausdorff metric.*

Assumption 2. *Multi-valued function $X(t, x, u)$ satisfies the growth property: $\exists M > 0$ such that*

$$||X(t, x, u)||_+ \leq M(1 + ||x||)\ \forall (t, x, u) \in Q,$$

where $||X(t, x, u)||_+ = \sup_{\xi \in X(t,x,u)} ||\xi||$, $||\xi||$ is the Euclidian norm of $\xi \in \mathbb{R}^n$.

Assumption 3. *Multi-valued function $X(t, x, u)$ satisfies the Lipschitz condition: $\exists \lambda > 0$ such that*

$$dist_H(X(t, x_1, u_1), X(t, x_2, u_2)) \leq \lambda(||x_1 - x_2|| + ||u_1 - u_2||)$$

Assumption 4. *$(x, u) \mapsto L(t, x, u)$ is a continuous mapping, moreover, function $t \mapsto L(t, x, u)$ is measurable $\forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$, and*

$$|L(t, x, u)| \leq c(t)(1 + ||u||),$$

where $c(\cdot) \in L^2(0, T)$ is a given function.

Assumption 5. *$\Phi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.*

Assumption 6. *$U \subset L^2(0, T)$ is a compact set.*

Remark 1. *Under the Assumptions 1–3 for all $u \in L^2(0,T)$, the Cauchy problem*

$$\begin{cases} \dot{x} \in X\left(\frac{t}{\varepsilon}, x, u\right), t \in (0,T) \\ x(0) = x_0 \end{cases} \quad (4)$$

has a solution (e.g., [15]); that is, there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ satisfying inclusion (4) a.e.

Remark 2. *Under condition (2), the multi-valued mapping Y satisfies Assumptions 2 and 3; hence, $\forall u \in L^2(0,T)$, the Cauchy problem*

$$\begin{cases} \dot{y} \in Y(y,u), t \in (0,T) \\ y(0) = x_0 \end{cases} \quad (5)$$

has a solution.

We will consider the next multi-valued analogue of the Krasnoselsky–Krein theorem.

Theorem 1 ([2,5,11,16])**.** *Suppose the following conditions are fulfilled for the differential inclusion*

$$\dot{x} \in F(t,x,\lambda), \quad (6)$$

with multi-valued mapping $F(t,x,\lambda)$ taking values in $conv(\mathbb{R}^n)$ (that is, the subspace from $conv(\mathbb{R}^n)$, which consists of convex sets), defined for $0 \le t \le T$; $x \in D$, D is a bounded domain in $\mathbb{R}^n$; $\lambda \in \Lambda$, where Λ is a set of values for parameter λ for which $\lambda_0 \in \Lambda$ is the limit point.

(1) Multi-valued mapping $F(t,x,\lambda)$ is uniformly bounded, continuous on t, uniformly continuous on x with respect to t, and λ: $\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0 : \forall t \in [0,T], x \in D, x' \in D$ and $\lambda \in \Lambda$; we have

$$dist_H\big(F(t,x',\lambda) - F(t,x,\lambda)\big) < \varepsilon$$

once $|x' - x| < \delta$.

(2) Multi-valued mapping $F(t,x,\lambda)$ is integrally continuous on λ at point λ_0; that is, for $0 \le t_1 \le t_2 \le T$ and for any $x \in D$, we have the following:

$$dist_H\left(\int_{t_1}^{t_2} F(s,x,\lambda)ds, \int_{t_1}^{t_2} F(s,x,\lambda_0)ds\right) \to 0, \lambda \to \lambda_0,$$

where we consider integrals in the sense of Aumann [14].

(3) Solutions $x(t,\lambda_0)$ of the inclusion

$$\dot{x} \in F(t,x,\lambda_0), \quad (7)$$

satisfying the condition $x(0,\lambda_0) = x_0 \in D^1 \subset D$ are defined for $0 \le t \le T$ and belong to domain D together with some ρ-neighborhood.

