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Direct Constructions of Uniform Designs under the Weighted Discrete Discrepancy †

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Abstract: Uniform designs are widely used in many fields in view of their ability to reduce experimental costs. In a lot of practical cases, different factors may take different numbers of values, so a mixed-level uniform design is needed. Since it is not reasonable to use the uniformity measure with the same weight for factors with different levels, the weighted discrete discrepancy was proposed in the existing literature. This paper discusses the construction method of mixed-level uniform designs under the weighted discrete discrepancy. The underlying method is to utilize some properties of partitioned difference families (PDFs) to obtain an infinite class of uniformly resolvable weighted balanced designs (URWBDs), which can directly produce corresponding uniform designs. Some examples are presented to illustrate the methods.

Keywords: partitioned difference family; uniform design; uniformly resolvable weighted balanced design; weighted discrete discrepancy

MSC: 62K15

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1. Introduction

In the past several decades, uniform designs have played an important role in various experiments with many factors. Such designs reduce the experimental cost substantially since they can determine the superior level combination of multiple factors with the small number of runs. Due to this benefit, uniform designs are increasingly used in many fields, and the corresponding theoretic results are also investigated by a vast amount of research.

During the development of uniform design theories, there are two aspects that have always been of interest to researchers. One is related to the uniformity measure. When Wang and Fang [1] first proposed the uniform design, the uniformity measure they used was the star discrepancy. However, Hickernell [2,3] showed some shortcomings of the star discrepancy and proposed several modified criteria, such as wrap-around L_2-discrepancy (WD) and centered L_2-discrepancy (CD). Later on, Hickernell and Liu [4] proposed the discrete discrepancy (DD), which is suitable for the experimental domain with discrete points. After that, DD has been further discussed by many existing papers; see, for example, Qin and Fang [5], Qin and Ai [6], and Chatterjee and Qin [7,8]. Moreover, in practical applications, there often exist factors with different levels that are more numerous than three, and thus, mixed high-level designs have recieved more and more attention. However, the factors with different levels are assumed to be of equal importance according to the definition of DD, so Tang et al. [9] proposed the weighted discrete discrepancy, which changes the kernel function of the discrete discrepancy into the weighted form. Another is related to the construction of uniform designs under the specific uniformity measure. Currently, there are three main approaches for constructing uniform designs: the quasi-Monte Carlo approach (see, for example, Fang and Li [10], and Zhou and Xu [11], the numerical search approach (see, for example, Winker and Fang [12,13], Fang, Tang and Yin [14], Zhou, Fang and Ning [15] and Zhou and Fang [16]) and the combinatorial design approach (see, for example, Fang, Ge and Liu [17,18], Fang et al. [19], Tang et al. [9] and Huang et al. [20]). Compared with the first two, the designs constructed by the third approach can often reach the lower bound of a certain specific discrepancy in most cases. This paper provides a combinatorial design approach to constructing uniform designs under the weighted discrete discrepancy.

The rest of this paper is organized as follows. Some concepts and notations are introduced in Section 2. Section 3 illustrates a construction method of uniform designs under the weighted discrete discrepancy by using a special combinatorial configuration named uniformly resolvable weighted balanced design (URWBD). Section 4 fully investigates the properties of a special class of partitioned difference families (PDFs) and then uses them to construct URWBDs. Some comments are presented in Section 5.

2. Preliminaries

In this section, some basic concepts and notations are introduced.

2.1. Multiset and PDF

Following Miyamoto [21], we first give the definition of a multiset as follows.

Definition 1. *Let U be a non-empty set; then, a multiset on U is a collection of repeated elements of U.*

Similar to the set case, operations related to multisets can be defined accordingly.

Definition 2. *Let U be a non-empty set. Then, for any multisets A and B on U, the union of A and B is denoted as $A \uplus B = \{e : e \in A \text{ or } e \in B\}$. Furthermore, suppose "$+$" and "$\cdot$" are two well-defined operations on U, i.e, addition and multiplication. Then, for any multiset H and any element e on U, the cosets of H related to "$+$" and "$\cdot$" are denoted as $H + e = e + H = \{h + e : h \in H\}$ and $eH = He = \{e \cdot h : h \in H\}$, respectively.*

In addition, for any element d on U and any positive integer n, let $\{d\}^*_n$ represent a multiset containing unique element d exactly n times; for any set (or multiset) A on U, let $\{A\}^*_n$ denote the set (or multiset) formed by n repetitions of A.

Throughout this paper, we assume all operations are conducted based on multisets.

Definition 3. *Let G be an Abelian group of order v whose operation is written additively, and let C_i, $i = 1, 2, \ldots, u$, be a c_i-subset of G (base blocks) respectively. If the following condition is satisfied,*

$$\biguplus_{1 \le i \le u} \{x - y : x, y \in C_i, x \neq y\} = \{G \backslash \{0\}\}^*_\lambda$$

then $\mathcal{C} = \{C_i : i = 1, 2, \ldots, u\}$ is called a difference family over G and is denoted by a $(v, \{c_0, c_1, \ldots, c_u\}, \lambda)$–DF or a (v, K, λ)–DF, where K is the set of sizes of the base blocks. If some c_i are equal, the above difference family is abbreviated as $(v, \{c_1^{s_1}, c_2^{s_2}, \ldots, c_m^{s_m}\}, \lambda)$–DF, where $1 \le m \le u$, s_i is a positive integer, and $s_1 + \ldots + s_m = u$.

A (v, K, λ)–DF whose base blocks are pairwise disjointed is called a disjoint difference family (DDF for short). There are a number of papers on the construction of DDFs. In particular, some notable results on DDFs are concerning $(v, k, k-1)$–DDFs (see Momihara [22], Buratti [23,24] and Kaspers and Pott [25]).

