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Algebraic Basis of the Algebra of All Symmetric Continuous Polynomials on the Cartesian Product of ℓ_p-Spaces

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Abstract: We construct a countable algebraic basis of the algebra of all symmetric continuous polynomials on the Cartesian product $\ell_{p_1} \times \ldots \times \ell_{p_n}$, where $p_1, \ldots, p_n \in [1, +\infty)$, and ℓ_p is the complex Banach space of all p-power summable sequences of complex numbers for $p \in [1, +\infty)$.

Keywords: symmetric polynomial on a Banach space; continuous polynomial on a Banach space; algebraic basis; space of p-summable sequences

MSC: 46G25; 47H60; 46B45; 46G20

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1. Introduction

For classical results on symmetric polynomials on finite dimensional spaces, we refer to [1–3]. Symmetric polynomials on infinite dimensional Banach spaces were studied, firstly, by Nemirovski and Semenov in [4]. In particular, in [4] the authors constructed a countable algebraic basis (see definition below) of the algebra of symmetric continuous real-valued polynomials on the real Banach space ℓ_p and a finite algebraic basis of the algebra of symmetric continuous real-valued polynomials on the real Banach space $L_p[0,1]$, where $1 \le p < +\infty$.

In [5], these results were generalized to separable sequence real Banach spaces with symmetric basis (see, e.g., ([6], Definition 3.a.1, p. 113) for the definition of a Banach space with symmetric basis) and to separable rearrangement invariant function real Banach spaces (see, e.g., ([7], Definition 2.a.1, p. 117) for the definition of a rearrangement invariant function Banach space) resp. In [8], it was shown that there are only trivial symmetric continuous polynomials on the space ℓ_∞. Consequently, the results of [5] cannot be generalized to nonseparable sequence Banach spaces. The most general approach to the studying of symmetric functions on Banach spaces was introduced in [9–13].

Note that the existence of a finite or countable algebraic basis in some algebra of symmetric continuous polynomials gives us the opportunity to obtain some information or, even, to describe spectra of topological algebras of symmetric holomorphic functions, which contain the algebra of symmetric continuous polynomials as a dense subalgebra. For example, in [14], the authors constructed an algebraic basis of the algebra of symmetric continuous complex-valued polynomials on the complex Banach space $L_\infty[0,1]$ of complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$.

This result gave us the opportunity to describe the spectrum of the Fréchet algebra $H_{bs}(L_\infty[0,1])$ of symmetric analytic entire functions, which are bounded on bounded sets on the complex Banach space $L_\infty[0,1]$ (see [14]) and to show that the algebra $H_{bs}(L_\infty[0,1])$ is isomorphic to the algebra of all analytic functions on the strong dual of the topological vector space of entire functions on the complex plane $\mathbb{C}$ (see [15]).

In [16,17], there were constructed algebraic bases of algebras of symmetric continuous polynomials on Cartesian powers of complex Banach spaces $L_p[0,1]$ and $L_p[0,+\infty)$ of all complex-valued Lebesgue integrable in a power p functions on $[0,1]$ and $[0,+\infty)$ resp., where $1 \leq p < +\infty$. These results gave us the opportunity to represent Fréchet algebras of symmetric entire analytic functions of bounded type on these Cartesian powers as Fréchet algebras of entire analytic functions on their spectra (see [18]).

The spectra of algebras with countable algebraic bases and completions of such algebras also were studied in [19–21]. Symmetric analytic functions of unbounded type were studied in [22–25]. Applications of symmetric analytic functions to the spectra of linear operators were introduced in [26].

Symmetric polynomials and symmetric holomorphic functions on spaces ℓ_p were studied by a number of authors [22,27–41] (see also the survey [42]). Symmetric polynomials and symmetric holomorphic functions on Cartesian powers of spaces ℓ_p were studied in [43–47]. In particular, in [46] there was constructed a countable algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on the Cartesian power of the complex Banach space ℓ_p. This result was generalized to the real case in [47]. In this work, we generalize the results of the work [46] to the algebra of symmetric continuous polynomials on the arbitrary Cartesian product $\ell_{p_1} \times \ldots \times \ell_{p_n}$.

2. Preliminaries

We denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the set of all positive integers and the set of all nonnegative integers resp.

2.1. Polynomials

Let X be a complex Banach space with norm $\|\cdot\|_X$. A function $P : X \to \mathbb{C}$ is called an N-homogeneous polynomial if there exist $N \in \mathbb{N}$ and an N-linear form $A_P : X^N \to \mathbb{C}$ such that P is the restriction of A_P to the diagonal, i.e.,

$$P(x) = A_P(\underbrace{x,\ldots,x}_{N})$$

for all $x \in X$.

