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Coincidence Theory of a Nonlinear Periodic Sturm–Liouville System and Its Applications

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Abstract: Based on the second derivative, this paper directly establishes the coincidence degree theory of a nonlinear periodic Sturm–Liouville (SL) system. As applications, we study the existence of periodic solutions to the S–L system with some special nonlinear functions by applying Mawhin's continuation theorem. Some examples and simulations are furnished to inspect the correctness and availability of the chief findings.

Keywords: nonlinear S–L system; periodic solution; existence; coincidence degree theory

MSC: 34A34; 34K13; 54H25

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1. Introduction

The S–L equation is one of the most important second-order ODEs, which includes famous physical equations such as the Helmholtz equation, Bessel equation and Legendre equation. Meanwhile, any second-order linear ODE can be transformed into the S–L equation by an appropriate transformation. Therefore, it is of great significance to reveal the various dynamic properties of S–L equation. This manuscript mainly considers the following nonlinear periodic S–L system:

$$-[\alpha(t)\mathcal{S}'(t)]' + \beta(t)\mathcal{S}(t) = f(t, \mathcal{S}(t)), \tag{1}$$

where $\alpha \in C^1(\mathbb{R}, (0, +\infty))$, $\beta \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R}^2, \mathbb{R})$, and there is a constant $\varpi > 0$ such that $\alpha(t + \varpi) = \alpha(t)$, $\beta(t + \varpi) = \beta(t)$ and $f(t + \varpi, \cdot) = f(t, \cdot)$, for all $t \in \mathbb{R}$.

The S–L equation is a famous mathematical and physical equation with a history of more than 200 years. Many scholars have conducted extensive and in-depth research on it from aspects of theory and application, and have achieved fruitful results. However, we will not repeat the early research results of the S–L equation. We only review some of the latest research achievements and progress of the S–L equation in recent years. The latest research trends on the S–L equation mainly include the following aspects. The first involves the theoretical and numerical methods and applications of inverse S–L problems (see [1–16]). The second focuses on the investigation of some generalized S—L equations, such as the fractional differential S–L equation (see [13–15,17–24]) and the S–L equation on time scales (see [25–34]). The third deals with the S–L problems with certain singularity (see [35–38]) or discontinuity (see [11,12]).

This manuscript focuses on the following novel works: (a) We have established the coincidence degree theory for system (1) based on the second derivative. Since the zero-index Fredholm operator $\mathscr{L}$ that we constructed involves the second derivative, it will be complex and difficult to construct the inverse operator $\mathscr{K}_{\mathscr{P}}$ of $\mathscr{L}$ and projection operators $\mathscr{P}, \mathscr{Q}$. (b) For some special forms of $f(t, \mathcal{S}(t), \mathcal{S}'(t))$, we employ Mawhin's continuation theorem to prove that system (1) has a periodic positive solution. In addition, with the exception that $\alpha(t)$ is required to be a positive periodic function, other conditions for the existence of periodic solutions of (1) are not affected by $\alpha(t)$. As far as we know, our research topics and findings have not been seen in previous papers. Therefore, our work expands the application scope of coincidence degree theory and tries a new research method for the S–L equation.

The remaining composition of the manuscript as follows. Section 2 mainly establishes the coincidence degree theory for the S–L system (1). In Section 3, we apply our coincidence degree theory to study the existence of periodic positive solutions for two kinds of special nonlinear S–L system. Section 4 provides some examples and simulations to examine the correctness of our results. In the last Section 5, we briefly summarize the problem and method of the paper, and offer a simple outlook for future research directions.

2. Preliminaries and Coincidence Theory of System (1)

In this section, we attempt to build the coincidence degree theory for system (1). To this end, the following concepts and lemmas are essential.

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, and define a linear operator $\mathscr{L} : \mathrm{Dom}(\mathscr{L}) \subset \mathbb{X} \to \mathbb{Y}$ and a continuous operator $\mathscr{N} : \mathbb{X} \times [0,1] \to \mathbb{Y}$. $\mathscr{L}$ is called a zero-index Fredholm operator iff $\dim \mathrm{Ker}(\mathscr{L}) = \mathrm{codim}\, \mathrm{Im}(\mathscr{L}) < \infty$, and $\mathrm{Im}(\mathscr{L})$ is closed in $\mathbb{Y}$. Assuming that $\mathscr{L}$ is a zero-index Fredholm operator, then there are continuous projectors $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Ker}(\mathscr{L}) = \mathrm{Im}(\mathscr{P})$, $\mathrm{Im}(\mathscr{L}) = \mathrm{Im}(\mathscr{I} - \mathscr{Q}) = \mathrm{Ker}(\mathscr{Q})$, and $\mathbb{X} = \mathrm{Ker}(\mathscr{L}) \oplus \mathrm{Ker}(\mathscr{P})$, $\mathbb{Y} = \mathrm{Im}(\mathscr{L}) \oplus \mathrm{Im}(\mathscr{Q})$, where $\mathscr{I}$ is an identity operator. It infers that $\mathscr{L}|_{\mathrm{Dom}(\mathscr{L})\cap\mathrm{Ker}(\mathscr{P})} : (\mathscr{I} - \mathscr{P})\mathbb{X} \to \mathrm{Im}(\mathscr{L})$ is invertible and its inverse is denoted by $\mathscr{K}_P$. Let $\Omega \subset \mathbb{X}$ be a bounded open set, if $\mathscr{Q}\mathscr{N}(\overline{\Omega} \times [0,1])$ is bounded and $\mathscr{K}_P(\mathscr{I} - \mathscr{Q})\mathscr{N} : \overline{\Omega} \times [0,1] \to \mathbb{X}$ is compact, then $\mathscr{N}$ is called $\mathscr{L}$-compact on $\overline{\Omega} \times [0,1]$. Since $\mathrm{Ker}(\mathscr{L})$ and $\mathrm{Im}(\mathscr{Q})$ are isomorphic, there is an isomorphism $\mathscr{J} : \mathrm{Im}(\mathscr{Q}) \to \mathrm{Ker}(\mathscr{L})$. Furthermore, the below Mawhin continuation theorem [39] is extremely important in the subsequent discussion.

