Article

# A Novel Approach for the Approximate Solution of Wave Problems in Multi-Dimensional Orders with Computational Applications 

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#### Abstract

The main goal of this paper is to introduce a new scheme, known as the Aboodh homotopy integral transform method (AHITM), for the approximate solution of wave problems in multidimensional orders. The Aboodh integral transform (AIT) removes the restriction of variables in the recurrence relation, whereas the homotopy perturbation method (HPM) derives the successive iterations using the initial conditions. The convergence analysis is provided to study a wave equation with multiple dimensions. Some computational applications are considered to show the efficiency of this scheme. Graphical representation between the approximate and the exact solution predicts the high rate of convergence of this approach.


Keywords: Aboodh integral transform; homotopy perturbation method; wave problems; series results

MSC: 35L05; 35A22

## 1. Introduction

In the real world, partial differential equations (PDEs) are used to analyze a wide range of physical phenomena that occur in different branches of applied sciences, including fluid dynamics, mathematical biology, quantum physics, chemical kinetics, and linear optics [1-3]. Various approaches have been introduced to obtain the analytical solutions of these PDEs. Although the calculations for these strategies are pretty straightforward, their limitations are predicated on the assumption of small parameters. As a result, many researchers developed some novel methods to get around these restrictions. Numerous scientists have studied multiple innovative and unique methods to obtain analytical solutions that are reasonably close to the precise solutions, such as the homotopy analysis method [4], modified extended tanh method [5], new Kudryashov method [6], Chun-Hui He's iteration method [7], the sub-equation method [8], Exp-function method [9], modified exponential rational method [10], homotopy asymptotic method [11], modified extended tanh expansion [12], fractal variational iteration transform method [13], residual power series (RPS) method [14] and Adomian decomposition method [15]. In the past, many experts and researchers established the application of the homotopy perturbation method (HPM) [16-18] in various physical problems and showed the performance of this approach in consistently transforming the challenging issues into a straightforward resolution.

The wave equation is a partial differential equation for a scalar function that describes the propagation phenomenon in different areas of engineering, physics, and scientific applications [19,20]. Wazwaz [21] studied linear and nonlinear problems in bounded and unbounded domains using the variational iteration method. Ghasemi et al. [22] employed the homotopy perturbation method to derive the numerical solution of two-dimensional nonlinear differential equations. Keskin and Oturanc [23] applied the reduced differential transform method to various wave equations. Ullah et al. [24] proposed the optimal homotopy asymptotic method to obtain the analytic series solution of wave equations. Adwan et al. [25] presented the numerical solutions of multi-dimensional wave equations and showed the accuracy of the proposed techniques. Jleli et al. [26] studied the framework of the homotopy perturbation transform method for analytic treatment of wave equations. Mullen and Belytschko [27] provided the finite element scheme for the examination of two-dimensional wave equations and considered some semi-discretizations. These schemes have many limitations and assumptions in finding the approximate solutions of the problems. To overcome these limitations and restrictions of variables, we introduce a new iterative strategy for the approximate solutions of multi-dimensional wave problems.

The variational iteration method (VIM), Laplace transform and homotopy analysis method (HAM) have some limitations, such as the VIM involving integration and producing the constant of integration, the Laplace transform involving the convolution theorem and the HAM also considering some assumptions. The Aboodh integral transform is very easy to implement for differential problems. The purpose of this paper is to apply the AHITM with a combination of the Abdooh integral transform and the HPM for wave problems of different dimensions. Less computations, fast convergence and significant results make this scheme unique and different from other approaches in the literature. This strategy derives a series of solutions with fast convergence and yields an approximate solution very close to the precise solution. This paper is structured as follows. In Section 2, we give brief details about the Abdooh integral transform. In Section 3, we present the formulation of the $\mathbb{A}$ HITM for solving multi-dimension problems. We provide the convergence analysis in Section 4. Some computational applications are demonstrated to show the effectiveness in Section 5, and finally, we discuss the conclusions in Section 6.

## 2. Preliminary Definitions of $\mathbb{A} I T$

In this section, we describe a few fundamental characteristics and concepts of $\mathbb{A} I T$ that are very helpful in the formulation of this scheme:

Definition 1. If we let $\vartheta(\phi)$ be a function precise for $\sigma \geq 0$, then

$$
\begin{equation*}
\mathcal{L}\{\vartheta(\phi)\}=F(s)=\theta \int_{0}^{\infty} \vartheta(\phi) e^{-\sigma \phi} d \phi, \tag{1}
\end{equation*}
$$

is called a Laplace transform.
Definition 2. The $\mathbb{A} I T$ of a function $\vartheta(\phi)$ is defined as [28]

$$
\begin{equation*}
\mathbb{A}[\vartheta(\phi)]=R(\sigma)=\frac{1}{\sigma} \int_{0}^{\infty} \vartheta(\phi) e^{-\sigma \phi} d \phi . \quad \phi \geq 0, \quad k_{1} \leq \sigma \leq k_{2} \tag{2}
\end{equation*}
$$

where $\mathbb{A}$ represents the symbol of $\mathbb{A} I T, k_{1}$ and $k_{2}$ are constants and $\sigma$ is the independent variable of the transformed function $\phi$. Conversely, since $R(\sigma)$ is the $\mathbb{A} I T$ of function $\vartheta(\phi)$, then

$$
\mathbb{A}^{-1}[R(\sigma)]=\vartheta(\phi), \quad \mathbb{A}^{-1}
$$

is called the inverse $\mathbb{A I T}$.