A function $P : X \to \mathbb{C}$, which can be represented in the form

$$P = P_0 + P_1 + \ldots + P_N, \tag{1}$$

where P_0 is a constant function on X and $P_j : X \to \mathbb{C}$ is a j-homogeneous polynomial for every $j \in \{1,\ldots,N\}$, which is called a polynomial of degree at most N.

It is known that a polynomial $P : X \to \mathbb{C}$ is continuous if and only if its norm

$$\|P\| = \sup_{\|x\|_X \leq 1} |P(x)|$$

is finite. Consequently, if $P : X \to \mathbb{C}$ is a continuous N-homogeneous polynomial, then we have

$$|P(x)| \leq \|P\| \|x\|_X^N \tag{2}$$

for every $x \in X$.

For details on polynomials on Banach spaces, we refer the reader to [48] or [49,50].

2.2. Algebraic Combinations and Algebraic Basis

Let functions $f, f_1, \ldots, f_m$ act from T to $\mathbb{C}$, where T is an arbitrary nonempty set. If there exists a polynomial $Q : \mathbb{C}^m \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x), \ldots, f_m(x))$$

for every $x \in T$, then the function f is called an algebraic combination of functions $f_1, \ldots, f_m$. A set $\{f_1, \ldots, f_m\}$ is called algebraically independent if the fact that

$$Q(f_1(x), \ldots, f_m(x)) = 0$$

for all $x \in T$ implies that the polynomial Q is identically equal to zero. An infinite set of functions is called algebraically independent if every finite subset is algebraically independent. Note that the algebraic independence implies the uniqueness of the representation in the form of an algebraic combination.

Let $\mathcal{A}$ be an algebra of functions. A subset $\mathcal{B}$ of $\mathcal{A}$ is called an algebraic basis of $\mathcal{A}$ if each element of $\mathcal{A}$ can be uniquely represented as an algebraic combination of some elements of $\mathcal{B}$.

2.3. Symmetric Polynomials on the Space $c_{00}(\mathbb{C}^n)$

For $m \in \mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the space of all sequences $x = (x_1, \ldots, x_m, 0, \ldots)$, where $x_1, \ldots, x_m \in \mathbb{C}^n$ and $0 = (0, \ldots, 0) \in \mathbb{C}^n$. Note that $c_{00}(\mathbb{C}^n)$ is isomorphic to $(\mathbb{C}^n)^m$. Let $c_{00}(\mathbb{C}^n) = \bigcup_{m=1}^{\infty} c_{00}^{(m)}(\mathbb{C}^n)$.

A function $f : c_{00}(\mathbb{C}^n) \to \mathbb{C}$ is called symmetric if

$$f(x \circ \sigma) = f(x)$$

for every $x = (x_1, x_2, \ldots) \in c_{00}(\mathbb{C}^n)$ and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, where

$$x \circ \sigma = \Big(x_{\sigma(1)}, x_{\sigma(2)}, \ldots\Big).$$

For $\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$, let us define a polynomial $H_{\boldsymbol{k}} : c_{00}(\mathbb{C}^n) \to \mathbb{C}$ by

$$H_{\boldsymbol{k}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s > 0}}^{n} \Big(x_j^{(s)}\Big)^{k_s}, \tag{3}$$

where

$$x = \Big(\Big(x_1^{(1)}, \ldots, x_1^{(n)}\Big), \Big(x_2^{(1)}, \ldots, x_2^{(n)}\Big), \ldots\Big) \in c_{00}(\mathbb{C}^n).$$

Let M be a nonempty finite subset of $\mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$. Let $\mathbb{C}^M$ be the vector space of all functions $\xi : M \to \mathbb{C}$. Elements of the space $\mathbb{C}^M$ can be considered as $|M|$-dimensional complex vectors $\xi = (\xi_{\boldsymbol{k}})_{\boldsymbol{k} \in M}$, indexed by elements of M, where $|M|$ is the cardinality of M. Thus, $\mathbb{C}^M$ is isomorphic to $\mathbb{C}^{|M|}$. The space $\mathbb{C}^M$ we endow with norm $\|\xi\|_\infty = \max_{\boldsymbol{k} \in M} |\xi_{\boldsymbol{k}}|$, where $\xi = (\xi_{\boldsymbol{k}})_{\boldsymbol{k} \in M} \in \mathbb{C}^M$.