Lemma 1. *[39] For the given real Banach spaces $\mathbb{X}, \mathbb{Y}$, and a bounded open subset $\mathbb{X} \supset \Omega \neq \phi$, let $\mathscr{L} : \mathbb{X} \to \mathbb{Y}$ be a zero-index Fredholm operator, and an operator $\mathscr{N} : \mathbb{X} \times [0,1] \to \mathbb{Y}$ be $\mathscr{L}$-compact on $\overline{\Omega} \times [0,1]$. Assuming that*

(i) *every solution $\mathcal{W}$ of $\mathscr{L}\mathcal{W} = \mu\mathscr{N}(\mathcal{W}, \mu)$ possesses $\mathcal{W} \notin \partial\Omega \cap \mathrm{Dom}(\mathscr{L})$, $\forall\, \mu \in (0,1)$;*
(ii) *$\mathscr{Q}\mathscr{N}(\mathcal{W}, 0)\mathcal{W} \neq 0$, $\forall\, \mathcal{W} \in \partial\Omega \cap \mathrm{Ker}(\mathscr{L})$;*
(iii) *$\deg(\mathscr{J}\mathscr{Q}\mathscr{N}(\mathcal{W}, 0), \Omega \cap \mathrm{Ker}(\mathscr{L}), 0) \neq 0$.*

then $\mathscr{L}\mathcal{W} = \mathscr{N}(\mathcal{W}, 1)$ has a solution in $\overline{\Omega} \cap \mathrm{Dom}(\mathscr{L})$.

Let $\mathbb{X} = \mathbb{Y} = \{\mathcal{W} \in C^2(\mathbb{R}, \mathbb{R}) : \mathcal{W}(t + \varpi) = \mathcal{W}(t),\, t \in \mathbb{R}\}$ be equipped with the norm $\|\mathcal{W}\| = \max\{|\mathcal{W}(t)| : 0 \leq t \leq \varpi\}$, then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces.

Lemma 2. *For a given $\alpha \in C^1(\mathbb{R}, (0, +\infty))$ with $\alpha(t + \varpi) = \alpha(t)$, define $\mathscr{L} : \mathbb{X} \to \mathbb{Y}$ by*

$$(\mathscr{L}\mathcal{W})(t) = \frac{d}{dt}\left[\alpha(t)\frac{d\mathcal{W}}{dt}\right], \tag{2}$$

then $\mathscr{L}$ is a zero-index Fredholm operator.

Proof. Obviously, $\mathscr{L}$ is linear. From $(\mathscr{L}\mathcal{W})(t) = 0$ and (2), one has

$$\mathcal{W}(t) = c + d\int_0^t \frac{ds}{\alpha(s)}, \tag{3}$$

where c and d are arbitrary real constants. By $\alpha(t + \varpi) = \alpha(t) > 0$ and (3), one obtains

$$\mathcal{W}(t + \varpi) = c + d\int_0^{t+\varpi} \frac{ds}{\alpha(s)} = c + d\int_0^t \frac{ds}{\alpha(s)} + d\int_t^{t+\varpi} \frac{ds}{\alpha(s)} = \mathcal{W}(t) + d\int_0^{\varpi} \frac{ds}{\alpha(s)}. \tag{4}$$

It follows from $\mathcal{W}(t+\varpi) = \mathcal{W}(t)$ and (4) that $d \equiv 0$, which implies that $\text{Ker}(\mathscr{L}) = \mathbb{R}$. Thus, $\dim \text{Ker}(\mathscr{L}) = \text{codim}\, \text{Im}(\mathscr{L}) = 1$. In addition, for any $\{z_n\} \subset \text{Im}(\mathscr{L})$ such that $\lim_{n\to\infty} z_n = z$, there exists $\mathcal{W}_n \in \mathbb{X}$ such that $\mathscr{L}\mathcal{W}_n = z_n$. Therefore, one has

$$\lim_{n\to\infty} (\mathscr{L}\mathcal{W}_n) = \mathscr{L}(\lim_{n\to\infty} \mathcal{W}_n) = \lim_{n\to\infty} z_n = z,$$

which implies that $\lim_{n\to\infty} \mathcal{W}_n = \mathcal{W} \in \mathbb{X}$. Thus, one concludes that $\mathscr{L}\mathcal{W} = z \in \text{Im}(\mathscr{L})$, i.e., $\text{Im}(\mathscr{L})$ is closed in $\mathbb{Y}$. So $\mathscr{L}$ is a zero-index Fredholm operator. The proof is complete. $\square$

Lemma 3. *Define the operators $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ by*

$$\mathscr{P}\mathcal{W}(t) = \mathscr{Q}\mathcal{W}(t) = \frac{1}{\varpi} \int_0^{\varpi} \mathcal{W}(s)ds, \ \forall \mathcal{W} \in \mathbb{X} = \mathbb{Y}. \tag{5}$$

Then, $\mathscr{P}$ and $\mathscr{Q}$ are all continuous projectors such that

$$\text{Ker}(\mathscr{L}) = \text{Im}(\mathscr{P}), \quad \text{Im}(\mathscr{L}) = \text{Ker}(\mathscr{Q}),$$

$$\mathbb{X} = \text{Ker}(\mathscr{L}) \oplus \text{Ker}(\mathscr{P}), \quad \mathbb{Y} = \text{Im}(\mathscr{L}) \oplus \text{Im}(\mathscr{Q}).$$

Proof. It is easy to verify that $\mathscr{P}$ and $\mathscr{Q}$ are all continuous. From (5), one has

$$\mathscr{P}^2\mathcal{W}(t) = \mathscr{P}(\mathscr{P}\mathcal{W}(t)) = \frac{1}{\varpi}\int_0^{\varpi} \mathscr{P}\mathcal{W}(\tau)d\tau = \frac{1}{\varpi}\int_0^{\varpi} \left[\frac{1}{\varpi}\int_0^{\varpi} \mathcal{W}(s)ds\right] d\tau$$
$$= \left[\frac{1}{\varpi}\int_0^{\varpi} \mathcal{W}(s)ds\right] \times \frac{1}{\varpi}\int_0^{\varpi} d\tau = \frac{1}{\varpi}\int_0^{\varpi} \mathcal{W}(s)ds = \mathscr{P}\mathcal{W}(t), \ \forall \mathcal{W} \in \mathbb{X},$$

which implies $\mathscr{P}^2 = \mathscr{P}$. Similarly, $\mathscr{Q}^2 = \mathscr{Q}$. Thus, one knows that $\mathscr{P}$ and $\mathscr{Q}$ are all continuous projectors. For each $\mathcal{W} \in \mathbb{X}$, it follows from (5) that $\mathscr{P}\mathcal{W}(t)$ is a real constant, which indicates $\text{Im}(\mathscr{P}) \subset \mathbb{R}$. For any constant $c \in \mathbb{R}$, we have $c \in \mathbb{X}$ and $\mathscr{P}c = c$. This leads to $\mathbb{R} \subset \text{Im}(\mathscr{P})$. So $\text{Ker}(\mathscr{L}) = \text{Im}(\mathscr{P}) = \mathbb{R}$, and $\mathbb{X} = \text{Im}(\mathscr{P}) \oplus \text{Ker}(\mathscr{P}) = \text{Ker}(\mathscr{L}) \oplus \text{Ker}(\mathscr{P})$. For any $z \in \text{Ker}(\mathscr{Q})$, then $z \in \mathbb{Y}$ and $\int_0^{\varpi} z(s)ds = 0$. Denote $h(t) = \int_0^t z(s)ds$, then $h(t+\varpi) = h(t)$. Now, we solve $\mathscr{L}\mathcal{W}(t) = z(t)$ to obtain

$$\mathcal{W}(t) = c_1 \int_0^t \frac{d\tau}{\alpha(\tau)} + \int_0^t \frac{h(\tau)}{\alpha(\tau)}d\tau + c_0, \tag{6}$$

where c_0, c_1 are any real constants. Similar to (4), we derive from (6) that

$$\mathcal{W}(t+\varpi) = \mathcal{W}(t) + c_1 \int_0^{\varpi} \frac{d\tau}{\alpha(\tau)} + \int_0^{\varpi} \frac{h(\tau)}{\alpha(\tau)}d\tau. \tag{7}$$