Proposition 1. If we let $\mathbb{A}\left\{\vartheta_{1}(\phi)\right\}=R_{1}(\sigma)$ and $\mathbb{A}\left\{\vartheta_{2}(\phi)\right\}=R_{2}(\sigma)$, then [29]

$$
\begin{align*}
\mathbb{A}\left\{a u_{1}(\phi)+b u_{2}(\phi)\right\} & =a S\left\{\vartheta_{1}(\phi)\right\}+b S\left\{\vartheta_{2}(\phi)\right\},  \tag{3}\\
\Rightarrow \quad \mathbb{A}\left\{a u_{1}(\phi)+b u_{2}(\phi)\right\} & =a R_{1}(\sigma)+b R_{2}(\sigma) .
\end{align*}
$$

Proposition 2. If $\mathbb{A}\{\vartheta(\phi)\}=R(\sigma)$, then the differential properties are defined as follows [29,30]:

$$
\begin{equation*}
\mathbb{A}\left\{\vartheta^{\prime}(\phi)\right\}=\sigma R(\sigma)-\frac{\vartheta(0)}{\sigma} \tag{1}
\end{equation*}
$$

(2) $\mathbb{A}\left\{\vartheta^{\prime \prime}(\phi)\right\}=\sigma^{2} R(\sigma)-\vartheta(0)-\frac{\vartheta^{\prime}(0)}{\sigma}$;

$$
\begin{equation*}
\mathbb{A}\left\{\vartheta^{m}(\phi)\right\}=\sigma^{m} R(\sigma)-\frac{\vartheta(0)}{\sigma^{2-m}}-\frac{\vartheta^{\prime}(0)}{\sigma^{3-m}}-\cdots-\frac{\vartheta^{m-1}(0)}{\sigma} \tag{4}
\end{equation*}
$$

## 3. Formulation of AHITM

In this segment, we formulate the strategy of the $\mathbb{A}$ HITM for finding the approximate solutions of 1D, 2D and 3D wave equation flows. We observe that this strategy is independent of integration and any hypotheses during the formulation of this scheme. We consider a differential problem such that

$$
\begin{equation*}
\vartheta^{\prime \prime}(\varsigma, \phi)=\vartheta(\varsigma, \phi)+g(\vartheta)+g(\varsigma, \phi) \tag{5}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\vartheta(\varsigma, 0)=a_{1}, \quad \vartheta_{\phi}(\varsigma, 0)=a_{2} \tag{6}
\end{equation*}
$$

where $\vartheta$ denotes the function in a region of time $\phi$ and $g(\vartheta)$ is considered a nonlinear term with the source term $g(\varsigma, \phi)$ of arbitrary constat $a$. Employing the AIT in Equation (5) yields

$$
\mathbb{A}\left[\vartheta^{\prime \prime}(\varsigma, \phi)\right]=\mathbb{A}[\vartheta(\varsigma, \phi)+g(\vartheta)+g(\varsigma, \phi)] .
$$

Using the proposition in Equation (4) for the $\mathbb{A} I T$, we obtain

$$
\sigma^{2} R(\sigma)-\vartheta(\varsigma, 0)-\frac{\vartheta^{\prime}(\varsigma, 0)}{\sigma}=\mathbb{A}[\vartheta(\varsigma, \phi)+g(\vartheta)+g(\varsigma, \phi)] .
$$

Hence, $R(\sigma)$ is evaluated such that

$$
\begin{equation*}
R[\sigma]=\frac{\vartheta(\zeta, 0)}{\sigma^{2}}+\frac{\vartheta^{\prime}(\zeta, 0)}{\sigma^{3}}+\frac{1}{\sigma^{2}} \mathbb{A}[\vartheta(\varsigma, \phi)+g(\vartheta)+g(\zeta, \phi)] . \tag{7}
\end{equation*}
$$

By using the inverse $\mathbb{A}$ IT on Equation (7), we obtain

$$
\vartheta(\varsigma, \phi)=\vartheta(\varsigma, 0)+\phi \vartheta^{\prime}(\varsigma, 0)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\{\vartheta(\varsigma, \phi)+g(\vartheta)+g(\varsigma, \phi)\}\right]
$$

Using the initial conditions, we obtain

$$
\vartheta(\varsigma, \phi)=a_{1}+\phi a_{2}+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\{\vartheta(\varsigma, \phi)+g(\vartheta)+g(\varsigma, \phi)\}\right] .
$$

Using the proposition in Equation (3), we obtain

$$
\vartheta(\varsigma, \phi)=a_{1}+\phi a_{2}+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\{g(\zeta, \phi)\}\right]+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}[\vartheta(\varsigma, \phi)+g(\vartheta)]\right]
$$