Then, for each $\eta > 0$, there is a neighborhood $U(\lambda_0)$ of point λ_0 such that for $\lambda \in U(\lambda_0)$ and for an arbitrary solution $x(t,\lambda)$ of the inclusion (6) defined on $0 \le t \le T$ and satisfying the initial condition $x(0,\lambda) = x_0$, there exists a solution $x(t,\lambda_0)$ of inclusion (7), and inequality $||x(t,\lambda) - x(t,\lambda_0)|| < \eta, 0 \le t \le T$, holds.

Remark 3. *The concept of an integral continuity plays a key role in the investigation of the considered optimal control problem using an averaging method. It is known [17] that (2) is equivalent to the integral continuity.*

4. Main Results

Theorem 2. *Under Assumptions 1–6, problem (1) (resp. the problem (3)) has the solution* $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ *(resp.* $\{\overline{y}, \overline{u}\}$*).*

Proof. Fix $\varepsilon > 0$ and suppress it in what follows. Under the conditions on L and Φ, the cost functional in (1) reaches its finite extremum. Now deduce a priory estimate for $x(t)$. Since x is an absolutely continuous function, then $t \mapsto ||x(t)||$ is absolutely continuous too and

$$\frac{d}{dt}||x(t)|| \leq ||\dot{x}(t)|| \text{ a.e.}$$

Then,

$$\frac{d}{dt}||x(t)|| \leq ||\dot{x}(t)|| \leq ||X(t,x,u)||_+ \leq M(1+||x||)$$

and

$$||x(t)|| \leq ||x(0)|| + \int_0^t M(1+||x||)ds = ||x(0)|| + MT + \int_0^t M||x||ds$$

Taking into account Gronwall's inequality, we have

$$||x(t)|| \leq (||x(0)|| + MT)e^{\int_0^t Mds} = (||x(0)|| + MT)e^{Mt} \leq (||x(0)|| + MT)e^{MT} \tag{8}$$

Let $\{x_n, u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for problem (1) and $J(x_n, u_n,) \leq \overline{J} + \frac{1}{n}$. Due to (8), we have the uniform boundedness of sequence $\{x_n\}_{n\in\mathbb{N}}$ on every finite interval $[0, T]$, i.e., $\exists L > 0$.

$$\sup_{t\in[0,T]} ||x_n(t)|| \leq L, t \in [0,T]$$

Moreover,

$$\sup_{t\in[0,T]} ||\dot{x}_n(t)|| \leq L, t \in [0,T] \tag{9}$$

and

$$||x_n(t_2) - x_n(t_1)|| \leq \int_{t_1}^{t_2} M(1+L)ds = M(1+L)(t_2 - t_1)$$

Thus, sequence $\{x_n\}_{n\in\mathbb{N}}$ is precompact in $\mathbb{C}([0,T])$. Due to the Arzelà–Ascoli theorem, $x_n \to \overline{x}$ in $C([0,T])$ up to a subsequence.

From [13] and (9), we deduce that $\overline{x}$ is absolutely continuous and $\dot{x}_n \to \dot{\overline{x}}$ $*$-weakly as $n \to \infty$ in $L^\infty(0,T)$. Since $\forall\varepsilon > 0$ for a.e. t, there exists n_0 such that $\forall n \geq n_0$

$$\lambda(||x_n(t) - \overline{x}(t)|| + ||u_n(t) - \overline{u}(t)||) < \varepsilon,$$

Then, by the Assumption 3, we have

$$\dot{x}_n(t) \in X\left(\frac{t}{\varepsilon}, x_n(t), u_n(t)\right) \subset O_\varepsilon\left(X\left(\frac{t}{\varepsilon}, x_n(t), u_n(t)\right)\right).$$

Taking into account the convergence theorem ([18], p. 60) for a.e. t, we have

$$\dot{\overline{x}}(t) \in X\left(\frac{t}{\varepsilon}, \overline{x}(t), \overline{u}(t)\right).$$

By Assumption 6, we obtain convergence $u_n \to \overline{u}, n \to \infty$ in $L^2[0,T]$ up to a subsequence.