Taking $c_1^* = -\frac{\int_0^{\varpi} \frac{h(\tau)}{\alpha(\tau)}d\tau}{\int_0^{\varpi} \frac{d\tau}{\alpha(\tau)}}$, we know from (6) and (7) that $\mathcal{W}^*(t) = c_1^* \int_0^t \frac{d\tau}{\alpha(\tau)} + \int_0^t \frac{h(\tau)}{\alpha(\tau)}d\tau + c_0 \in \mathbb{X}$ such that $\mathscr{L}\mathcal{W}^*(t) = z(t)$, which means that $z(t) \in \text{Im}(\mathscr{L})$, that is, $\text{Ker}(\mathscr{Q}) \subset \text{Im}(\mathscr{L})$. Conversely, for each $z \in \text{Im}(\mathscr{L}) \subset \mathbb{Y}$, there exists a $\mathcal{W}(t) \in \mathbb{X}$ such that $\mathscr{L}\mathcal{W}(t) = z(t)$. By $\mathcal{W}(t+\varpi) = \mathcal{W}(t)$, $\alpha(t+\varpi) = \alpha(t) > 0$ and (6), we obtain

$$0 = c_1 \int_t^{t+\varpi} \frac{d\tau}{\alpha(\tau)} + \int_t^{t+\varpi} \frac{h(\tau)}{\alpha(\tau)}d\tau = c_1 \int_t^{\varpi} \frac{d\tau}{\alpha(\tau)} + \int_t^{t+\varpi} \frac{h(\tau)}{\alpha(\tau)}d\tau. \tag{8}$$

Taking the derivative of two sides of (8) with respect to t, we apply $\alpha(t+\varpi) = \alpha(t) > 0$ to obtain

$$\frac{h(t+\varpi)}{\alpha(t+\varpi)} - \frac{h(t)}{\alpha(t)} = 0 \Rightarrow h(t+\varpi) = h(t) \Rightarrow h(\varpi) = \int_0^{\varpi} z(s)ds = 0. \tag{9}$$

(9) means that $z \in \mathrm{Ker}(\mathscr{Q})$, namely $\mathrm{Im}(\mathscr{L}) \subset \mathrm{Ker}(\mathscr{Q})$. Thus, $\mathrm{Im}(\mathscr{L}) = \mathrm{Ker}(\mathscr{Q})$, and $\mathbb{Y} = \mathrm{Ker}(\mathscr{Q}) \oplus \mathrm{Im}(\mathscr{Q}) = \mathrm{Im}(\mathscr{L}) \oplus \mathrm{Im}(\mathscr{Q})$. The proof is complete. □

From Lemmas 2 and 3, $\mathscr{L}|_{\mathrm{Dom}(\mathscr{L})\cap\mathrm{Ker}(\mathscr{P})} : (\mathscr{I} - \mathscr{P})\mathbb{X} \to \mathrm{Im}(\mathscr{L})$ is invertible. Its inverse $\mathscr{K}_P$ is given as follows.

Lemma 4. *For all $u \in \mathrm{Im}(\mathscr{L})$, $\mathscr{K}_P$ is defined by*

$$\mathscr{K}_P u(t) = \int_0^t \frac{1}{\alpha(s)}\left[\int_0^s u(\tau)d\tau\right]ds - A_u \int_0^t \frac{ds}{\alpha(s)} - B_u, \tag{10}$$

where $A_u = \frac{\int_0^{\varpi} \frac{1}{\alpha(\tau)}[\int_0^{\tau} u(\xi)d\xi]d\tau}{\int_0^{\varpi} \frac{d\tau}{\alpha(\tau)}}$, *and* $B_u = \frac{1}{\varpi}\int_0^{\varpi}\left\{\int_0^s\left[\frac{1}{\alpha(\xi)}\int_0^{\xi} u(\tau)d\tau + \frac{A_u}{\alpha(\xi)}\right]d\xi\right\}ds$.

Proof. It suffices to prove that $\mathscr{L}(\mathscr{K}_P u(t)) = u(t), \forall\, u \in \mathrm{Im}(\mathscr{L})$, and $\mathscr{K}_P(\mathscr{L}w(t)) = w(t)$, $\forall\, w \in \mathrm{Dom}(\mathscr{L}) \cap \mathrm{Ker}(\mathscr{P})$. In fact, from (2) and (10), we have

$$\begin{aligned}\mathscr{L}(\mathscr{K}_P u(t)) =& \frac{d}{dt}\left[\alpha(t)\frac{d(\mathscr{K}_P u(t))}{dt}\right] = \frac{d}{dt}\left[\alpha(t)\left(\frac{1}{\alpha(t)}\int_0^t u(\tau)d\tau - \frac{A_u}{\alpha(t)}\right)\right]\\ =& \frac{d}{dt}\left[\int_0^t u(\tau)d\tau - A_u\right] = u(t),\ \forall\, u \in \mathrm{Im}(\mathscr{L}).\end{aligned}$$

Let $\mathscr{L}w(t) = \frac{d}{dt}\left[\alpha(t)\frac{dw}{dt}\right] = z(t)$, then $\int_0^t z(\tau)d\tau = \alpha(t)w'(t) - \alpha(0)w'(0)$, and

$$w(t) = w(0) + \alpha(0)w'(0)\int_0^t \frac{1}{\alpha(s)}ds + \int_0^t \frac{1}{\alpha(s)}\left[\int_0^s z(\tau)\right]ds. \tag{11}$$

From (11) and $w(t) = w(t+\varpi)$, we obtain

$$\alpha(0)w'(0) = -\frac{\int_0^{\varpi}\frac{1}{\alpha(s)}[\int_0^s z(\tau)]ds}{\int_0^{\varpi}\frac{1}{\alpha(s)}ds} = -A_z. \tag{12}$$