For a nonempty finite subset M of $\mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$, let us define a mapping $\pi_M : c_{00}(\mathbb{C}^n) \to \mathbb{C}^M$ by

$$\pi_M(x) = (H_{\boldsymbol{k}}(x))_{\boldsymbol{k} \in M}, \tag{4}$$

where $x \in c_{00}(\mathbb{C}^n)$.

Theorem 1 ([46], Theorem 9)**.** *Let $P : c_{00}(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric N-homogeneous polynomial. Let $M_N = \{\boldsymbol{k} \in \mathbb{Z}_+^n : 1 \le |\boldsymbol{k}| \le N\}$. There exists a polynomial $q : \mathbb{C}^{M_N} \to \mathbb{C}$ such that $P = q \circ \pi_{M_N}$, where the mapping π_{M_N} is defined by (4).*

We shall use the following lemma.

Lemma 1 ([46], Lemma 11)**.** *Let $K \subset \mathbb{C}^m$ and $\varkappa : K \to \mathbb{C}^{m-1}$ be an orthogonal projection: $\varkappa\big((x_1, x_2, \ldots, x_m)\big) = (x_2, \ldots, x_m)$. Let $K_1 = \varkappa(K)$, $\operatorname{int} K_1 \neq \emptyset$ and for every open set $U \subset K_1$ a set $\varkappa^{-1}(U)$ is unbounded. If polynomial $Q(x_1, \ldots, x_m)$ is bounded on K, then Q does not depend on x_1.*

3. The Main Result

Let $n \in \mathbb{N}$, $p_1, \ldots, p_n \in [1, +\infty)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$. We shall consider the Cartesian power $\ell_{p_1} \times \ldots \times \ell_{p_n}$ of the complex spaces $\ell_{p_1}, \ldots, \ell_{p_n}$ as the space of all sequences

$$x = (x_1, x_2, \ldots), \tag{5}$$

where $x_j = \left(x_j^{(1)}, \ldots, x_j^{(n)}\right) \in \mathbb{C}^n$ for $j \in \mathbb{N}$, such that the sequence $\left(x_1^{(s)}, x_2^{(s)}, \ldots\right)$ belongs to ℓ_{p_s} for every $s \in \{1, \ldots, n\}$. We endow $\ell_{p_1} \times \ldots \times \ell_{p_n}$ with norm

$$\|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}} = \left(\sum_{s=1}^{n} \left\| \left(x_1^{(s)}, x_2^{(s)}, \ldots \right) \right\|_{p_s}^{\max \boldsymbol{p}} \right)^{1/\max \boldsymbol{p}},$$

where $\|\cdot\|_{p_s}$ is the norm of the space ℓ_{p_s}. Note that $c_{00}(\mathbb{C}^n)$ is a dense subspace of $\ell_{p_1} \times \ldots \times \ell_{p_n}$.

Analogically to the definition of symmetric functions on $c_{00}(\mathbb{C}^n)$, a function $f : \ell_{p_1} \times \ldots \times \ell_{p_n} \to \mathbb{C}$ is called symmetric if

$$f(x \circ \sigma) = f(x)$$

for every $x = (x_1, x_2, \ldots) \in \ell_{p_1} \times \ldots \times \ell_{p_n}$ and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$, where

$$x \circ \sigma = \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots \right).$$

Let $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$ be such that $k_1/p_1 + \ldots + k_n/p_n \geq 1$. Let us define a polynomial $H_{\boldsymbol{p},\boldsymbol{k}} : \ell_{p_1} \times \ldots \times \ell_{p_n} \to \mathbb{C}$ by

$$H_{\boldsymbol{p},\boldsymbol{k}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s > 0}}^{n} \left(x_j^{(s)} \right)^{k_s}. \tag{6}$$

Note that the polynomial $H_{\boldsymbol{p},\boldsymbol{k}}$ is symmetric. Let us show that $H_{\boldsymbol{p},\boldsymbol{k}}$ is well-defined and continuous.