Now we will show that $\{\overline{x},\overline{u}\}$ is the solution of (1). Since $u_n(t) \to \overline{u}(t)$ and $x_n(t) \to \overline{x}(t), n \to \infty$ a.e., by Assumption 4, we obtain

$$L(t,x_n(t),u_n(t)) \to L(t,\overline{x}(t),\overline{u}(t)) \text{ a.e.}, n \to \infty,$$

and $\{L(t,x_n,u_n)\}$ is bounded in $L^2(0,T)$. Therefore, by Lemma 1.3 in [19], we have it that $L(t,x_n,u_n) \to L(t,\overline{x},\overline{u})$ is weak in $L^2(0,T)$ for $n \to \infty$.

Hence,

$$\int_0^T L(t,x_n(t),u_n(t))dt \to \int_0^T L(t,\overline{x}(t),\overline{u}(t))dt, n \to \infty,$$

With convergence $\Phi(x_n(T)) \to \Phi(\overline{x}(T))$, we have $\lim_{n\to\infty} J[x_n,u_n] = J[\overline{x},\overline{u}] = \overline{J}$; therefore, $\{\overline{x},\overline{u}\}$ is the solution of (1). □

Consider the following assumption.

Assumption 7. *Suppose that for all $u(\cdot) \in U$, problem (5) has a unique solution.*

Theorem 3. *Suppose Assumptions 1–6 and (2) hold. Under Assumption 7,*

$$\overline{J}_{\varepsilon_n} = J[\overline{x}_{\varepsilon_n},\overline{u}_{\varepsilon_n}] \to \overline{J} := J[\overline{y},\overline{u}], \text{ as } \varepsilon_n \to 0,$$

and up to a subsequence

$$\begin{aligned} &\overline{u}_{\varepsilon_n} \to \overline{u} \text{ in } L^2(0,T), \varepsilon_n \to 0, \\ &\overline{x}_{\varepsilon_n} \to \overline{y} \text{ in } C(0,T), \varepsilon_n \to 0, \end{aligned}$$

where $\{\overline{x}_{\varepsilon_n},\overline{u}_{\varepsilon_n}\}$ is the solution of (1), and $\{\overline{y},\overline{u}\}$ is the solution of (3).

Proof. (1) First, we prove that if $u_n \to \hat{u}$ in $L^2(0,T)$, x_n is the solution of (4) with $\varepsilon = \varepsilon_n$, $u = u_n$; then,

$$x_n \to \hat{y} \text{ in } C([0,T]), \tag{10}$$

where $\hat{y}$ is the solution of (5) with $u = \hat{u}$.

Let $\hat{y}$ be the unique solution of (5) with control $u = \hat{u}$. Then,

$$\begin{aligned} &||x_n(t) - \hat{y}(t)|| \le \text{dist}_H\left(\int_0^t X\left(\tfrac{s}{\varepsilon_n}, x_n(s), u_n(s)\right)ds, \int_0^t Y(\hat{y}(s),\hat{u}(s))ds\right) \\ &\le \ \text{dist}_H\left(\int_0^t X\left(\tfrac{s}{\varepsilon_n}, x_n(s), u_n(s)\right)ds, \int_0^t X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(s), \hat{u}(s)\right)ds\right) \\ &+ \text{dist}_H\left(\int_0^t X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(s), \hat{u}(s)\right)ds, \int_0^t Y(s,\hat{y}(s),\hat{u}(s))ds\right) \\ &\le \lambda \int_0^t (||x_n(s) - \hat{y}(s)|| + ||u_n(s) - \hat{u}(s)||)ds + I_n, \end{aligned}$$

where

$$I_n = \text{dist}_H\left(\int_0^t X\left(\frac{s}{\varepsilon_n}, \hat{y}(s), \hat{u}(s)\right)ds, \int_0^t Y(\hat{y}(s),\hat{u}(s))ds\right).$$

It is known that every function in $L^2(0,T)$ can be approximated with continuous functions in L^2-norm, and any continuous function can be approximated by a piecewise constant function in the continuous norm. Then, for any $\eta > 0$, let $\hat{\hat{u}} \in C([0,T])$ be such that the following is the case.