Noticing that $\int_0^{\varpi} w(s)ds = 0$, we derive from (10) and (12) that

$$\begin{aligned}&\mathscr{K}_P(\mathscr{L}w(t)) = \int_0^t \frac{1}{\alpha(s)}\left[\int_0^s z(\tau)d\tau\right]ds - A_z\int_0^t \frac{ds}{\alpha(s)} - B_z\\ =& \int_0^t \frac{1}{\alpha(s)}\Big[\alpha(s)w'(s) - \alpha(0)w'(0)\Big]ds - \frac{\int_0^{\varpi}\frac{1}{\alpha(\tau)}[\alpha(\tau)w'(\tau) - \alpha(0)w'(0)]d\tau}{\int_0^{\varpi}\frac{d\tau}{\alpha(\tau)}}\int_0^t \frac{ds}{\alpha(s)}\\ &- \frac{1}{\varpi}\int_0^{\varpi}\left\{\int_0^s\left[\frac{1}{\alpha(\xi)}[\alpha(\xi)w'(\xi) - \alpha(0)w'(0)] + \frac{A_z}{\alpha(\xi)}\right]d\xi\right\}ds\\ =& w(t) - w(0) - A_z\int_0^t \frac{ds}{\alpha(s)} - \frac{w(\varpi) - w(0) - A_z\int_0^{\varpi}\frac{ds}{\alpha(s)}}{\int_0^{\varpi}\frac{ds}{\alpha(s)}}\int_0^t \frac{ds}{\alpha(s)} - \frac{1}{\varpi}\int_0^{\varpi}\left\{\int_0^s w'(\xi)d\xi\right\}ds\\ =& w(t) - w(0) - \frac{1}{\varpi}\int_0^{\varpi}\{w(s) - w(0)\}ds\\ =& w(t) - w(0) - \frac{1}{\varpi}\int_0^{\varpi} w(s)ds + w(0) = w(t),\ \ \forall\, w \in \mathrm{Dom}(\mathscr{L}) \cap \mathrm{Ker}(\mathscr{P}).\end{aligned}$$

The proof is complete. □

3. Existence of Periodic Solution

Section 2 has basically established the coincidence degree theory corresponding to system (1). As applications, this section stresses the existence of periodic solutions for (1) with $f(t,\mathcal{S}) = \mathcal{S}[r_1(t) - c_1(t)\mathcal{S}^\theta] - h_1(t)$ and $f(t,\mathcal{S}) = \mathcal{S}[r_2(t) - c_2(t)\mathcal{S}] - h_2(t)$, where $r_i(t), c_i(t), h_i(t) > 0 (i = 1,2)$ and $0 < \theta \neq 1$. These two special forms of $f(t,\mathcal{S})$ are derived from the ecosystem model. The former comes from the Gilpin–Ayala ecosystem; the latter comes from the Lotka–Volterra ecosystem. $r_i(t)$, $c_i(t)$ and $h_i(t)$ stand for the natural growth rate, intraspecific competition rate and artificial harvest of species, respectively. For convenience, we denote $\overline{F} = \max_{0\le t\le\varpi} F(t)$ and $\underline{F} = \min_{0\le t\le\varpi} F(t)$, where $F(t) : \mathbb{R} \to \mathbb{R}$ is a continuous ϖ-periodic function.

Theorem 1. *In system* (1)*, let* $f(t,\mathcal{S}) = \mathcal{S}[r_1(t) - c_1(t)\mathcal{S}^\theta] - h_1(t)$*. If the following conditions* (B_1) *and* (B_2) *hold, then system* (1) *contains at least one* ϖ*-periodic positive solution in* $\mathbb{X}$*.*

(B_1) *Assume that* $\alpha \in C^1(\mathbb{R}, (0, +\infty))$, $\beta \in C(\mathbb{R}, \mathbb{R})$, $r_1, c_1, h_1 \in C(\mathbb{R}, (0, +\infty))$*, and* $0 < \theta \neq 1$ *is a constant. Moreover,* α, β, r_1, c_1 *and* h_1 *are* ϖ*-periodic functions.*

(B_2) $\underline{c_1} > 0$, $\underline{r_1 - \beta} > 0$*, and* $\theta\overline{c_1}^{-\frac{1}{\theta}}\left(\frac{\underline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} > \overline{h_1}$.

Proof. The proof of this assertion is mainly completed by applying Lemma 1. To do so, the operators $\mathscr{L}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{K}_P$ are defined by (2), (5) and (10) based on Lemmas 2–4. In addition, the operator $\mathscr{N} : \mathbb{X} \times [0,1] \to \mathbb{Y}$ is given by

$$\mathscr{N}(\mathcal{S}, \mu) = \beta(t)\mathcal{S} - \mathcal{S}[r_1(t) - c_1(t)\mathcal{S}^\theta] + h_1(t). \tag{13}$$

Clearly, $\mathscr{Q}\mathscr{N}$ and $\mathscr{K}_P(\mathscr{I} - \mathscr{Q})\mathscr{N}$ are continuous. For any open-bounded subset Ω of $\mathbb{X}$, we easily apply the Arzela–Ascoli theorem to show that $\mathscr{K}_P(\mathscr{I} - \mathscr{Q})\mathscr{N}(\overline{\Omega})$ is compact, and $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded. Thus, $\mathscr{N}$ is $\mathscr{L}$-compact on $\overline{\Omega}$.

Consider an operator equation $\mathscr{L}\mathcal{S} = \mu\mathscr{N}(\mathcal{S}, \mu)$, i.e.,

$$\frac{d}{dt}\left[\alpha(t)\frac{d\mathcal{S}}{dt}\right] = \mu\Big[\beta(t)\mathcal{S} - \mathcal{S}[r_1(t) - c_1(t)\mathcal{S}^\theta] + h_1(t)\Big]. \tag{14}$$

If Equation (14) contains an ϖ-periodic solution $\mathcal{S} \in \mathbb{X}$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $\mathcal{S}(t_1) = \overline{\mathcal{S}}$, $\mathcal{S}(t_2) = \underline{\mathcal{S}}$, $\mathcal{S}'(t_1) = \mathcal{S}'(t_2) = 0$, $\mathcal{S}''(t_1) < 0$ and $\mathcal{S}''(t_2) > 0$. Noticing that $[\alpha(t)\mathcal{S}']' = \alpha'(t)\mathcal{S}' + \alpha(t)\mathcal{S}''$, it follows from (B_1) that

$$\begin{cases} 0 > \alpha(t_1)\mathcal{S}''(t_1) = \mu\big[\beta(t_1)\mathcal{S}(t_1) - \mathcal{S}(t_1)[r_1(t_1) - c_1(t_1)\mathcal{S}^\theta(t_1)] + h_1(t_1)\big], \\ 0 < \alpha(t_2)\mathcal{S}''(t_2) = \mu\big[\beta(t_2)\mathcal{S}(t_2) - \mathcal{S}(t_2)[r_1(t_2) - c_1(t_2)\mathcal{S}^\theta(t_2)] + h_1(t_2)\big]. \end{cases} \tag{15}$$