This implies that

$$
\begin{equation*}
\vartheta(\varsigma, \phi)=G(\varsigma, \phi)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}[\vartheta(\varsigma, \phi)+g(\vartheta)]\right] \tag{8}
\end{equation*}
$$

where

$$
G(\zeta, \phi)=a_{1}+\phi a_{2}+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\{g(\zeta, \phi)\}\right]
$$

Equation (8) is called the recurrence relation, which is now suitable for the implementation of the HPM such that

$$
\begin{equation*}
\vartheta(\zeta, \phi)=\sum_{i=0}^{\infty} p^{i} \vartheta_{i}(\zeta, \phi)=\vartheta_{0}+p^{1} \vartheta_{1}+p^{2} \vartheta_{2}+\cdots \tag{9}
\end{equation*}
$$

The nonlinear terms $g(\vartheta)$ are evaluated by considering the algorithm

$$
\begin{equation*}
g(\vartheta)=\sum_{i=0}^{\infty} p^{i} H_{i}(\vartheta)=H_{0}+p^{1} H_{1}+p^{2} H_{2}+\cdots \tag{10}
\end{equation*}
$$

where the $H_{n}$ polynomials are derived as follows:

$$
\begin{equation*}
H_{n}\left(\vartheta_{0}+\vartheta_{1}+\cdots+\vartheta_{n}\right)=\frac{1}{n!} \frac{\partial^{n}}{\partial p^{n}}\left(g\left(\sum_{i=0}^{\infty} p^{i} \vartheta_{i}\right)\right)_{p=0}, \quad n=0,1,2, \cdots \tag{11}
\end{equation*}
$$

We use Equations (9)-(11) in Equation (8) to compare the identical power of $p$ such that

$$
\begin{aligned}
& p^{0}: \vartheta_{0}(\varsigma, \phi)=G(\varsigma, \phi), \\
& p^{1}: \vartheta_{1}(\varsigma, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\vartheta_{0}(\varsigma, \phi)+H_{0}(\vartheta)\right\}\right], \\
& p^{2}: \vartheta_{2}(\varsigma, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\vartheta_{1}(\varsigma, \phi)+H_{1}(\vartheta)\right\}\right], \\
& p^{3}: \vartheta_{3}(\varsigma, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\vartheta_{2}(\varsigma, \phi)+H_{2}(\vartheta)\right\}\right],
\end{aligned}
$$

Proceeding with this process yields

$$
\begin{equation*}
\vartheta(\varsigma, \phi)=\vartheta_{0}+\vartheta_{1}+\vartheta_{2}+\cdots=\sum_{i=0}^{\infty} \vartheta_{i} . \tag{12}
\end{equation*}
$$

Thus, Equation (12) is the approximate result of the differential problem in Equation (5).

## 4. Convergence Analysis

Statement: Let $P$ and $Q$ be Banach spaces where $X: P \rightarrow Q$ is a nonlinear mapping. If the series produced by HPM is

$$
\vartheta_{n}(P, \varsigma)=X\left(\vartheta_{n-1}(P, \varsigma)\right)=\sum_{i=0}^{n-1} \vartheta_{i}(P, \varsigma), \quad n=1,2,3 \ldots
$$

then the following conditions must be true:
(1) $\quad\left\|\vartheta_{n}(P, \varsigma)-\vartheta(P, \varsigma)\right\| \leq \varphi^{n}\|\vartheta(P, \varsigma)-\vartheta(P, \varsigma)\|$;
(2) $\vartheta_{n}(P, \varsigma)$ is forever in the neighborhood of $\vartheta(P, x)$ meaning $\vartheta_{n}(P, \varsigma) \in B(\vartheta(P, \varsigma), r)=$ $\left\{\vartheta^{*}(P, \varsigma) /\left\|\vartheta^{*}(P, \varsigma)-\vartheta(P, \varsigma)\right\|\right\} ;$
(3) $\lim _{n \rightarrow \infty} \vartheta_{n}(P, x)=\vartheta(P, \varsigma)$.

## Proof.

(1) Consider condition (1) by recognition of $n$ such that $\left\|\vartheta_{1}-\vartheta\right\|=\left\|T\left(\vartheta_{0}\right)-\vartheta\right\|$, and the Banach fixed point theorem states that $X$ has a fixed point $\vartheta$ (i.e., $X(\vartheta)=\vartheta$ ). Therefore, we have

$$
\left\|\vartheta_{1}-\vartheta\right\|=\left\|G\left(\vartheta_{0}\right)-\vartheta\right\|=\left\|G\left(\vartheta_{0}\right)-G(\vartheta)\right\| \leq \varphi\left\|\vartheta_{0}-\vartheta\right\|=\varphi\|\vartheta(P, \varsigma)-\vartheta\| .
$$

where $X$ is a nonlinear mapping. By considering that $\left\|\vartheta_{n-1}-\vartheta\right\| \leq \varphi^{n-1} \| \vartheta(P, 0)-$ $\vartheta(P, x) \|$ is an induction hypothesis, then

$$
\left\|\vartheta_{n}-\vartheta\right\|=\left\|G\left(\vartheta_{n-1}\right)-G(\vartheta)\right\| \leq \varphi\left\|\vartheta_{n-1}-\vartheta\right\| \leq \varphi \varphi^{n-1}\|\vartheta(P, \varsigma)-\vartheta\| .
$$