A partitioned difference family (PDF), denoted as a (v, K, λ)–PDF, is a (v, K, λ)–DDF whose blocks partition G. PDFs were introduced by Ding and Yin [26] in view of their applications to constant-composition codes. PDFs are an important tool in the construction of various structures, including optimal frequency hopping sequences (see Fuji-Hara, Miao and Mishima [27]), optimal difference systems of sets (see Wang and Wang [28]) and optimal constant-composition codes (see Ding and Yin [26]).

For the construction in Section 4, we only discuss PDFs with base blocks in the form of $\{(k-1)^m, k^n\}$ in this paper. Buratti, Yan and Wang [29] have proved that such a PDF can exist only when m is equal to 1 and λ is equal to $k-1$. In that paper, they constructed several infinite classes of $(v, \{k-1, k^n\}, k-1)$–PDFs. Fuji-Hara, Miao and Mishima [27] also proposed a method of constructing a special class of PDFs. In this paper, we will first investigate some properties of $(q^2, \{q-1, q^{q-1}\}, q-1)$–PDFs and use them to construct required experimental designs.

2.2. Weighted Discrete Discrepancy

In this subsection, we will briefly introduce some basic concepts related to uniform design. Let $\mathcal{O}$ be the set of n different points on domain $\mathcal{F}$, and let $M(\mathcal{O})$ represent a uniformity measure of $\mathcal{O}$ on $\mathcal{F}$. Among all possible sets consisting of n distinct points on $\mathcal{F}$, the set that makes M take the minimum value is the uniform design under the uniformity measure.

A uniformity measure is essential for uniform design. In this paper, we will use the weighted discrete discrepancy as the uniformity measure. The discrete discrepancy was introduced in Hickernell and Liu [4] Many studies have applied existing combinatorial structures to construct uniform designs under the discrete discrepancy; see, for example, Fang et al. [30], Huang et al. [20], Qin [31], and references therein.

Following Hickernell [2,3], let F be a uniform distribution function on the point set $\mathcal{X}$, where $\mathcal{X}$ is a measurable subset on $\mathbb{R}^n$. Let F_n be the relevant empirical distribution function on design point P, which is the subset of $\mathcal{X}$. Then, we define a symmetric and nonnegative definite kernel function $K(\mathbf{x}, \mathbf{y})$ on $\mathcal{X} \times \mathcal{X}$. Namely, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x}) \tag{1}$$

and for any $a_i, a_j \in \mathbb{R}, \mathbf{x_i}, \mathbf{x_j} \in \mathcal{X}$,

$$\sum_{i,j=1}^{n} a_i a_j K(\mathbf{x_i}, \mathbf{x_j}) \geq 0. \tag{2}$$

Therefore, the discrepancy on P can be expressed as

$$\begin{aligned} D(P; K) &= \left\{ \int_{\mathcal{X}^2} K(\mathbf{x}, \mathbf{y}) d[F(\mathbf{x}) - F_n(\mathbf{x})] d[F(\mathbf{y}) - F_n(\mathbf{y})] \right\}^{\frac{1}{2}} \\ &= \left\{ \int_{\mathcal{X}^2} K(\mathbf{x}, \mathbf{y}) dF(\mathbf{x}) dF(\mathbf{y}) - \frac{2}{n} \sum_{\mathbf{z} \in P} \int_{\mathcal{X}} K(\mathbf{x}, \mathbf{z}) dF(\mathbf{x}) \right. \\ &\quad \left. + \frac{2}{n^2} \sum_{\mathbf{z}, \mathbf{z}' \in P} K(\mathbf{z}, \mathbf{z}') \right\}^{\frac{1}{2}}. \end{aligned} \tag{3}$$

Different discrepancies can be obtained using different kernel functions and different design point domains. Let $\mathcal{X} = A_1 \times \ldots \times A_m$, where m refers to the number of all factors and A_j is the set of levels for factor j. Thus, for any $x_j, y_j \in A_j$, $a > b > 0$, let

$$\widetilde{K}_j(x_j, y_j) = \begin{cases} a & if\ x_j = y_j, \\ b & if\ x_j \neq y_j, \end{cases} \tag{4}$$

and for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, let

$$K(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{m} \widetilde{K}_j(x_j, y_j).$$

Then, $K(\mathbf{x}, \mathbf{y})$ satisfies conditions (1) and (2) and is a kernel function. The discrete discrepancy is defined by (3) when $A_j = \{1, 2, \ldots, q\}$ for any $j \in \{1, 2, \ldots, m\}$. According to Tang et al. [9], let $A_j = \{1, 2, \ldots, q_j\}$ for any $j \in \{1, 2, \ldots, m\}$. The weighted discrete discrepancy is defined by modifying Equation (4), which is changed to the weighted form

$$\widetilde{K}_j(x_j, y_j) = \begin{cases} f(q_j) & if\ x_j = y_j, \\ g(q_j) & if\ x_j \neq y_j. \end{cases}$$

Throughout this paper, we only consider the above $\widetilde{K}_j(x_j, y_j)$ with natural weights $f(q_j) = a^{q_j}$ and $g(q_j) = b^{q_j}$.

3. The Construction of a Class of Uniform Designs

Under the weighted discrete discrepancy, Tang et al. [9] proposed a new combinatorial configuration, uniformly resolvable weighted balanced design (URWBD), to construct the uniform design.

Definition 4. *Let n be a positive integer and K be a set of positive integers. A weighted balanced design with a weight of W and an index of λ_w denoted as an $(n, K, \lambda_w; W)$–WBD is an ordered pair $(V, \mathcal{B})$, satisfying that*

1. *V is an n-element set of distinct points;*
2. *$\mathcal{B}$ is a family of subsets(called blocks) in V, and K is the set of the length k_B for each block $B \in \mathcal{B}$;*
3. *W is the weight set of K, and each element k in K corresponds to a weight $\omega(k)$ in W;*
4. *For all blocks containing any pair of points (x, y) in V, the sum of weights corresponding to their lengths is λ_w, namely $\sum_{\{x,y\} \subseteq B} \omega(k_B) = \lambda_w$.*

When the weight is inversely proportional to the length of the block, i.e., $\omega(k) = \frac{c}{k}$ (c is a constant), we call it a natural weight. The weighted balanced design with the natural weight is denoted as an (n, K, λ_w)–WBD. In addition, if $\mathcal{B}$ can be divided into several parallel classes, and blocks belong to the same parallel class are with identical lengths, then the design is called a uniformly resolvable weighted balanced design, which is abbreviated as an (n, K, λ_w)–URWBD.