Let $\mathcal{S}(t_1) = e^{\overline{U}}$, $\mathcal{S}(t_2) = e^{\underline{U}}$, then inequalities (15) become

$$\begin{cases} c_1(t_1)e^{(1+\theta)\overline{U}} - [r_1(t_1) - \beta(t_1)]e^{\overline{U}} + h_1(t_1) < 0, \\ c_1(t_2)e^{(1+\theta)\underline{U}} - [r_1(t_2) - \beta(t_2)]e^{\underline{U}} + h_1(t_2) > 0, \end{cases}$$

which implies that

$$\begin{cases} \underline{c_1}e^{(1+\theta)\overline{U}} - \overline{r_1 - \beta}e^{\overline{U}} + \underline{h_1} < 0, \\ \overline{c_1}e^{(1+\theta)\underline{U}} - \underline{r_1 - \beta}e^{\underline{U}} + \overline{h_1} > 0. \end{cases} \tag{16}$$

Let $\phi(z) = \underline{c_1}e^{(1+\theta)z} - \overline{r_1 - \beta}e^z + \underline{h_1}$, $\psi(z) = \overline{c_1}e^{(1+\theta)z} - \underline{r_1 - \beta}e^z + \overline{h_1}$. According to (B_2) and Lemma 2.2 in [40,41], we know that the unique minimum points of $\phi(z)$ and $\psi(z)$ are, respectively, given by

$$\overline{U_0} = \frac{1}{\theta}\ln\left[\frac{\overline{r_1 - \beta}}{\underline{c_1}(1+\theta)}\right], \ \underline{U_0} = \frac{1}{\theta}\ln\left[\frac{\underline{r_1 - \beta}}{\overline{c_1}(1+\theta)}\right]. \tag{17}$$

The minimums are

$$\phi(\overline{U_0}) = -\theta\underline{c_1}^{-\frac{1}{\theta}}\left(\frac{\overline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} + \underline{h_1},\ \psi(\underline{U_0}) = -\theta\overline{c_1}^{-\frac{1}{\theta}}\left(\frac{\underline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} + \overline{h_1}.$$

Since

$$\theta\underline{c_1}^{-\frac{1}{\theta}}\left(\frac{\overline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} > \theta\overline{c_1}^{-\frac{1}{\theta}}\left(\frac{\underline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} > \overline{h_1} \geq \underline{h_1},$$

we yield that $\phi(\overline{U_0}) < 0$, $\psi(\underline{U_0}) < 0$ and there exist only four real constants $\overline{U_1}$, $\overline{U_2}$, $\underline{U_1}$ and $\underline{U_2}$ such that

$$\overline{U_1} < \overline{U_0} < \overline{U_2},\ \underline{U_1} < \underline{U_0} < \underline{U_2},\ \phi(\overline{U_1}) = \phi(\overline{U_2}) = \psi(\underline{U_1}) = \psi(\underline{U_2}) = 0. \tag{18}$$

Combined with the above arguments and (18), the solutions of inequalities (16) are

$$\overline{U_1} < \overline{U} < \overline{U_2},\quad \underline{U_1} > \underline{U} \text{ or } \underline{U} > \underline{U_2}. \tag{19}$$

From the expressions of $\phi(z)$ and $\psi(z)$, $\forall z \in \mathbb{R}$, we have $\phi(z) < \psi(z)$. Thus, we obtain $\psi(\overline{U_2}) > \phi(\overline{U_2}) = 0 = \psi(\underline{U_2})$. By (17) and (18), we obtain $\underline{U_0} < \overline{U_0} < \overline{U_2}$ and $\underline{U_0} < \underline{U_2}$. Noting that $\psi(z)$ is strictly increasing in $[\underline{U_0}, +\infty)$, $\psi(\overline{U_2}) > \psi(\underline{U_2})$ leads to

$$\underline{U_2} < \overline{U_2}. \tag{20}$$

In light of (19), (20) and $\underline{U} \leq \overline{U}$, choose

$$\Omega = \{\mathcal{S}(t) \in \mathbb{X} : e^{\underline{U_2}} < \mathcal{S}(t) < e^{\overline{U_2}}\}.$$

Obviously, $\Omega \subset \mathbb{X}$ is open-bounded such that Lemma 1(i) is true.

Noting that $\partial\Omega = \{e^{\underline{U_2}}, e^{\overline{U_2}}\}$, we derive from (16)–(20) that $\mathscr{Q}\mathscr{N}(e^{\underline{U_2}}, 0) \neq 0$ and $\mathscr{Q}\mathscr{N}(e^{\overline{U_2}}, 0) \neq 0$. Thus, Lemma 1(ii) is true.

Choosing $\mathscr{J} = \mathscr{I}$ as the identity operator, and noting that $\mathscr{N}(\mathcal{S}_*, 0) = \beta(t)\mathcal{S}_* - \mathcal{S}_*[r_1(t) - c_1(t)\mathcal{S}_*\theta] + h_1(t) = 0$, a direct calculation gives

$$\begin{aligned}\deg\{\mathscr{J}\mathscr{Q}\mathscr{N}(\mathcal{S},0), \Omega\cap\mathrm{Ker}(\mathscr{J}), 0\} &= \mathrm{sgn}\left(\frac{\partial}{\partial\mathcal{S}}\mathscr{N}(\mathcal{S},0)\bigg|_{\mathcal{S}=\mathcal{S}_*}\right)\\ =&\mathrm{sgn}\left(-\theta c_1(t)\mathcal{S}_*^{\theta} - \frac{h_1(t)}{\mathcal{S}_*}\right) = -1.\end{aligned}$$

Thus, Lemma 1 (iii) is also true. It follows from Lemma 1 that system (1) has at least an ϖ-periodic positive solution $\widetilde{\mathcal{S}}(t)$ satisfying $e^{\underline{U_2}} < \widetilde{\mathcal{S}}(t) < e^{\overline{U_2}}$. The proof is complete. □

Theorem 2. *In system* (1), *let* $f(t,\mathcal{S}) = \mathcal{S}[r_2(t) - c_2(t)\mathcal{S}] - h_2(t)$. *If the following conditions* (B_3) *and* (B_4) *hold, then system* (1) *contains at least one* ϖ*-periodic positive solution in* $\mathbb{X}$.