(2) Our initial challenge is to demonstrate $\vartheta(P, \varsigma) \in B(\vartheta(P, \varsigma), r)$, which is attained by replacing $m$. Thus, for $m=1,\|\vartheta(P, \varsigma)-\vartheta(P, \varsigma)\|=\|\vartheta(P, 0)-\vartheta(P, \varsigma)\| \leq r$ with $\vartheta(P, 0)$ as an initial condition. Consider that $\|\vartheta(P, x)-\vartheta(P, \varsigma)\| \leq r$ for $m-2$ is an induction theory. Thus, we have

$$
\begin{aligned}
\|\vartheta(P, \varsigma)-\vartheta(P, \varsigma)\| & =\vartheta_{m-2}(P, \varsigma)-\frac{f_{m}(P)}{\Gamma(\delta-m+1)} x^{\delta-m} \| \\
& \leq\left\|\vartheta_{m-1}(P, \varsigma)-\vartheta(P, \varsigma)\right\|+\left\|\frac{f_{m}(P)}{\Gamma(\delta-m+1)} x^{\delta-m}\right\| \\
& =r .
\end{aligned}
$$

Now, $\forall n \geq 1$, using (1), we obtain

$$
\left\|\vartheta_{n}-\vartheta\right\| \leq \varphi^{n}\|\vartheta(P, \varsigma)-\vartheta\| \leq \varphi^{n} r \leq r .
$$

(3) Using condition (2) and $\lim _{n \rightarrow \infty} \varphi^{n}=0$, it follows that $\lim _{n \rightarrow \infty}\left\|\vartheta_{n}-\vartheta\right\|=0$, and hence

$$
\lim _{n \rightarrow \infty} \vartheta_{n}=\vartheta
$$

Thus, $\vartheta$ converges.

## 5. Computational Applications

We illustrate some computational applications to check the validity and authenticity of the $\mathbb{A}$ HITM. We observe that this strategy is extremely convenient to utilize and generates the series of convergence much easier than other schemes. We also study the physical behaviors of the these surface solutions. The error distribution is obtained graphically to show that the results obtained by the $\mathbb{A}$ HITM are very close to the precise results.

### 5.1. Example 1

Suppose a one-dimensional wave equation

$$
\begin{equation*}
\frac{\partial^{2} \vartheta}{\partial \phi^{2}}=\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}-3 \vartheta, \tag{13}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\vartheta(\varsigma, 0)=0, \quad \vartheta_{\phi}(\varsigma, 0)=2 \cos (\varsigma), \tag{14}
\end{equation*}
$$

and boundary condition

$$
\begin{equation*}
\vartheta(0, \phi)=\sin (2 \phi), \quad \vartheta_{\zeta}(\pi, \phi)=-\sin (2 \phi) . \tag{15}
\end{equation*}
$$

Using the $\mathbb{A}$ IT on Equation (13), we obtain $R(\sigma)$ such that

$$
R[\sigma]=\frac{\vartheta(\varsigma, 0)}{\sigma^{2}}+\frac{\vartheta^{\prime}(\varsigma, 0)}{\sigma^{3}}+\frac{1}{\sigma^{2}} \mathbb{A}\left[\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}-3 \vartheta\right] .
$$

Using the inverse $\mathbb{A} I T$ yields

$$
\vartheta(\varsigma, \phi)=\vartheta(\varsigma, 0)+\phi \vartheta_{\phi}(\varsigma, 0)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}-3 \vartheta\right\}\right] .
$$

Now, we apply the HPM to obtain a relation such that

$$
\begin{equation*}
\sum_{i=0}^{\infty} p^{i} \vartheta_{i}(\varsigma, \phi)=2 \phi \cos (\varsigma)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\sum_{i=0}^{\infty} p^{i} \frac{\partial^{2} \vartheta_{i}}{\partial \varsigma^{2}}-3 \sum_{i=0}^{\infty} p^{i} \vartheta\right\}\right] . \tag{16}
\end{equation*}
$$

When evaluating similar components of $p$, we obtain

$$
\begin{aligned}
& p^{0}: \vartheta_{0}(\zeta, \phi)=\vartheta(\varsigma, 0)=2 \phi \cos (\varsigma), \\
& p^{1}: \vartheta_{1}(\varsigma, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{0}}{\partial \varsigma^{2}}-3 \vartheta_{0}\right\}\right]=-\frac{(2 \phi)^{3}}{3!} \cos (\varsigma), \\
& p^{2}: \vartheta_{2}(\zeta, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{1}}{\partial \varsigma^{2}}-3 \vartheta_{1}\right\}\right]=\frac{(2 \phi)^{5}}{5!} \cos (\varsigma), \\
& p^{3}: \vartheta_{3}(\zeta, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{2}}{\partial \varsigma^{2}}-3 \vartheta_{2}\right\}\right]=-\frac{(2 \phi)^{7}}{7!} \cos (\zeta), \\
& p^{4}: \vartheta_{4}(\zeta, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{3}}{\partial \varsigma^{2}}-3 \vartheta_{3}\right\}\right]=\frac{(2 \phi)^{9}}{9!} \cos (\varsigma),
\end{aligned}
$$