$$||\hat{u} - \hat{u}||_{L^2(0,T)} \le \frac{\eta}{3}.$$

Denote $\underline{u}(t) = \sum_{k=1}^{N} \alpha_k \cdot \chi_{[t_k,t_{k+1}]}(t)$ (N depends only on η) such that

$$\sup_{t\in[0,T]} ||\hat{u}(u) - \underline{u}(t)|| \le \frac{\eta}{3},$$

then

$$\begin{aligned} I_n &\le \lambda \int_0^t ||\hat{u} - \hat{u}||ds + \lambda \int_0^t ||\hat{u} - \underline{u}||ds \\ &+ \mathrm{dist}_H\left(\int_0^t X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(s), \underline{u}(s)\right), \int_0^t Y(\hat{y}(s), \underline{u}(s))ds\right). \end{aligned}$$

Let $\tau_i = \frac{T\cdot i}{m}, i = \overline{0,m}$ and choose an m that is large enough so that $\forall k \in \overline{1,N}$ at least one of $\{\tau_i\}$ belongs to $[t_k, t_{k+1})$. By joining sets $\{t_k\}$ and $\{\tau_i\}$ and denoting the resulting set as $\{t_i\}_{i=0}^{\overline{N}}$ with $\overline{N} \le N + m$, we obtain the following.

$$|t_{i+1} - t_i| \le \frac{1}{m},\ \underline{u}(t) = \sum_{i=0}^{\overline{N}} \alpha_i \cdot \chi_{[t_i,t_{i+1}]}(t).$$

Then,

$$\begin{aligned} I_n &\le \lambda \cdot T^{1/2}\tfrac{\eta}{3} + \lambda T \cdot \tfrac{\eta}{3} \\ &+ \textstyle\sum_{i=0}^{\overline{N}} \mathrm{dist}_H\left(\int_{t_i}^{t_{i+1}} X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(s), \alpha_i\right), \int_{t_i}^{t_{i+1}} X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(t_i), \alpha_i\right)\right) \\ &+ \textstyle\sum_{i=0}^{\overline{N}} \mathrm{dist}_H\left(\int_{t_i}^{t_{i+1}} X\left(\tfrac{s}{\varepsilon_n}, \hat{y}(t_i), \alpha_i\right), \int_{t_i}^{t_{i+1}} Y(\hat{y}(t_i), \alpha_i)\right) \\ &+ \textstyle\sum_{i=0}^{\overline{N}} \int_{t_i}^{t_{i+1}} ||Y(\hat{y}(t_i), \alpha_i) - Y(\hat{y}(s), \alpha_i)||ds \\ &= \lambda T^{1/2} \cdot \tfrac{\eta}{3} + \lambda T\tfrac{\eta}{3} + I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned}$$

Now, we derive the upper bound for $I_n^{(1)}$:

$$I_n^{(1)} \le \sum_{i=0}^{\overline{N}} \lambda \int_{t_i}^{t_{i+1}} ||\hat{y}(s) - \hat{y}(t_i)||ds.$$

Since

$$\hat{y}(s) - \hat{y}(t_i) = \int_{t_i}^{s} Y(\hat{y}(\tau), \hat{u}(\tau))d\tau$$

and taking into account the boundedness of $\hat{y}$, we have the boundedness of the multi-valued function, Y, with constant C; therefore, we obtain

$$||\hat{y}(s) - \hat{y}(t_i)|| \le C|t_{i+1} - t_i| \le \frac{C}{m}.$$

Finally,

$$I_n^{(1)} \le (N + m)\frac{C}{m^2}$$

Using similar arguments, we derive the same upper bound for $I_n^{(3)}$.

Now, taking into account (2) and Theorem 1, we chose n_0 such that $\forall n \ge n_0$

$$I_n^{(2)} \le \overline{N} \cdot \frac{\eta}{3\cdot\overline{N}} = \frac{\eta}{3}.$$

Therefore,

$$I_n \leq \lambda \cdot T^{1/2}\frac{\eta}{3} + \lambda T\frac{\eta}{3} + \frac{2C(N+m)}{m^2} + \frac{\eta}{3}.$$

By choosing m such that $\frac{2C(N+m)}{m^2} < \frac{\eta}{3}$, we obtain $I_n \to 0, n \to \infty$, which proves convergence (10).