Example 1. *Let $V = \{\mathbb{Z}_8 \times \mathbb{Z}_3, \infty_0, \infty_1, \infty_2\}$ be the point set. Denote*

$$\mathcal{B}_+ = \left\{ \begin{array}{l} \{\infty_0, \infty_1, \infty_2\}, \{(0,0), (0,1), (0,2)\}, \{(1,0), (1,1), (1,2)\}, \\ \{(2,0), (2,1), (2,2)\}, \{(3,0), (3,1), (3,2)\}, \{(4,0), (4,1), (4,2)\}, \\ \{(5,0), (5,1), (5,2)\}, \{(6,0), (6,1), (6,2)\}, \{(7,0), (7,1), (7,2)\} \end{array} \right\},$$

$$\mathcal{B}_- = \left\{ \begin{array}{l} \{\infty_0, (0,0), (1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0)\}, \\ \{\infty_1, (0,1), (1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (7,1)\}, \\ \{\infty_2, (0,2), (1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (7,2)\} \end{array} \right\}$$

and

$$\mathcal{B}_i = \left\{ \begin{array}{l} \{\infty_0, (i,0), (4+i,0), (1+i,1), (2+i,1), (7+i,1), \\ (3+i,2), (5+i,2), (6+i,2)\}, \\ \{\infty_1, (i,1), (4+i,1), (1+i,2), (2+i,2), (7+i,2), \\ (3+i,0), (5+i,0), (6+i,0)\}, \\ \{\infty_2, (i,2), (4+i,2), (1+i,0), (2+i,0), (7+i,0), \\ (3+i,1), (5+i,1), (6+i,1)\} \end{array} \right\},$$

where $i = 0, 1, \ldots, 7$, and the operation "+" is performed in $\mathbb{Z}_8$. Then, $\mathcal{B}_+ \cup \mathcal{B}_- \cup_{i=0}^{7} \mathcal{B}_i$ forms a $(27, \{3, 9\}, 3)$–URWBD with the natural weight. Notice that Example 1 in Tang et al. [9] is similar to our example, but their construction is based on different point sets. Although the ideas are same, *our paper aims to propose the direct construction method of an infinite class of URWBDs rather than a few examples.*

U-type designs can be derived from URWBDs. Using the same notation as in Tang et al. [9], let $(V, \mathcal{B})$ be a URWBD on the n-element set of distinct points. Suppose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_m$, where each $\mathcal{B}_i = \{B_{i1}, B_{i2}, \ldots, B_{iq_i}\}$ represents the i-th parallel class for $i = 1, 2, \ldots, m$. Note that the length of the blocks contained in each $\mathcal{B}_i$ is k_i, so we must have $k_i \times q_i = n$ for any $i = 1, 2, \ldots, m$. Then, the following steps can produce a U-type design.

1. Let the q_i blocks of each parallel class $\mathcal{B}_i$ be naturally sequenced for $i = 1, 2, \ldots, m$.
2. For $i = 1, 2, \ldots, m$, define an n-dimensional column vector $\mathbf{x}_i = (d_{ij})$ for each $\mathcal{B}_i$ so that $d_{ij} = l$ if the point j is contained in the l-th block of $\mathcal{B}_i$.
3. Combine all $\mathbf{x}_i$ to form an $n \times m$ matrix $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$.

Example 2. *The U-type design $U(27; 3^9 9^1)$ in Table 1 can be constructed using the $(27, \{3, 9\}, 3)$–URWBD in Example 1 according to the above construction.*

Table 1. A U-type design $U(27; 3^9 9^1)$.

Run	Row Label	1	2	3	4	5	6	7	8	9	10
1	∞_0	1	1	1	1	1	1	1	1	1	1
2	(0, 0)	2	1	1	3	2	2	1	2	3	3
3	(1, 0)	3	1	3	1	3	2	2	1	2	3
4	(2, 0)	4	1	3	3	1	3	2	2	1	2
5	(3, 0)	5	1	2	3	3	1	3	2	2	1
6	(4, 0)	6	1	1	2	3	3	1	3	2	2
7	(5, 0)	7	1	2	1	2	3	3	1	3	2
8	(6, 0)	8	1	2	2	1	2	3	3	1	3
9	(7, 0)	9	1	3	2	2	1	2	3	3	1
10	∞_1	1	2	2	2	2	2	2	2	2	2
11	(0, 1)	2	2	2	1	3	3	2	3	1	1
12	(1, 1)	3	2	1	2	1	3	3	2	3	1
13	(2, 1)	4	2	1	1	2	1	3	3	2	3
14	(3, 1)	5	2	3	1	1	2	1	3	3	2
15	(4, 1)	6	2	2	3	1	1	2	1	3	3
16	(5, 1)	7	2	3	2	3	1	1	2	1	3
17	(6, 1)	8	2	3	3	2	3	1	1	2	1
18	(7, 1)	9	2	1	3	3	2	3	1	1	2
19	∞_2	1	3	3	3	3	3	3	3	3	3
20	(0, 2)	2	3	3	2	1	1	3	1	2	2
21	(1, 2)	3	3	2	3	2	1	1	3	1	2
22	(2, 2)	4	3	2	2	3	2	1	1	3	1
23	(3, 2)	5	3	1	2	2	3	2	1	1	3
24	(4, 2)	6	3	3	1	2	2	3	2	1	1
25	(5, 2)	7	3	1	3	1	2	2	3	2	1
26	(6, 2)	8	3	1	1	3	1	2	2	3	2
27	(7, 2)	9	3	2	1	1	3	1	2	2	3

Because of the special structure of URWBDs, the U-type designs obtained via them should have good properties. The following theorem is given by Tang et al. [9].