Lemma 2. *Let $\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$ be such that $k_1/p_1 + \ldots + k_n/p_n \geq 1$. Let $z^{(1)}, \ldots, z^{(n)} \in \mathbb{C}$ be such that $\left|z^{(1)}\right| \leq 1, \ldots, \left|z^{(n)}\right| \leq 1$. Then,*

$$\left|z^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|z^{(n)}\right|^{k_n} \leq \left|z^{(1)}\right|^{p_1} + \ldots + \left|z^{(n)}\right|^{p_n}.$$

Proof. Note that

$$\left|z^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|z^{(n)}\right|^{k_n} \leq \left(\left|z^{(1)}\right|^{p_1} \right)^{k_1/p_1} \cdot \ldots \cdot \left(\left|z^{(n)}\right|^{p_n} \right)^{k_n/p_n} \leq \left(\max_{1 \leq s \leq n} \left|z^{(s)}\right|^{p_s} \right)^{k_1/p_1 + \ldots + k_n/p_n}.$$

Since $\max_{1 \leq s \leq n} \left|z^{(s)}\right|^{p_s} \leq 1$, taking into account the inequality $k_1/p_1 + \ldots + k_n/p_n \geq 1$,

$$\left(\max_{1 \leq s \leq n} \left|z^{(s)}\right|^{p_s} \right)^{k_1/p_1 + \ldots + k_n/p_n} \leq \max_{1 \leq s \leq n} \left|z^{(s)}\right|^{p_s}.$$

Note that

$$\max_{1 \leq s \leq n} \left|z^{(s)}\right|^{p_s} \leq \left|z^{(1)}\right|^{p_1} + \ldots + \left|z^{(n)}\right|^{p_n}.$$

Thus,

$$\left|z^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|z^{(n)}\right|^{k_n} \leq \left|z^{(1)}\right|^{p_1} + \ldots + \left|z^{(n)}\right|^{p_n}.$$

This completes the proof. □

Proposition 1. *The polynomial $H_{\boldsymbol{p},\boldsymbol{k}}$, defined by (6), is well-defined and $\|H_{\boldsymbol{p},\boldsymbol{k}}\| \leq n$.*

Proof. Let us show that $H_{\boldsymbol{p},\boldsymbol{k}}$ is well-defined. Let $x \in \ell_{p_1} \times \ldots \times \ell_{p_n}$ be of the form (5). Since the series $\sum_{j=1}^{\infty} \left|x_j^{(1)}\right|^{p_1}, \ldots, \sum_{j=1}^{\infty} \left|x_j^{(n)}\right|^{p_n}$ are convergent, it follows that there exists $M \in \mathbb{N}$ such that $\left|x_j^{(1)}\right| \leq 1, \ldots, \left|x_j^{(n)}\right| \leq 1$ for every $j \geq M$. Therefore, for $j \geq M$, taking into account the inequality $k_1/p_1 + \ldots + k_n/p_n \geq 1$, by Lemma 2,

$$\left|x_j^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|x_j^{(n)}\right|^{k_n} \leq \left|x_j^{(1)}\right|^{p_1} + \ldots + \left|x_j^{(n)}\right|^{p_n}.$$

Consequently,

$$\begin{aligned}\sum_{j=M}^{\infty} \left|x_j^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|x_j^{(n)}\right|^{k_n} &\leq \sum_{j=1}^{\infty} \left(\left|x_j^{(1)}\right|^{p_1} + \ldots + \left|x_j^{(n)}\right|^{p_n} \right) \\ &= \left\| \left(x_1^{(1)}, x_2^{(1)}, \ldots\right) \right\|_{p_1}^{p_1} + \ldots + \left\| \left(x_1^{(n)}, x_2^{(n)}, \ldots\right) \right\|_{p_n}^{p_n} < \infty.\end{aligned}$$

Therefore, the series $\sum_{j=1}^{\infty} \prod_{s=1}^{n} \left(x_j^{(s)}\right)^{k_s}$ is absolutely convergent. Thus, $H_{\boldsymbol{p},\boldsymbol{k}}$ is well-defined.

Let us show that $\|H_{\boldsymbol{p},\boldsymbol{k}}\| \leq n$. Let $x \in \ell_{p_1} \times \ldots \times \ell_{p_n}$ be such that $\|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}} \leq 1$. Then, $\left|x_j^{(1)}\right| \leq 1, \ldots, \left|x_j^{(n)}\right| \leq 1$ for every $j \in \mathbb{N}$. Therefore, for every $j \in \mathbb{N}$, by Lemma 2,

$$\left|x_j^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|x_j^{(n)}\right|^{k_n} \leq \left|x_j^{(1)}\right|^{p_1} + \ldots + \left|x_j^{(n)}\right|^{p_n}.$$