(B_3) *Assume that* $\alpha \in C^1(\mathbb{R}, (0,+\infty))$, $\beta \in C(\mathbb{R},\mathbb{R})$, $r_2, c_2, h_2 \in C(\mathbb{R}, (0,+\infty))$, *and* α, β, r_2, c_2 *and* h_2 *are* ϖ*-periodic functions.*

(B_4) $\underline{c_2} > 0$, $\underline{r_2 - \beta} > 2\sqrt{\overline{c_2}\,\overline{h_2}}$.

Proof. Similar to the proof of Theorem 1, the operators $\mathscr{L}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{K}_P$ are defined by (2), (5) and (10) based on Lemmas 2–4. In addition, the operator $\mathscr{N} : \mathbb{X} \times [0,1] \to \mathbb{Y}$ is given by

$$\mathscr{N}(\mathcal{S}, \mu) = \beta(t)\mathcal{S} - \mathcal{S}[r_2(t) - c_2(t)\mathcal{S}] + h_2(t). \tag{21}$$

Clearly, $\mathscr{Q}\mathscr{N}$ and $\mathscr{K}_P(\mathscr{I}-\mathscr{Q})\mathscr{N}$ are continuous. For any open-bounded subset Ω of $\mathbb{X}$, we easily apply the Arzela–Ascoli theorem to show that $\mathscr{K}_P(\mathscr{I}-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is compact, and $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded. Thus, $\mathscr{N}$ is $\mathscr{L}$-compact on $\overline{\Omega}$.

Consider an operator equation $\mathscr{L}\mathcal{S} = \mu\mathscr{N}(\mathcal{S},\mu)$, i.e.,

$$\frac{d}{dt}\left[\alpha(t)\frac{d\mathcal{S}}{dt}\right] = \mu[\beta(t)\mathcal{S} - \mathcal{S}[r_2(t) - c_2(t)\mathcal{S}] + h_2(t)]. \tag{22}$$

Assuming that Equation (22) has an ω-periodic solution $\mathcal{S} \in \mathbb{X}$, then there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $\mathcal{S}(\tau_1) = \overline{\mathcal{S}}$, $\mathcal{S}(\tau_2) = \underline{\mathcal{S}}$, $\mathcal{S}'(\tau_1) = \mathcal{S}'(\tau_2) = 0$, $\mathcal{S}''(\tau_1) < 0$ and $\mathcal{S}''(\tau_2) > 0$. Noticing that $[\alpha(t)\mathcal{S}']' = \alpha'(t)\mathcal{S}' + \alpha(t)\mathcal{S}''$, we derive from (B_3) that

$$\begin{cases} 0 > \alpha(\tau_1)\mathcal{S}''(\tau_1) = \mu[\beta(\tau_1)\mathcal{S}(\tau_1) - \mathcal{S}(\tau_1)[r_2(\tau_1) - c_2(\tau_1)\mathcal{S}(\tau_1)] + h_2(\tau_1)], \\ 0 < \alpha(\tau_2)\mathcal{S}''(\tau_2) = \mu[\beta(\tau_2)\mathcal{S}(\tau_2) - \mathcal{S}(\tau_2)[r_2(\tau_2) - c_2(\tau_2)\mathcal{S}(\tau_2)] + h_2(\tau_2)]. \end{cases} \tag{23}$$

From Inequalities (23), one has

$$\begin{cases} c_2(\tau_1)\overline{\mathcal{S}}^2 - [r_2(\tau_1) - \beta(\tau_1)]\overline{\mathcal{S}} + h_2(\tau_1) < 0, \\ c_2(\tau_2)\underline{\mathcal{S}}^2 - [r_2(\tau_2) - \beta(\tau_2)]\underline{\mathcal{S}} + h_2(\tau_2) > 0, \end{cases}$$

which implies that

$$\begin{cases} \underline{c_2}\,\overline{\mathcal{S}}^2 - \overline{r_2 - \beta}\,\overline{\mathcal{S}} + \underline{h_2} < 0, \\ \overline{c_2}\,\underline{\mathcal{S}}^2 - \underline{r_2 - \beta}\,\underline{\mathcal{S}} + \overline{h_2} > 0. \end{cases} \tag{24}$$

According to (B_4), one has $\overline{r_2 - \beta} \geq \underline{r_2 - \beta} > 2\sqrt{\overline{c_2}\,\overline{h_2}} \geq 2\sqrt{\underline{c_2}\,\underline{h_2}}$. Thus, the inequalities (24) are solved as

$$\widehat{l}^- < \overline{\mathcal{S}} < \widehat{l}^+, \quad \underline{\mathcal{S}} > l^+ \text{ or } \underline{\mathcal{S}} < l^-, \tag{25}$$

where

$$\widehat{l}^{\pm} = \frac{\overline{r_2 - \beta} \pm \sqrt{(\overline{r_2 - \beta})^2 - 4\underline{c_2}\,\underline{h_2}}}{2\underline{c_2}}, \; l^{\pm} = \frac{\underline{r_2 - \beta} \pm \sqrt{(\underline{r_2 - \beta})^2 - 4\overline{c_2}\,\overline{h_2}}}{2\overline{c_2}}. \tag{26}$$

From (26), one obtains

$$l^+ = \frac{\underline{r_2 - \beta} + \sqrt{(\underline{r_2 - \beta})^2 - 4\overline{c_2}\,\overline{h_2}}}{2\overline{c_2}} < \frac{\overline{r_2 - \beta} + \sqrt{(\overline{r_2 - \beta})^2 - 4\underline{c_2}\,\underline{h_2}}}{2\underline{c_2}} = \widehat{l}^+, \tag{27}$$

and

$$\begin{aligned} \widehat{l}^- &= \frac{\overline{r_2 - \beta} - \sqrt{(\overline{r_2 - \beta})^2 - 4\underline{c_2}\,\underline{h_2}}}{2\underline{c_2}} = \frac{2\underline{h_2}}{\overline{r_2 - \beta} + \sqrt{(\overline{r_2 - \beta})^2 - 4\underline{c_2}\,\underline{h_2}}} \\ &< \frac{2\overline{h_2}}{\underline{r_2 - \beta} - \sqrt{(\underline{r_2 - \beta})^2 - 4\overline{c_2}\,\overline{h_2}}} = \frac{\underline{r_2 - \beta} - \sqrt{(\underline{r_2 - \beta})^2 - 4\overline{c_2}\,\overline{h_2}}}{2\overline{c_2}} = l^-. \end{aligned} \tag{28}$$

Together with (25), (27), (28) and $\underline{\mathcal{S}} \leq \overline{\mathcal{S}}$, we choose

$$\Omega = \left\{\mathcal{S}(t) \in \mathbb{X} : l^+ < \mathcal{S}(t) < \widehat{l}^+\right\}.$$

Apparently, $\Omega \subset \mathbb{X}$ is open-bounded such that Lemma 1 (i) holds. Additionally, $\partial\Omega = \{l^+, \widehat{l}^+\}$; we know from (24) and (25) that $\mathscr{Q}\mathscr{N}(l^+, 0) \neq 0$ and $\mathscr{Q}\mathscr{N}(\widehat{l}^+, 0) \neq 0$. Thus, Lemma 1(ii) holds.