Theorem 1. *The U-type design derived from the (n, K, λ_w)–URWBD with the natural weight by the above construction is a uniform design under the weighted discrete discrepancy.*

According to the above theorem, the U-type design in Example 2 is a uniform design under the weighted discrete discrepancy. Generally, we have the following corollary.

Corollary 1. *If there exists a $(p^3, \{p, p^2\}, p)$–URWBD, then there exists a uniform design under the weighted discrete discrepancy.*

4. URWBDs from PDFs

This section aims to provide a direct construction method for $(p^3, \{p, p^2\}, p)$–URWBDs, which requires a special type of $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDFs.

4.1. Properties of a Class of PDFs

Following Fuji-Hara, Miao and Mishima [27], we first give the definition of an affine geometry.

Definition 5. *Let q be a prime power, n a positive integer and let $V_n(q)$ denote the n-dimensional vector space over $GF(q)$. When T is a t-dimensional linear subspace of $V_n(q)$, a coset (called a t-flat) of T is a set of form $T + \boldsymbol{a}$ for any vector $\boldsymbol{a} \in V_n(q)$. Elements of a 0-flat and of a 1-flat are called points and lines, respectively. A system consisting of all the vectors, all the t-flats of $V_n(q)$, and their incidence relation is called an affine geometry, denoted by $AG(n, q)$.*

According to Lemma 3.1 in Fuji-Hara, Miao and Mishima [27], we can directly obtain the following corollary.

Corollary 2. *Let q be a prime power; then, a parallel class of lines in an $AG(2, q)$ forms a $(q^2, \{q-1, q^{q-1}\}, q-1)$–PDF over $\mathbb{Z}_{q^2-1}$.*

The proof of Lemma 3.1 in Fuji-Hara, Miao and Mishima [27] gives a method to construct $(q^2, \{q-1, q^{q-1}\}, q-1)$–PDFs. However, Fuji-Hara, Miao and Mishima [27] proposed the PDF in order to construct optimal frequency hopping (FH) sequences, and there is no further discussion about the additional properties of the PDFs. In what follows, we will first consider some related properties of $(q^2, \{q-1, q^{q-1}\}, q-1)$–PDFs when q is a prime number. We will take a special parallel class to construct $(q^2, \{q-1, q^{q-1}\}, q-1)$–PDFs according to the above corollary and propose some further properties of them.

Let p be a prime number and α be the primitive element of $GF(p^2)$; then, the elements in $GF(p^2)$ can be expressed as $\alpha^0, \alpha^1, \dots, \alpha^{p^2-2}, \alpha^\infty$ $(= 0)$ or $a\alpha + b$ $(0 \le a, b \le p)$. Then, according to the construction method in Fuji-Hara, Miao and Mishima [27], there exists a bijection τ, which maps the point $a\alpha + b$ in $AG(2, p)$ $(0 \le a, b \le p)$ to the point in $\mathbb{Z}_{p^2-1} \cup \infty$ one-to-one. The detailed definition of bijection τ is as follows:

$$\begin{aligned} &\tau : AG(2, p) \to \mathbb{Z}_{p^2-1} \cup \infty. \\ &\tau : a\alpha + b\ (= \alpha^m) \mapsto m\ (m \in \mathbb{Z}_{p^2-1} \cup \infty). \end{aligned} \tag{5}$$

Denote $T_0 = \{0, 1, 2, \dots, p-1\}$ and $T_i = T_0 + i\alpha$, where $i = 0, 1, \dots, p-1$. Let $\Pi(T_0) = \{T_0, T_1, \dots, T_{p-1}\}$; then, $\Pi(T_0)$ is a parallel class containing T_0. Now denote $Q_0 = \tau(T_0)\backslash\infty$, $Q_i = \tau(T_i)$ for any $i \in \{1, 2, \dots, p-1\}$ and $\mathcal{P} = \{Q_0, Q_1, \dots, Q_{p-1}\}$; then, $\mathcal{P}$ is a $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDF over $\mathbb{Z}_{p^2-1}$.

Remark 1. *It is clear that $\mathcal{P}$ contains p sets. In addition, the sets in $\mathcal{P}$ are arranged in their natural order according to the above construction. Notice that the order cannot be arbitrarily changed in order to facilitate the following proof of some properties of $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDFs.*

Example 3. *Take $p = 3$; then, $T_0 = \{0, 1, 2\}$. Let α be a primitive element of $GF(9)$ and $x^2 + 2x + 2$ be the primitive polynomial of α. By (5), we have $Q_0 = \tau(T_0)\backslash\infty = \{0, 4\}$, $Q_1 = \tau(T_0 + \alpha) = \{1, 2, 7\}$ and $Q_2 = \tau(T_0 + 2\alpha) = \{3, 5, 6\}$. Consequently, $\{Q_0, Q_1, Q_2\}$ is a $(9, \{2, 3^2\}, 2)$–PDF over $\mathbb{Z}_8$.*

Lemma 1. *Let p be a prime number. Denote $T_0 = \{0, 1, 2, \ldots, p-1\}$ and $Q_0 = \tau(T_0)\backslash\infty$; then, $Q_0 = \{0, p+1, \ldots, (p-2)(p+1)\}$.*

Proof. Denote the primitive element of $GF(p^2)$ as α. Since $GF(p)$ is a subfield of $GF(p^2)$, the primitive element of $GF(p)$ is α^{p+1}. Thus, the elements of $GF(p)$, $T_0 = \{0, 1, 2, \ldots, p-1\}$, can also be written in the following form:

$$\{0, \alpha^0, \alpha^{p+1}, \alpha^{2(p+1)}, \ldots, \alpha^{(p-2)(p+1)}\}.$$

Notice that τ in (5) is a bijection, so we have $Q_0 = \{0, p+1, \ldots, (p-2)(p+1)\}$. □

The following corollary follows immediately from the proof of Lemma 1.