Consequently,

$$\begin{aligned}|H_{\boldsymbol{p},\boldsymbol{k}}(x)| &\leq \sum_{j=1}^{\infty} \left|x_j^{(1)}\right|^{k_1} \cdot \ldots \cdot \left|x_j^{(n)}\right|^{k_n} \leq \sum_{j=1}^{\infty} \left(\left|x_j^{(1)}\right|^{p_1} + \ldots + \left|x_j^{(n)}\right|^{p_n} \right) \\ &= \left\| \left(x_1^{(1)}, x_2^{(1)}, \ldots\right) \right\|_{p_1}^{p_1} + \ldots + \left\| \left(x_1^{(n)}, x_2^{(n)}, \ldots\right) \right\|_{p_n}^{p_n} \\ &\leq \|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}^{p_1} + \ldots + \|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}^{p_n} \leq n.\end{aligned}$$

Thus, $\|H_{\boldsymbol{p},\boldsymbol{k}}\| \leq n$. This completes the proof. □

For arbitrary $x = (x_1, \ldots, x_m, 0, \ldots), y = (y_1, \ldots, y_s, 0, \ldots) \in c_{00}(\mathbb{C}^n)$, we set

$$x \oplus y = (x_1, \ldots, x_m, y_1, \ldots, y_s, 0, \ldots).$$

For $x^{(1)}, \ldots, x^{(r)} \in c_{00}(\mathbb{C}^n)$, let

$$\bigoplus_{j=1}^{r} x^{(j)} = x^{(1)} \oplus \ldots \oplus x^{(r)}.$$

Note that

$$\left\| \bigoplus_{j=1}^{r} x^{(j)} \right\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}^{\max \boldsymbol{p}} = \sum_{j=1}^{r} \left\| x^{(j)} \right\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}^{\max \boldsymbol{p}}. \tag{7}$$

Note that for every $\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}$,

$$H_{\boldsymbol{k}}\left(\bigoplus_{j=1}^{r} x^{(j)} \right) = \sum_{j=1}^{r} H_{\boldsymbol{k}}\left(x^{(j)}\right). \tag{8}$$

For every $m \in \mathbb{N}$ and $j \in \{1, \dots, m\}$, we set

$$\gamma_{mj} = \frac{1}{m^{1/m}} \exp(2\pi i j/m). \tag{9}$$

We set $\gamma_{01} = 0$. For $\boldsymbol{l} = (l_1, \dots, l_n) \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$, let

$$a_{\boldsymbol{l}} = \bigoplus_{j_1=1}^{\widehat{l_1}} \dots \bigoplus_{j_n=1}^{\widehat{l_n}} ((\gamma_{l_1 j_1}, \dots, \gamma_{l_n j_n}), (0, \dots, 0), \dots), \tag{10}$$

where $\widehat{l_j} = \max\{1, l_j\}$ for $j \in \{1, \dots, n\}$.

Let us define a partial order on $\mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$ in the following way. For $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$, we set $\boldsymbol{k} \succeq \boldsymbol{l}$ if and only if there exists $\boldsymbol{m} \in \mathbb{Z}_+^n$ such that $k_s = m_s l_s$ for every $s \in \{1, \dots, n\}$. We write $\boldsymbol{k} \succ \boldsymbol{l}$, if $\boldsymbol{k} \succeq \boldsymbol{l}$ and $\boldsymbol{k} \neq \boldsymbol{l}$.

By ([46], Proposition 3), for every $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$,

$$H_{\boldsymbol{k}}(a_{\boldsymbol{l}}) = \begin{cases} \prod\limits_{\substack{s=1 \\ k_s > 0}}^{n} \frac{1}{l_s^{k_s/l_s - 1}} \prod\limits_{\substack{s=1 \\ k_s = 0}}^{n} \widehat{l_s}, & \text{if} \quad \boldsymbol{k} \succeq \boldsymbol{l} \\ 0, & \text{otherwise} \end{cases}, \tag{11}$$

where by the definition, the product of an empty set of multipliers is equal to 1. In particular,

$$H_{\boldsymbol{k}}(a_{\boldsymbol{k}}) = 1. \tag{12}$$

For $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $x = \left(\left(x_1^{(1)}, \dots, x_1^{(n)} \right), \left(x_2^{(1)}, \dots, x_2^{(n)} \right), \dots \right) \in c_{00}(\mathbb{C}^n)$, let

$$(\lambda_1, \dots, \lambda_n) \odot x = \left(\left(\lambda_1 x_1^{(1)}, \dots, \lambda_n x_1^{(n)} \right), \left(\lambda_1 x_2^{(1)}, \dots, \lambda_n x_2^{(n)} \right), \dots \right).$$