$$
\vdots
$$

In a similar way, we can consider the approximate series such that

$$
\begin{align*}
\vartheta(\varsigma, \phi) & =\vartheta_{0}(\varsigma, \phi)+\vartheta_{1}(\zeta, \phi)+\vartheta_{2}(\varsigma, \phi)+\vartheta_{3}(\zeta, \phi)+\vartheta_{4}(\varsigma, \phi)+\cdots \\
& =\cos (\varsigma)\left(2 \phi-\frac{(2 \phi)^{3}}{3!}+\frac{(2 \phi)^{5}}{5!}-\frac{(2 \phi)^{7}}{7!}+\frac{(2 \phi)^{9}}{9!}\right)+\cdots \tag{17}
\end{align*}
$$

which can approach

$$
\begin{equation*}
\vartheta(\varsigma, \phi)=\cos (\zeta) \sin (2 \phi) . \tag{18}
\end{equation*}
$$

Figure 1 contains two diagrams: (a) the $\mathbb{A}$ HITM results of $\vartheta(\varsigma, \phi)$ and (b) the exact results of $\vartheta(\varsigma, \phi)$ at $-2 \leq \varsigma \leq 2$ and $0 \leq \phi \leq 0.5$ for a 1D wave problem. Figure 2 represents the graphical error of the 1D wave equation between the approximate and precise solutions at $0 \leq \varsigma \leq 20$ with $\phi=0.5$. Table 1 presents the absolute error between the approximate solution obtained by the $\mathbb{A}$ HITM and the exact solution at $\varsigma=0.5,1$ and $0.25,0.50,0.75,1$. We observe that the current approach demonstrated strong agreement with a precise answer
to the problem (Section 5.1) only after a few iterations. The rate of convergence shows that the $\mathbb{A}$ HITM is a relatable approach for $\vartheta(\varsigma, \phi)$. This means that we can effectively model any surface in accordance with the desired physical processes appearing in science and engineering.


Figure 1. Surface solutions of 1D wave equation. (a) Surface plot for approximate results. (b) Surface plot for precise results.


Figure 2. Graphical error between the approximate and precise results of $\vartheta(\varsigma, \phi)$.
Table 1. Absolute error between the approximate and exact solutions for Example 1.

| $\boldsymbol{\zeta}$ | $\boldsymbol{\phi}$ | Approximate | Exact | Absolute Error |
| :---: | :---: | :---: | :---: | :---: |
|  | 0.25 | 0.420735 | 0.420735 | $1 \times 10^{-8}$ |
| 0.5 | 0.50 | 0.73846 | 0.73846 | $1.7 \times 10^{-7}$ |
|  | 0.75 | 0.875386 | 0.875384 | $2 \times 10^{-6}$ |
|  | 1.0 | 0.798027 | 0.797984 | $4.3 \times 10^{-5}$ |
|  | 0.25 | 0.259035 | 0.259035 | $1 \times 10^{-9}$ |
| 1.0 | 0.5 | 0.454649 | 0.454649 | $1.5 \times 10^{-8}$ |
|  | 0.75 | 0.53895 | 0.538949 | $2.3 \times 10^{-7}$ |
|  | 1.0 | 0.491323 | 0.491295 | $2.8 \times 10^{-6}$ |

### 5.2. Example 2

Suppose a two-dimensional wave equation

$$
\begin{equation*}
\frac{\partial^{2} \vartheta}{\partial \phi^{2}}=2\left(\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta}{\partial \xi^{2}}\right)+6 \phi+2 \varsigma+4 \xi, \tag{19}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\vartheta(\varsigma, \zeta, 0)=0, \quad \vartheta_{\phi}(\varsigma, \zeta, 0)=2 \sin (\varsigma) \sin (\xi) \tag{20}
\end{equation*}
$$

and boundary condition

$$
\begin{align*}
& \vartheta(0, \xi, \phi)=\phi^{3}+2 \phi^{2} \xi^{2}, \quad \vartheta_{\zeta}(\pi, \xi, \phi)=\phi^{3}+\pi \phi^{2}+2 \phi^{2} \xi, \\
& \vartheta(\varsigma, 0, \phi)=\phi^{3}+\phi^{2} \varsigma, \quad \vartheta_{\varsigma}(\varsigma, \pi, \phi)=\phi^{3}+2 \pi \phi^{2}+\phi^{2} \varsigma . \tag{21}
\end{align*}
$$

By using the $\mathbb{A}$ IT on Equation (19), we obtain $R(\sigma)$ such that

$$
R[\sigma]=\frac{6}{\sigma^{5}}+\frac{2 \varsigma}{\sigma^{4}}+\frac{4 \xi}{\sigma^{4}}+\frac{\vartheta(0)}{\sigma^{2}}+\frac{\vartheta^{\prime}(0)}{\sigma^{3}}+\frac{1}{\sigma^{2}} \mathbb{A}\left[2\left(\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta}{\partial \xi^{2}}\right] .\right.
$$