Corollary 3. *Let p be a prime number and α be a primitive element of $GF(p^2)$. Denote $T_0 = \{0, 1, 2, \ldots, p-1\}$ and $T_0' = \{0, \alpha^0, \alpha^{p+1}, \alpha^{2(p+1)}, \ldots, \alpha^{(p-2)(p+1)}\}$; then, $T_0 = T_0'$.*

Remark 2. *Note that the elements in the two sets $\{0, 1, 2, \ldots, p-1\}$ and $\{0, \alpha^0, \alpha^{p+1}, \alpha^{2(p+1)}, \ldots, \alpha^{(p-2)(p+1)}\}$ do not necessarily correspond to each other in the order in which they are written above.*

According to Corollary 3 and Remark 2, for each i in T_0 $(i = 1, 2, \ldots, p-1)$, we assume that

$$i = \alpha^{m_i \cdot (p+1)}, \tag{6}$$

where $0 \leq m_i < p-1$ and m_i are not equal to each other. More precisely, m_i runs over $\{0, 1, 2, \ldots, p-2\}$. Denote $T_i = T_0 + i\alpha = \{i\alpha, \alpha^0 + i\alpha, \alpha^{p+1} + i\alpha, \ldots, \alpha^{(p-2)(p+1)} + i\alpha\}$ for any $i \in \{1, 2, \ldots, p-1\}$. Then, according to Corollary 3, we have

$$\begin{aligned} T_i &= \{\alpha^{m_i \cdot (p+1)} \cdot \alpha, \alpha^0 + \alpha^{m_i \cdot (p+1)} \cdot \alpha, \alpha^{p+1} + \alpha^{m_i \cdot (p+1)} \cdot \alpha, \ldots, \alpha^{(p-2)(p+1)} + \alpha^{m_i \cdot (p+1)} \cdot \alpha\} \\ &= \alpha^{m_i \cdot (p+1)}\{T_0 + \alpha\}. \end{aligned} \tag{7}$$

According to the definition of τ and Q_i for any i in $\{0, 1, 2, \ldots, p-1\}$, we can have the following lemma.

Lemma 2. *If $T_i = \{\alpha^{c_{ij}} : j \in \mathbb{Z}_p\}$, then $Q_i = \{c_{ij} : j \in \mathbb{Z}_p\}$, where $i \in \{1, 2, \ldots, p-1\}$.*

In order to obtain more properties of the above PDFs, we will further investigate the t-apart difference of two blocks. Although the cyclic perfect Mendelsohn difference family (CPMDF) has been introduced and constructed by Fuji-Hara and Miao [32], Fan, Cai and Tang [33] and Xu et al. [34] to investigate the t-apart difference of two distinct elements lying in the same block, here, we still need to consider the t-apart difference of two distinct elements lying in the different blocks. Thus, we need the following definitions.

Definition 6. *Let G be an Abelian multiplicative group of order v. Both D_i and D_j are non-empty subsets of G. Then, the multiset*

$$\{x/y : x \in D_i, y \in D_j\}$$

is an ordered quotient multiset, which is denoted by ∂D_{ij}. In particular, when $i = j$, the ordered difference multiset is abbreviated as ∂D_i.

From the expression of T_i in (7), it is clear that ∂T_i is the same for any $i \in \{1, 2, \ldots, p-1\}$. Therefore, we have the following lemma.

Lemma 3. *For any $i, j \in \{1, 2, \ldots, p-1\}$, $\partial T_i = \partial T_j$.*

Definition 7. *Let G be an Abelian additive group of order v. Both D_i and D_j are non-empty subsets of G. Then the multiset*

$$\{x - y : x \in D_i, y \in D_j\}$$

is an ordered difference multiset which is denoted by ΔD_{ij}. In particular, when $i = j$, the ordered difference multiset is abbreviated as ΔD_i.

Combining Lemmas 2 and 3, Corollary 4 can be obtained directly.

Corollary 4. *For any $i, j \in \{1, 2, \ldots, p-1\}$, $\Delta Q_i = \Delta Q_j$.*

Lemma 4. *For any $i \in \{1, 2, \ldots, p-1\}$, $\Delta Q_i \uplus Q_0 = \mathbb{Z}_{p^2-1}$.*

Proof. Since $Q_0 = \{0, p+1, \ldots, (p-2)(p+1)\}$, then $Q_0 \cong \mathbb{Z}_{p-1}$. Thus, Q_0 is an Abelian additive group, which implies ΔQ_0 contains only the elements in Q_0, and Q_0 is a $(p, p, p-1)$ difference set. In addition, $\{Q_0, Q_1, \ldots, Q_{p-1}\}$ is a $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDF, which means for any $x \in \mathbb{Z}_{p^2-1} \backslash \{0\}$, x occurs $p-1$ times in $\uplus_{i=0}^{p-1} \Delta Q_i$. Therefore, for any $x \in \mathbb{Z}_{p^2-1} \backslash \{Q_0\}$, x occurs $p-1$ times in $\uplus_{i=1}^{p-1} \Delta Q_i$. Applying Corollary 4, we know that, for any $x \in \mathbb{Z}_{p^2-1} \backslash \{Q_0\}$ and $i \in \{1, 2, \ldots, p-1\}$, x occurs once in any ΔQ_i. □

Lemma 5. *Let i, j, s be positive integers; then, for any $i \neq j, 0 < i, j, s \leq p-1$, $\Delta Q_{i,j} = \Delta Q_s \uplus \{(m_i - m_j)(p+1)\}_p^*$.*

Proof. According to Lemma 3 and (9), for any $0 < s \leq p-1$, there exists $n_{ij} = m_i - m_j$, such that $\partial T_{i,j} \backslash \{\alpha^{(m_i - m_j)(p+1)}\}_p^* = \partial T_s \cdot \alpha^{n_{ij} \cdot (p+1)}$. By Lemma 2, we have

$$\Delta Q_{i,j} \backslash \{(m_i - m_j)(p+1)\}_p^* = \Delta Q_s + n_{ij}(p+1).$$