Taking the identity operator $\mathscr{J} = \mathscr{I}$, and noticing that $\mathscr{N}(\mathcal{S}^*, 0) = \beta(t)\mathcal{S}^* - \mathcal{S}^*[r_2(t) - c_2(t)\mathcal{S}^*] + h_2(t) = 0$, we have

$$\begin{aligned}
&\deg\{\mathscr{J}\mathscr{Q}\mathscr{N}(\mathcal{S},0), \Omega \cap \mathrm{Ker}(\mathscr{J}), 0\} = \mathrm{sgn}\left(\frac{\partial}{\partial \mathcal{S}}\mathscr{N}(\mathcal{S},0)\bigg|_{\mathcal{S}=\mathcal{S}^*}\right) \\
&= \mathrm{sgn}\left(-c_2(t)\mathcal{S}^* - \frac{h_2(t)}{\mathcal{S}^*}\right) = -1.
\end{aligned}$$

Thus, Lemma 1(iii) also holds. From Lemma 1, we conclude that system (1) has at least an ϖ-periodic positive solution $\widetilde{\mathcal{S}}(t)$ satisfying $l^+ < \widetilde{\mathcal{S}}(t) < \widehat{l}^+$. The proof is complete. $\square$

4. Illustrative Examples and Simulations

Since Equation (1) is a second-order ODE, it is necessary to convert it into a system of first-order ODEs for numerical simulation. Let $u(t) = \mathcal{S}(t)$ and $v(t) = \alpha(t)\mathcal{S}'(t)$, then, Equation (1) becomes

$$\begin{cases} \frac{du(t)}{dt} = \frac{1}{\alpha(t)}v(t), \\ \frac{dv(t)}{dt} = \beta(t)u(t) - f(t, u(t)). \end{cases} \tag{29}$$

Example 1. *Consider the following ODE*

$$-[\alpha(t)\mathcal{S}'(t)]' + \beta(t)\mathcal{S}(t) = \mathcal{S}(t)[r_1(t) - c_1(t)\mathcal{S}^\theta(t)] - h_1(t), \tag{30}$$

where $\theta = \frac{1}{2}$, $\alpha(t) = 3 + \sin(t)$, $\beta(t) = -\sin(2t)$, $r_1(t) = 5 + \cos(2t)$, $c_1(t) = 2 + \sin(3t)$, $h_1(t) = \frac{3+2\sin(t)}{10}$.

Obviously, $\varpi = 2\pi$ and the condition (B_1) holds. By a simple calculation, we have $\overline{c_1} = 3$, $\underline{c_1} = 1$, $\overline{r_1 - \beta} = 5 + \sqrt{2}$, $\underline{r_1 - \beta} = 5 - \sqrt{2}$, $\underline{h_1} = 0.1$, and $\theta \overline{c_1}^{-\frac{1}{\theta}} \left(\frac{\underline{r_1 - \beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} \approx 0.5908 > \overline{h_1} = 0.5$. Therefore, the condition (B_2) also holds. By solving the following algebraic equation

$$\begin{cases} \underline{c_1} e^{(1+\theta)\overline{U}} - \overline{r_1 - \beta} e^{\overline{U}} + \underline{h_1} = 0, \\ \overline{c_1} e^{(1+\theta)\underline{U}} - \underline{r_1 - \beta} e^{\underline{U}} + \overline{h_1} = 0, \end{cases}$$

we obtain $\overline{U}_1 \approx -4.1412$, $\overline{U}_2 \approx 3.7163$, $\underline{U}_1 \approx -1.4509$ and $\underline{U}_2 \approx 0.0816$. Thus,

$$\Omega = \{\mathcal{S}(t) \in \mathbb{X} : e^{\underline{U}_2} < \mathcal{S}(t) < e^{\overline{U}_2}\} = \{\mathcal{S}(t) \in \mathbb{X} : 1.0850 < \mathcal{S}(t) < 41.1120\}.$$

Therefore, we conclude from Theorem 1 that (30) has at least a 2π-periodic positive solution $\widetilde{\mathcal{S}}(t) \in \Omega$.

Example 2. *Consider the following ODE*

$$-[\alpha(t)\mathcal{S}'(t)]' + \beta(t)\mathcal{S}(t) = \mathcal{S}(t)[r_1(t) - c_1(t)\mathcal{S}^\theta(t)] - h_1(t), \tag{31}$$

where $\theta = \sqrt{3}$, $\alpha(t) = \frac{1}{3+\cos(2t)}$, $\beta(t) = \cos(t)$, $r_1(t) = 8 + 2\cos(t)$, $c_1(t) = 4 + \cos(3t)$, $h_1(t) = \frac{5+3|\sin(2t)|}{10}$.

Obviously, $\varpi = 2\pi$ and the condition (B_1) holds. We simply compute that $\overline{c_1} = 5$, $\underline{c_1} = 3$, $\overline{r_1 - \beta} = 9$, $\underline{r_1 - \beta} = 7$, $\underline{h_1} = 0.5$, and $\theta\overline{c_1}^{-\frac{1}{\theta}}\left(\frac{\underline{r_1-\beta}}{1+\theta}\right)^{\frac{1+\theta}{\theta}} \approx 3.0167 > \overline{h_1} = 0.8$. Therefore, the condition (B_2) also holds. By solving the following algebraic equation

$$\begin{cases} \underline{c_1}e^{(1+\theta)\overline{U}} - \overline{r_1 - \beta}e^{\overline{U}} + \underline{h_1} = 0, \\ \overline{c_1}e^{(1+\theta)\underline{U}} - \underline{r_1 - \beta}e^{\underline{U}} + \overline{h_1} = 0, \end{cases}$$

we obtain $\overline{U}_1 \approx -2.8881$, $\overline{U}_2 \approx 0.6167$, $\underline{U}_1 \approx -2.1517$ and $\underline{U}_2 \approx 0.1334$. Thus,

$$\Omega = \{\mathcal{S}(t) \in \mathbb{X} : e^{\underline{U}_2} < \mathcal{S}(t) < e^{\overline{U}_2}\} = \{\mathcal{S}(t) \in \mathbb{X} : 1.1427 < \mathcal{S}(t) < 1.8528\}.$$

Therefore, we conclude from Theorem 1 that (31) has at least a 2π-periodic positive solution $\widetilde{\mathcal{S}}(t) \in \Omega$.