Using the inverse AIT yields
$\vartheta(\varsigma, \xi, \phi)=\phi^{3}+\varsigma \phi^{2}+2 \xi \phi^{2}+\vartheta(\varsigma, 0)+\phi \vartheta_{\phi}(\varsigma, 0)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{2\left(\frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta}{\partial \xi^{2}}\right\}\right]\right.$.
Now, we apply the HPM to obtain a relation such that

$$
\begin{equation*}
\sum_{i=0}^{\infty} p^{i} \vartheta_{i}(\varsigma, \xi, \phi)=\phi^{3}+\varsigma \phi^{2}+2 \xi \phi^{2}+2 \phi \sin (\varsigma) \sin (\xi)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{2\left(\sum_{i=0}^{\infty} p^{i} \frac{\partial^{2} \vartheta_{i}}{\partial \varsigma^{2}}+\sum_{i=0}^{\infty} p^{i} \frac{\partial^{2} \vartheta_{i}}{\partial \xi^{2}}\right)\right\}\right] \tag{22}
\end{equation*}
$$

By evaluating similar components of $p$, we obtain

$$
\begin{aligned}
& p^{0}: \vartheta_{0}(\varsigma, \xi, \phi)=\vartheta(\varsigma, 0)=\phi^{3}+\varsigma \phi^{2}+2 \xi \phi^{2}+2 \phi \sin (\varsigma) \sin (\xi), \\
& p^{1}: \vartheta_{1}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{0}}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta_{0}}{\partial \xi^{2}}\right\}\right]=-\frac{(2 \phi)^{3}}{3!} \sin (\varsigma) \sin (\xi), \\
& p^{2}: \vartheta_{2}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{1}}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta_{1}}{\partial \xi^{2}}\right\}\right]=\frac{(2 \phi)^{5}}{5!} \sin (\varsigma) \sin (\xi), \\
& p^{3}: \vartheta_{3}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{2}}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta_{2}}{\partial \xi^{2}}\right\}\right]=-\frac{(2 \phi)^{7}}{7!} \sin (\varsigma) \sin (\xi), \\
& p^{4}: \vartheta_{4}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\partial^{2} \vartheta_{3}}{\partial \varsigma^{2}}+\frac{\partial^{2} \vartheta_{3}}{\partial \xi^{2}}\right\}\right]=\frac{(2 \phi)^{9}}{9!} \sin (\varsigma) \sin (\xi), \\
& \therefore
\end{aligned}
$$

In a similar way, we can consider the approximate series such that

$$
\begin{align*}
\vartheta(\varsigma, \xi, \phi) & =\vartheta_{0}(\varsigma, \xi, \phi)+\vartheta_{1}(\varsigma, \xi, \phi)+\vartheta_{2}(\varsigma, \xi, \phi)+\vartheta_{3}(\varsigma, \xi, \phi)+\vartheta_{4}(\varsigma, \xi, \phi)+\cdots \\
& =\phi^{3}+\varsigma \phi^{2}+2 \xi \phi^{2}+\sin (\varsigma) \sin (\xi)\left(2 \phi-\frac{(2 \phi)^{3}}{3!}+\frac{(2 \phi)^{5}}{5!}-\frac{(2 \phi)^{7}}{7!}+\frac{(2 \phi)^{9}}{9!}\right)+\cdots \tag{23}
\end{align*}
$$

which can approach

$$
\begin{equation*}
\vartheta(\varsigma, \xi, \phi)=\phi^{3}+\varsigma \phi^{2}+2 \xi \phi^{2}+\sin (\varsigma) \sin (\xi) \sin (2 \phi) . \tag{24}
\end{equation*}
$$

Figure 3 contains two diagrams: (a) the $\mathbb{A} H I T M$ results of $\vartheta(\varsigma, \xi, \phi)$ and (b) the exact results of $\vartheta(\varsigma, \xi, \phi)$ at $-1 \leq \varsigma \leq 1$ and $0 \leq \phi \leq 0.1$ with $\xi=0.5$ for the 2 D wave problem. Figure 4 represents the graphical error of the 2 D wave equation between the approximate and precise solutions at $0 \leq \varsigma \leq 20$ with $\xi=0.01$ and $\phi=0.01$. Table 2 presents the absolute error between the approximate solution obtained by the AHITM and the exact solution at $\varsigma=0.5,1$ and $0.25,0.50,0.75,1$, where $\xi=0.5$. We observe that the current approach demonstrated strong agreement with the precise answer to the problem (Section 5.2) only after a few iterations. The rate of convergence shows that the $\mathbb{A}$ HITM is a
reliable approach for $\vartheta(\varsigma, \xi, \phi)$. This means that we can effectively model any surface in accordance with the desired physical processes appearing in nature.


Figure 3. Surface solutions of 2D wave equation. (a) Surface plot for approximate results. (b) Surface plot for precise results.