From Lemma 4, $\Delta Q_s = \mathbb{Z}_{p^2-1} \backslash Q_0$. Thus, with the fact that $Q_0 \subseteq \mathbb{Z}_{p^2-1}$ and $n_{ij}(p+1) \in Q_0$,

$$\begin{aligned}
\Delta Q_s + n_{ij}(p+1) &= \mathbb{Z}_{p^2-1} \backslash Q_0 + n_{ij}(p+1) \\
&= (\mathbb{Z}_{p^2-1} + n_{ij}(p+1)) \backslash (Q_0 + n_{ij}(p+1)) \\
&= \mathbb{Z}_{p^2-1} \backslash Q_0 \\
&= \Delta Q_s.
\end{aligned}$$

Thus, $\Delta Q_{i,j} \backslash \{(m_i - m_j)(p+1)\}_p^* = \Delta Q_s$. □

Lemma 6. *Let i be an integer; then, for any $0 < i \leq p-1$, $\mathbb{Z}_{p^2-1} = \Delta Q_{i,0} \uplus Q_0$.*

Proof. Denote $T_{0'} = T_0 \backslash \{0\}$. By (7), for any i and $d \in T_{0'}$, there exists an integer $j \in \{1, 2, \ldots, p-1\}$ such that $d^{-1} T_i = T_j$, and as d runs over $T_{0'}$, j also runs over $\{1, 2, \ldots, p-1\}$. Then, for any $i \in \{1, 2, \ldots, p-1\}$, we have $\partial T_{i,0'} = T_1 \uplus T_2 \uplus \cdots \uplus T_{p-1}$. Applying Lemma 2, we have

$$\Delta Q_{i,0} = Q_1 \uplus Q_2 \uplus \cdots \uplus Q_{p-1}.$$

Therefore, $\mathbb{Z}_{p^2-1} = \Delta Q_{i,0} \uplus Q_0$ due to the fact that $\mathcal{P} = \{Q_0, Q_1, \ldots, Q_{p-1}\}$ is a partition of $\mathbb{Z}_{p^2-1}$. □

As the operations are well-defined on $\mathbb{Z}_{p^2-1}$, we have $Q_0 = \{0, p+1, \ldots, (p-2)(p+1)\} = -Q_0$. Thus, $\Delta Q_{0,i} = -\Delta Q_{i,0} = \mathbb{Z}_{p^2-1} \backslash Q_0$ according to Lemma 6, and then we can obtain the following corollary.

Corollary 5. *Let i be an integer; then, for any $0 < i \leq p-1$, $\mathbb{Z}_{p^2-1} = \Delta Q_{0,i} \uplus Q_0$.*

For any $s \in \{1, 2, \ldots, p-1\}$ and $r \in \{1, 2, \ldots, p-1\}$, their operations are well-defined on $\mathbb{Z}_p$. Then, by (6), we can have

$$sr = \alpha^{m_{sr}(p+1)}.$$

Moreover, let r be fixed and let s run over $\{1, 2, \ldots, p-1\}$. Then, we know that m_{sr} also runs over $\{0, 1, 2, \ldots, p-2\}$.

Lemma 7. *Let $C = \{\alpha^{(m_r - m_{2r})(p+1)}, \alpha^{(m_{2r} - m_{3r})(p+1)}, \ldots, \alpha^{(m_{(p-2)r} - m_{(p-1)r})(p+1)}\}$ for any fixed $r \in \{1, 2, \ldots, p-1\}$; then, $C = T_0 \backslash \{0, 1\}$.*

Proof. By (7), $T_0 \backslash \{0, 1\} = \{2, 3, \ldots, p-1\} = \{\alpha^{p+1}, \alpha^{2(p+1)}, \ldots, \alpha^{(p-2)(p+1)}\}$. Then, for any $i \in \{1, 2, \ldots, p-2\}$, we have

$$\alpha^{(m_i - m_{(i+1)r})(p+1)} \in T_0 \backslash \{0, 1\}.$$

Therefore, with the fact that $|C| = p-2$, we know that $C = T_0 \backslash \{0, 1\}$ if and only if any two elements in set C are not equal. Thus, in the following, proof by contradiction is used to prove this.

Suppose that any two elements in C are equal, i.e,

$$\alpha^{(m_{ir} - m_{(i+1)r})(p+1)} = \alpha^{(m_{jr} - m_{(j+1)r})(p+1)}, \tag{8}$$

where i and j are integers, $0 < i, j < p-1$ and $i \neq j$. Then, multiplying both sides of (8) by $\alpha^{(m_{(i+1)r} + m_{(j+1)r})(p+1)}$, we obtain

$$\alpha^{(m_{ir} + m_{(j+1)r})(p+1)} = \alpha^{(m_{jr} + m_{(i+1)r})(p+1)}. \tag{9}$$

By applying (6),

$$\alpha^{m_{ir}(p+1)} + \alpha^{m_{(j+1)r}(p+1)} = ir + (j+1)r = (i+j+1)r,$$

$$\alpha^{m_{jr}(p+1)} + \alpha^{m_{(i+1)r}(p+1)} = jr + (i+1)r = (i+j+1)r,$$

thus $\alpha^{m_{ir}(p+1)} + \alpha^{m_{(j+1)r}(p+1)} = \alpha^{m_{jr}(p+1)} + \alpha^{m_{(i+1)r}(p+1)}$. By squaring each side of the above equation and utilizing (9), we can obtain

$$[\alpha^{m_{ir}(p+1)}]^2 + [\alpha^{m_{(j+1)r}(p+1)}]^2 = [\alpha^{m_{jr}(p+1)}]^2 + [\alpha^{m_{(i+1)r}(p+1)}]^2.$$

Then, by (6), $(ir)^2 + [(j+1)r]^2 \equiv (jr)^2 + [(i+1)r]^2 \pmod{p}$. Therefore,

$$2(i-j) \equiv 0 \pmod{p},$$

which is in contradiction with $i \neq j$, $p > 2$ and p being a prime number, so any two elements in the set C are not equal. □