(2) Let $\varepsilon_n \to 0, n \to \infty$, and $\overline{x}_n := \overline{x}_{\varepsilon_n}, \overline{u}_n := \overline{u}_{\varepsilon_n}$ is the optimal process in (1). Then, up to a subsequence $\overline{u}_n \to \overline{u}$ in $L^2(0,T)$ and using the result from part (1) of the proof, we have:

$$\overline{x}_n \to \overline{y} \text{ in } C([0,T]),$$

where $\{\overline{y}, \overline{u}\}$ is the admissible process in (3). Taking into account the convergence

$$L(t, \overline{x}_n(t), \overline{u}_n(t)) \to L(t, \overline{y}(t), \overline{u}(t)) \text{ a.e.}$$

and inequality

$$|L(t, \overline{x}_n(t), \overline{u}_n(t))| \leq c(t)(1 + ||\overline{u}_n(t)||),$$

where $c \in L^2(0,T)$ (due to Lemma 1.3 in [19]), we obtain

$$L(t, \overline{x}_n, \overline{u}_n) \to L(t, \overline{y}, \overline{u}) \text{ weakly in } L^2(0,T), n \to \infty.$$

Then, we have the following.

$$\lim_{n\to\infty} J[\overline{x}_n, \overline{u}_n] = \lim_{n\to\infty}\left(\int_0^T L(t, \overline{x}_n(t), \overline{u}_n(t))dt + \Phi(\overline{x}_n(T))\right) = J[\overline{y}, \overline{u}].$$

On the other hand, for any admissible pair $\{x_n, u\}$ in (1), we have the following.

$$J[\overline{x}_n, \overline{u}_n] \leq J(x_n, u).$$

Using the result from part (1) of the proof, we have $x_n \to y$ in $\mathbb{C}([0,T])$, where $\{y, u\}$ is an admissible pair in (3). In addition, we have

$$J(x_n, u) \to J[y, u], n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} J[\overline{x}_n, \overline{u}_n] = J[\overline{y}, \overline{u}] \leq J[y, u]$$

and we conclude that $\{\overline{y}, \overline{u}\}$ is the optimal process for (3). □

We will end this section with an example. Let us consider the following problem.

$$\begin{cases} \dot{x}(t) \in [1 - 2\sin^2\left(\frac{t}{\varepsilon} \cdot x\right), 1 + 2\sin^2\left(\frac{t}{\varepsilon} \cdot x\right)] \cdot u, t \in (0,T), \\ x(0) = x_0, u(\cdot) \in [-1,1], \\ J[x,u] = x(T) \to \inf. \end{cases} \tag{11}$$

The corresponding averaged problem is

$$\begin{cases} \dot{y}(t) = u(t), t \in (0,T), \\ y(0) = x_0, u(\cdot) \in [-1,1], \\ J[y,u] = y(T) \to \inf. \end{cases} \tag{12}$$

The assumptions of the Theorems 2 and 3 are fulfilled for problems (11) and (12). Therefore,

$$y(T) = x_0 + \int_0^T u(t)dt$$

and $u(t) \equiv -1$ is the approximate control.

5. Conclusions and Future Research

We sought to obtain a theoretical result that demonstrates the effectiveness of the averaging method of finding an approximate solution of the optimal control problem of a non-linear system of differential inclusions with fast-oscillating parameters. We proved that the optimal control of the problem with averaging coefficients can be considered as "approximately" optimal for the initial perturbed system. To demonstrate the effectiveness of the method, we plan to continue research focusing on the practical applications and simulation results using particular genetic algorithms.

Author Contributions: Conceptualization, T.Z., N.K. and A.R.; methodology, T.Z., N.K. and A.R.; formal analysis, T.Z., N.K. and A.R.; investigation, T.Z., N.K. and A.R.; writing—original draft preparation, T.Z., N.K. and A.R.; writing—review and editing, T.Z., N.K. and A.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

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