Figure 4. Graphical error between the approximate and precise results of $\vartheta(\varsigma, \xi, \phi)$.

Table 2. Absolute error between the approximate and exact solutions for Example 2.

| $\boldsymbol{\zeta}$ | $\boldsymbol{\phi}$ | Approximate | Exact | Absolute Error |
| :---: | :---: | :---: | :---: | :---: |
|  | 0.25 | 0.365286 | 0.365286 | $1 \times 10^{-7}$ |
| 0.5 | 0.50 | 1.08947 | 1.08947 | $1 \times 10^{-7}$ |
|  | 0.75 | 2.23054 | 2.23054 | $1.5 \times 10^{-6}$ |
|  | 1.0 | 3.86685 | 3.86683 | $2 \times 10^{-5}$ |
|  | 0.25 | 0.542593 | 0.542593 | $1 \times 10^{-8}$ |
| 1.0 | 0.5 | 1.47082 | 1.47082 | $1.2 \times 10^{-7}$ |
|  | 0.75 | 2.81568 | 2.81567 | $2.3 \times 10^{-6}$ |
|  | 1.0 | 4.64388 | 4.64385 | $3 \times 10^{-5}$ |

### 5.3. Example 3

Consider the three-dimensional wave problem

$$
\begin{equation*}
\frac{\partial^{2} \vartheta}{\partial \phi^{2}}=\frac{\varsigma^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \eta^{2}}-\vartheta, \tag{25}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
\vartheta(\varsigma, \xi, \eta, 0)=0, \quad \vartheta_{\phi}(\varsigma, \xi, \eta, 0)=\varsigma^{4} \xi^{4} \eta^{4} \tag{26}
\end{equation*}
$$

and boundary condition

$$
\begin{array}{ll}
\vartheta(0, \xi, \eta, \phi)=0, & \vartheta(1, \xi, \eta, \phi)=\zeta^{4} \eta^{4} \sinh (\phi), \\
\vartheta(\varsigma, 0, \eta, \phi)=0, & \vartheta(\varsigma, 1, \eta, \phi)=\varsigma^{4} \eta^{4} \sinh (\phi),  \tag{27}\\
\vartheta(\varsigma, \xi, 0, \phi)=0, & \vartheta(\varsigma, \xi, 1, \phi)=\varsigma^{4} \xi^{4} \sinh (\phi) .
\end{array}
$$

By using the AIT in Equation (25), we obtain $R(\sigma)$ such that

$$
R[\sigma]=\frac{\vartheta(\varsigma, 0)}{\sigma^{2}}+\frac{\vartheta^{\prime}(\varsigma, 0)}{\sigma^{3}}+\frac{1}{\sigma^{2}} \mathbb{A}\left[\frac{\zeta^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \eta^{2}}-\vartheta\right] .
$$

Using the inverse AIT yields

$$
\vartheta(\varsigma, \xi, \eta, \phi)=\vartheta(\varsigma, 0)+\phi \vartheta_{\phi}(\varsigma, 0)+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\varsigma^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta}{\partial \eta^{2}}-\vartheta\right\}\right] .
$$

Now, we apply the HPM to obtain a relation such that

$$
\begin{equation*}
\sum_{i=0}^{\infty} p^{i} \vartheta(\varsigma, \xi, \eta, \phi)=\phi \varsigma^{4} \xi^{4} \eta^{4}+\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\sum_{i=0}^{\infty} p^{i} \frac{\varsigma^{2}}{18} \frac{\partial^{2} \vartheta_{i}}{\partial \varsigma^{2}}+\sum_{i=0}^{\infty} p^{i} \frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta_{i}}{\partial \xi^{2}}+\sum_{i=0}^{\infty} p^{i} \frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta_{i}}{\partial \eta^{2}}-\sum_{i=0}^{\infty} p^{i} \vartheta\right\}\right] . \tag{28}
\end{equation*}
$$

By evaluating similar components of $p$, we obtain

$$
\begin{aligned}
& p^{0}: \vartheta_{0}(\varsigma, \xi, \eta, \phi)=\vartheta(\varsigma, \xi, \eta, 0)=\phi \varsigma^{4} \xi^{4} \eta^{4}, \\
& p^{1}: \vartheta_{1}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\zeta^{2}}{18} \frac{\partial^{2} \vartheta_{0}}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta_{0}}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta_{0}}{\partial \eta^{2}}-\vartheta_{0}\right\}\right]=\frac{\phi^{3}}{3!} \zeta^{4} \xi^{4} \eta^{4}, \\
& p^{2}: \vartheta_{2}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\varsigma^{2}}{18} \frac{\partial^{2} \vartheta_{1}}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta_{1}}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta_{1}}{\partial \eta^{2}}-\vartheta_{1}\right\}\right]=\frac{\phi^{5}}{5!} \varsigma^{4} \xi^{4} \eta^{4}, \\
& p^{3}: \vartheta_{3}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\varsigma^{2}}{18} \frac{\partial^{2} \vartheta_{2}}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta_{2}}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta_{2}}{\partial \eta^{2}}-\vartheta_{2}\right\}\right]=\frac{\phi^{7}}{7!} \zeta^{4} \xi^{4} \eta^{4}, \\
& p^{4}: \vartheta_{4}(\varsigma, \xi, \phi)=\mathbb{A}^{-1}\left[\frac{1}{\sigma^{2}} \mathbb{A}\left\{\frac{\zeta^{2}}{18} \frac{\partial^{2} \vartheta_{3}}{\partial \varsigma^{2}}+\frac{\xi^{2}}{18} \frac{\partial^{2} \vartheta_{3}}{\partial \xi^{2}}+\frac{\eta^{2}}{18} \frac{\partial^{2} \vartheta_{3}}{\partial \eta^{2}}-\vartheta_{3}\right\}\right]=\frac{\phi^{9}}{9!} \varsigma^{4} \xi^{4} \eta^{4}, \\
& \therefore
\end{aligned}
$$