Corollary 6. *Denote $C_r = \{(m_r - m_{2r})(p+1), (m_{2r} - m_{3r})(p+1), \ldots, (m_{(p-2)r} - m_{(p-1)r})(p+1)\}$. Then, for any $r \in \{1, 2, \ldots, p-1\}$, we have $C_r = Q_0 \backslash \{0\}$.*

Proof. The conclusion follows immediately from Lemmas 2 and 7. □

Theorem 2. *Let p be a prime number and denote $\uplus^r \Delta Q = \Delta Q_{0,r} \uplus \Delta Q_{r,2r} \uplus \cdots \uplus \Delta Q_{(p-2)r,(p-1)r} \uplus \Delta Q_{(p-1)r,0}$, where $r \in \{1, 2, \ldots, p-1\}$. Then, we have $\uplus^r \Delta Q = \left\{\mathbb{Z}_{p^2-1} \backslash \{0\}\right\}_p^*$ for any $r \in \{1, 2, \ldots, p-1\}$.*

Proof. The case $p = 2$ is trivial. When $p > 2$, applying Corollary 2, Lemma 5 and Corollary 6, for any $s \in \{1, 2, \ldots, p-1\}$, we have

$$\uplus^r \Delta Q = \{Q_0 \backslash \{0\}\}_p^* \uplus \{\Delta Q_s\}_{p-2}^* \Big(\Delta Q_{0,r} \uplus \Delta Q_{(p-1)r,0}\Big).$$

Then, by utilizing Lemma 6 and Corollary 5, we have $\Delta Q_{0,r} = \Delta Q_{(p-1)r,0} = \mathbb{Z}_{p^2-1} \backslash Q_0$. Thus,

$$\uplus^r \Delta Q = \{Q_0 \backslash \{0\}\}_p^* \uplus \{\Delta Q_s\}_{p-2}^* \uplus \{\mathbb{Z}_{p^2-1} \backslash Q_0\}_2^*.$$

According to Lemma 4, $\Delta Q_s = \mathbb{Z}_{p^2-1} \backslash Q_0$. Therefore,

$$\uplus^r \Delta Q = \{Q_0 \backslash \{0\}\}_p^* \uplus \{\mathbb{Z}_{p^2-1}\}_{p-2}^* \uplus \{\mathbb{Z}_{p^2-1} \backslash Q_0\}_2^* = \Big\{\mathbb{Z}_{p^2-1} \backslash \{0\}\Big\}_p^*.$$

□

4.2. The Construction of URWBDs

From the construction and theories of the $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDF demonstrated in Section 4.1, the $(p^3, \{p, p^2\}, p)$–URWBDs can be constructed according to the following method.

Let $\mathcal{P} = \{Q_0, Q_1, \ldots, Q_{p-1}\}$ be a $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDF over $\mathbb{Z}_{p^2-1}$, where p is a prime number. Denote $\Omega = \mathbb{Z}_{p^2-1} \times \mathbb{Z}_p$, and for any $b \in \mathbb{Z}_p$, $0 \le i, j \le p-1$, let

$$L_{Q_i,b} = \{(a, b) : a \in Q_i\}$$

and

$$\mathcal{L}_j = L_{Q_0,(0+j)} \cup L_{Q_1,(1+j)} \cup \ldots \cup L_{Q_{p-1},(p-1+j)}.$$

Then, we define a map φ_j:

$$\begin{aligned} \varphi_j &: \mathcal{P} \to \mathcal{L}_j \\ \varphi_j &: Q_i \mapsto L_{Q_i,(i+j)}. \end{aligned}$$

It is obvious that $\varphi_j(Q_i) = L_{Q_i,(i+j)}$, $\varphi_j^{-1}(L_{Q_i,(i+j)}) = Q_i$, and φ_j is a bijection. Let l be an integer and the point set $V = \{\mathbb{Z}_{p^2-1} \times \mathbb{Z}_p, \infty_0, \infty_1, \ldots, \infty_{(p-1)}\}$; then define

$$\begin{aligned} B_+ &= \Big\{\{\infty_0, \infty_1, \ldots, \infty_{(p-1)}\}\Big\} \cup \Big\{\{(a,0), (a,1), \ldots, (a,p)\} : a \in \mathbb{Z}_{p^2-1}\Big\}, \\ B_- &= \Big\{\{\infty_j, (0,j), (1,j), \ldots, (p^2-1, j)\} : j \in \mathbb{Z}_p\Big\}, \\ B_l &= \Big\{\Big\{\infty_j \cup L_{(Q_0+l),(0+j)} \cup L_{(Q_1+l)m,(1+j)} \cup \cdots \cup L_{(Q_{p-1}+l),(p-1+j)}\Big\} : j \in \mathbb{Z}_p\Big\}, \\ &\quad 0 \le l \le p^2-2. \end{aligned}$$

Clearly $|B_+| = p^2$ and the length of each block in B_+ is p. It is also easy to see that for any $0 \le l \le p^2-2$, $|B_-| = |B_l| = p$, and the length of every block in B_- or B_l is p^2. Moreover, B_+, B_- and B_l are all parallel classes partitioning V.

Theorem 3. *Let $\mathcal{B} = B_+ \cup B_- \cup \{B_l : l \in \mathbb{Z}_{p^2-1}\}$; then, $\mathcal{B}$ is a $(p^3, \{p, p^2\}, p)$–URWBD.*

Proof. Let m, n be all positive integers, where $0 \le m, n \le p-1$. Then, we consider four cases:

1. If the form of the point pair is (∞_m, ∞_n), then based on the construction above, for any $0 \le m, n \le p-1$, (∞_m, ∞_n) occurs once in the block B_+ and in no other blocks. Since the length of each block in B_+ is p, the sum of weights is $1 \cdot \frac{c}{p} = \frac{c}{p}$.