Example 3. *Consider the following ODE*

$$-[\alpha(t)\mathcal{S}'(t)]' + \beta(t)\mathcal{S}(t) = \mathcal{S}(t)[r_2(t) - c_2(t)\mathcal{S}(t)] - h_2(t), \tag{32}$$

where $\alpha(t) = \frac{5+2\cos(t)}{6}$, $\beta(t) = \sin(t) - \cos(t)$, $r_2(t) = 10 + 2\sin(t)$, $c_2(t) = 3 + \sin(t)$, $h_2(t) = \frac{7+3|\cos(t)|}{10}$.

Obviously, $\varpi = 2\pi$ and the condition (B_3) holds. A simple computation gives $\overline{c_2} = 4$, $\underline{c_2} = 2$, $\overline{h_2} = 1$, $\underline{h_2} = 0.7$, $\overline{r_2 - \beta} = 10 + \sqrt{2}$, $\underline{r_2 - \beta} = 10 - \sqrt{2} \approx 8.5858 > 2\sqrt{\overline{c_2}\,\overline{h_2}} = 4$. Therefore, the condition (B_4) also holds. By solving the following two quadratic equations

$$\begin{cases} \underline{c_2}\overline{\mathcal{S}}^2 - \overline{r_2 - \beta}\,\overline{\mathcal{S}} + \underline{h_2} = 0, \\ \overline{c_2}\underline{\mathcal{S}}^2 - \underline{r_2 - \beta}\,\underline{\mathcal{S}} + \overline{h_2} = 0. \end{cases}$$

we yield that $\widehat{l}^- \approx 0.0620$, $\widehat{l}^+ \approx 5.6451$, $l^- \approx 0.1236$ and $l^+ \approx 2.0229$. Thus,

$$\Omega = \{\mathcal{S}(t) \in \mathbb{X} : l^+ < \mathcal{S}(t) < \widehat{l}^+\} = \{\mathcal{S}(t) \in \mathbb{X} : 2.0229 < \mathcal{S}(t) < 5.6451\}.$$

Therefore, we conclude from Theorem 2 that (32) has at least a 2π-periodic positive solution $\widetilde{\mathcal{S}}(t) \in \Omega$.

Now, we apply (29) and ode45 function of MATLAB to simulate the phase portrait of (30), (31) and (32), respectively. It is easy to see from Figures 1–3 that there exist closed trajectories, which shows that (30), (31) and (32) have periodic solutions.

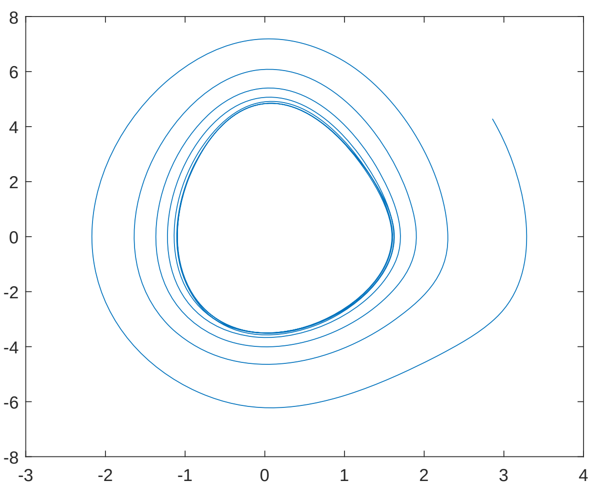

Figure 1. Phase portrait of (30) with $((u(0), v(0))^T = (2.856, 4.2840)^T$.

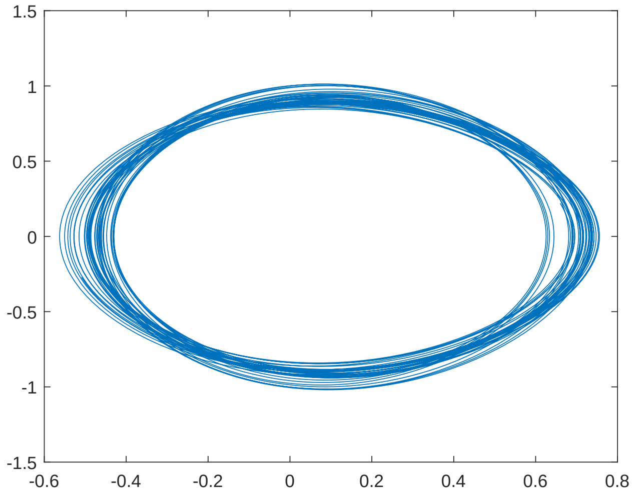

Figure 2. Phase portrait of (31) with $((u(0), v(0))^T = (0.66, 0.22)^T$.

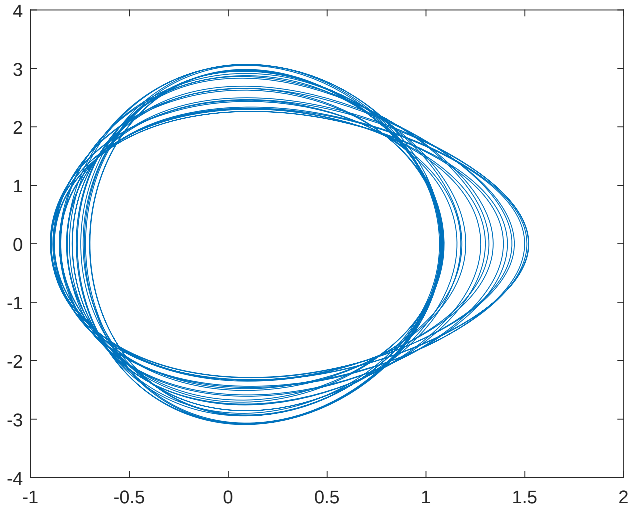

Figure 3. Phase portrait of (32) with $((u(0), v(0))^T = (1, 1.1667)^T$.

5. Conclusions

The Sturm–Liouville equation is a very famous differential equation. Many scholars have conducted extensive and in-depth research on its dynamics and have made many excellent achievements. In this manuscript, it is novel and interesting for us to establish the coincidence degree theory of Equation (1) and study the existence of its periodic solutions. We obtain some new and easily verifiable sufficient criteria for the existence of periodic solutions. Examples 1 and 2 and their simulations are applied to verify the correctness of Theorem 1 under the conditions of $0 < \theta < 1$ and $\theta > 1$, respectively. Using our method, we can estimate the existence region of periodic solutions. Our results are a useful supplement to the theory of periodic solutions of the Sturm–Liouville equation, and expand the application scope of coincidence degree theory. Based on this paper, we will further continue to study the dynamics of Equation (1) under pulse, delay and random effects. In addition, inspired by the papers [13–20,22–24,42–52], we will also study the Sturm–Liouville equation involving fractional differential as well as reaction–diffusion terms in the future.

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