In a similar way, we can consider the approximate series such that

$$
\begin{align*}
\vartheta(\varsigma, \xi, \eta, \phi) & =\vartheta_{0}(\varsigma, \xi, \eta, \phi)+\vartheta_{1}(\varsigma, \xi, \eta, \phi)+\vartheta_{2}(\varsigma, \xi, \eta, \phi)+\vartheta_{3}(\varsigma, \xi, \eta, \phi)+\vartheta_{4}(\varsigma, \xi, \eta, \phi)+\cdots, \\
& =\varsigma^{4} \zeta^{4} \eta^{4}\left(\phi+\frac{\phi^{3}}{3!}+\frac{\phi^{5}}{5!}+\frac{\phi^{7}}{7!}+\frac{\phi^{9}}{9!}\right)+\cdots . \tag{29}
\end{align*}
$$

which can approach

$$
\begin{equation*}
\vartheta(\varsigma, \xi, \eta, \phi)=\varsigma^{4} \tilde{\zeta}^{4} \eta^{4} \sinh (\phi) \tag{30}
\end{equation*}
$$

Figure 5 contains two diagrams: (a) the AHITM results of $\vartheta(\varsigma, \xi, \eta, \phi)$ and (b) the exact results of $\vartheta(\varsigma, \xi, \eta, \phi)$ at $-5 \leq \varsigma \leq 5$ and $0 \leq \phi \leq 0.05$ with $\xi=0.5$ and $\eta=0.5$ for the 3D wave problem. Figure 6 represents the graphical error of the 3D wave equation between the approximate and precise solutions at $0 \leq \varsigma \leq 10$ with $\varsigma=0.5, \xi=0.5$ and $\phi=0.1$. Table 3 presents the absolute error between the approximate solution obtained by the AHITM and the exact solution at $\varsigma=0.5,1$ and $0.25,0.50,0.75,1$, where $\xi=0.5$ and $\eta=0.5$. We observe
that the current approach demonstrated the strong agreement, with a precise answer to the problem (Section 5.3) only after a few iterations. The rate of convergence shows that the $\mathbb{A}$ HITM is a reliable approach for $\vartheta(\varsigma, \xi, \eta, \phi)$. This means that we can effectively model any surface in accordance with the desired physical processes appearing in nature.


Figure 5. Surface solutions of 3D wave equation. (a) Surface plot for approximate results. (b) Surface plot for precise results.


Figure 6. Graphical error between the approximate and precise results of $\vartheta(\varsigma, \xi, \eta, \phi)$.
Table 3. Absolute error between the approximate and exact solutions for Example 3.

| $\boldsymbol{\zeta}$ | $\boldsymbol{\phi}$ | Approximate | Exact | Absolute Error |
| :---: | :---: | :---: | :---: | :---: |
|  | 0.25 | 0.0157883 | 0.0157883 | $1 \times 10^{-9}$ |
| 0.5 | 0.50 | 0.0325685 | 0.0325685 | $1.2 \times 10^{-9}$ |
|  | 0.75 | 0.0513948 | 0.0513948 | $1.4 \times 10^{-8}$ |
|  | 1.0 | 0.0734501 | 0.0734501 | $2 \times 10^{-7}$ |
|  | 0.25 | 0.252612 | 0.252612 | $1 \times 10^{-9}$ |
| 1.0 | 0.5 | 0.521095 | 0.521095 | $1.8 \times 10^{-8}$ |
|  | 0.75 | 0.822317 | 0.822317 | $2.5 \times 10^{-7}$ |
|  | 1.0 | 1.1752 | 1.1752 | $2.9 \times 10^{-6}$ |

## 6. Conclusions

In this paper, we employed the $\mathbb{A}$ HITM for obtaining the approximate solutions to 1D, 2D and 3D wave equations. The main advantage of the AIT is that the recurrence relation produces the iteration without any assumption of a small parameter. The HPM helps to produce successive iterations in the recurrence relation. The obtained results show that this approach is very simple to utilize and derives the series solution in convergence form. The graphical error of plot distortion shows that the AHITM had the best agreement between the approximate solution and the exact solution. We encourage readers to extend this scheme for the numerical solution of a nonlinear coupled system of a fractional order in science and engineering for their future works.


#### Abstract

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