2. If the form of the point pair is $(\infty_m, (a, m))$, then for any $0 \le m \le p-1, 0 \le a \le p^2-2$, $(\infty_m, (a, m))$ occurs once in B_- and not in the block B_+. For each given $0 \le m \le p-1$, the point ∞_m occurs only in one block of each B_l in which the point (a, m) occurs only in $L_{(Q_0+l),(0+m)}$. Thus, as l traverses $\mathbb{Z}_{p^2-1}$, each point in $Q_0 + l$ can be equal to a when l takes a specific different value. Therefore, with $|Q_0 + l| = p-1$, $(\infty_m, (a, m))$ occurs $p-1$ times in $\cup_{l=0}^{p^2-2} B_l$. Since the length of every block in B_- or B_l is p^2, the sum of weights is $1 \cdot \frac{c}{p^2} + (p-1) \cdot \frac{c}{p^2} = \frac{c}{p}$.
3. If the form of the point pair is $(\infty_m, (a, m))$ and $m \ne b$, then for any $0 \le m, b \le p-1$, $0 \le a \le p^2-2$, neither B_+ nor B_- contains the point pair $(\infty_m, (a, m))$. For each given $0 \le m \le p-1$, the point ∞_m occurs only in one block of each B_l, and there exists only one integer $u \in \mathbb{Z}_p \backslash \{0\}$ such that $L_{(Q_u+l),b}$ in that block contains the point (a, b). Thus, as l traverses $\mathbb{Z}_{p^2-1}$, each point in $Q_u + l$ can be equal to a when l takes a specific different value. Since $|Q_u + l| = p$, the point pair $(\infty_m, (a, b))$ occurs p times in $\cup_{l=0}^{p^2-2} B_l$. Therefore, the sum of weights is $p \cdot \frac{c}{p^2} = \frac{c}{p}$.
4. When the form of the point pair is $((a_1, b_1), (a_2, b_2))$, where $a_1, a_2 \in \mathbb{Z}_{p^2-1}$ and $b_1, b_2 \in \mathbb{Z}_p$, the discussion is divided into three cases:
   (1) If $a_1 = a_2$ and $b_1 = b_2$, then $((a_1, b_1), (a_2, b_2))$ occurs once in B_+ and not in other blocks. with the length of each block in B_+ is p, the sum of weights is $1 \cdot \frac{c}{p} = \frac{c}{p}$.
   (2) If $a_1 \ne a_2$ and $b_1 = b_2$, then $((a_1, b_1), (a_2, b_2))$ occurs once in B_- and not in the block B_+. By the above construction of B_l, a_1 and a_2 are in the same $Q_i(0 \le i \le p-1)$. With the fact that $\{Q_0, Q_1, \ldots, Q_{p-1}\}$ is a $(p^2, \{p-1, p^{p-1}\}, p-1)$–PDF over $\mathbb{Z}_{p^2-1}$, for any $b_1 = b_2$, $d \equiv a_1 - a_2 (mod(p^2-1))$ occurs $p-1$ times. Thus, for any $a_1 \ne a_2$ and $b_1 = b_2$, $((a_1, b_1), (a_2, b_2))$ occurs $p-1$ times in $\cup_{l=0}^{p^2-2} B_l$. As the length of each block in B_- or B_l is p^2, the sum of weights is $1 \cdot \frac{c}{p^2} + (p-1) \cdot \frac{c}{p^2} = \frac{c}{p}$.
   (3) If $a_1 \ne a_2$ and $b_1 \ne b_2$, then $((a_1, b_1), (a_2, b_2))$ occurs only in B_l. By the above construction of B_l, a_1 and a_2 are not in the same $Q_i(0 \le i \le p-1)$. Therefore, by applying Theorem 3, for any $a_1 \ne a_2$ and $b_1 \ne b_2$, $d \equiv a_1 - a_2(mod(p^2-1))$ occurs p times in $\uplus^r \Delta Q$, which implies that $((a_1, b_1), (a_2, b_2))$ occurs p times in $\cup_{l=0}^{p^2-2} B_l$ based on the construction of B_l. Since the length of each block in B_l is p^2, the sum of weights is $p \cdot \frac{c}{p^2} = \frac{c}{p}$.

From the discussion above, $\mathcal{B}$ is a $(p^3, \{p, p^2\}, \frac{c}{p})$–URWBD. In addition, the natural weight $\omega(k) = \frac{c}{k}$ can be considered as $\frac{p^2}{k}$ on account of the fact that the value of c does not affect the nature of the design. Therefore, $\mathcal{B}$ is a $(p^3, \{p, p^2\}, p)$–URWBD. □

As an immediate consequence of the above construction and Theorem 3, we have the following corollary.

Corollary 7. *Let p be a prime number; then, for any p, there exists a* $(p^3, \{p, p^2\}, p)$*–URWBD.*

Example 4. *Take* $p = 3$*. Then, based on the* $(9, \{2, 3^2\}, 2)$*–PDF in Example 3, a* $(27, \{3, 9\}, 3)$*–URWBD is obtained through the above construction, and Example 1 is exactly constructed in this way.*

The following theorem can be obtained by applying Corollaries 1 and 7.

Theorem 4. *Let p be a prime number. Then, for any p, there exists a* $U(p^3; p^{(p^2)}(p^2)^1)$ *, which is a uniform design under the weighted discrete discrepancy.*

Example 5. *Take* $p = 3$*. A uniform design under the weighted discrete discrepancy* $U(27; 3^9 9^1)$ *(Table 1) is obtained from the* $(p^3, \{p, p^2\}, p)$*–URWBD in Example 4.*

5. Conclusions and Discussion

In this paper, based on a special kind of PDF constructed in Section 4.1, we propose and prove some of its properties, and then establish an equivalence between the PDFs and URWBDs with these properties. This equivalence allows us to obtain a class of URWBDs by way of the PDF. Moreover, the uniform design under the weighted discrete discrepancy is constructed by applying this kind of URWBD. However, the uniform designs in this paper are constructed when q is a prime number, so it is worthy of future research to consider the existence of uniform designs when q is a prime power.

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