It can be easily verified that

$$H_{\boldsymbol{k}}((\lambda_1, \dots, \lambda_n) \odot x) = H_{\boldsymbol{k}}(x) \prod_{\substack{s=1 \\ k_s > 0}}^{n} \lambda_s^{k_s}, \tag{13}$$

where $\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$. Note that

$$\begin{aligned} \|(\lambda_1, \dots, \lambda_n) \odot x\|_{\ell_{p_1} \times \dots \times \ell_{p_n}}^{\max \boldsymbol{p}} &= \sum_{s=1}^{n} \left\| \left(\lambda_s x_1^{(s)}, \lambda_s x_2^{(s)}, \dots \right) \right\|_{p_s}^{\max \boldsymbol{p}} \\ &= \sum_{s=1}^{n} |\lambda_s|^{\max \boldsymbol{p}} \left\| \left(x_1^{(s)}, x_2^{(s)}, \dots \right) \right\|_{p_s}^{\max \boldsymbol{p}} \le \sum_{s=1}^{n} |\lambda_s|^{\max \boldsymbol{p}} \left\| \left(x_1^{(s)}, x_2^{(s)}, \dots \right) \right\|_{1}^{\max \boldsymbol{p}} \\ &\le \sum_{s=1}^{n} |\lambda_s|^{\max \boldsymbol{p}} \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1}^{\max \boldsymbol{p}} = \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1}^{\max \boldsymbol{p}} \sum_{s=1}^{n} |\lambda_s|^{\max \boldsymbol{p}} \\ &\le \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1}^{\max \boldsymbol{p}} n \max_{s \in \{1 \dots, n\}} |\lambda_s|^{\max \boldsymbol{p}} = n \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1}^{\max \boldsymbol{p}} \left(\max_{s \in \{1 \dots, n\}} |\lambda_s| \right)^{\max \boldsymbol{p}}. \end{aligned} \tag{14}$$

For $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ let $\mathcal{V}(\boldsymbol{k}) = \{ s \in \{1, \dots, n\} : \ k_s \neq 0 \}$ and $\nu(\boldsymbol{k}) = |\mathcal{V}(\boldsymbol{k})|$.

Lemma 3. *Let $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$ be such that $\boldsymbol{l} \succeq \boldsymbol{k}$. If $k_s = 0$ for some $s \in \{1, \dots, n\}$, then $l_s = 0$. Consequently, $\mathcal{V}(\boldsymbol{l}) \subset \mathcal{V}(\boldsymbol{k})$.*

Proof. Since $\boldsymbol{l} \succeq \boldsymbol{k}$, there exists $m \in \mathbb{Z}_+^n$ such that $l_s = m_s k_s$ for every $s \in \{1, \dots, n\}$. Consequently, if $k_s = 0$, then $l_s = 0$.

If $s \in \mathcal{V}(\boldsymbol{l})$, then $l_s > 0$. Therefore, k_s cannot be equal to zero. Consequently, $s \in \mathcal{V}(\boldsymbol{k})$. Thus, $\mathcal{V}(\boldsymbol{l}) \subset \mathcal{V}(\boldsymbol{k})$. This completes the proof. $\square$

Lemma 4. *Let $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$ be such that $\boldsymbol{l} \succ \boldsymbol{k}$ and $\nu(\boldsymbol{l}) \geq \nu(\boldsymbol{k})$. Then,*

$$\sum_{s=1}^{n} \frac{l_s}{p_s} \geq \frac{1}{\max \boldsymbol{p}} + \sum_{s=1}^{n} \frac{k_s}{p_s}.$$

Proof. By Lemma 3, $\mathcal{V}(\boldsymbol{l}) \subset \mathcal{V}(\boldsymbol{k})$. On the other hand, $\nu(\boldsymbol{l}) \geq \nu(\boldsymbol{k})$, i.e., $|\mathcal{V}(\boldsymbol{l})| \geq |\mathcal{V}(\boldsymbol{k})|$. Therefore, $\mathcal{V}(\boldsymbol{l}) = \mathcal{V}(\boldsymbol{k})$. Consequently, $l_s = 0$ if and only if $k_s = 0$.

Since $\boldsymbol{l} \succ \boldsymbol{k}$, there exists $m \in \mathbb{Z}_+^n$ such that $l_s = m_s k_s$ for every $s \in \{1, \dots, n\}$. Since $l_s = 0$ if and only if $k_s = 0$, it follows that $m_s > 0$ for every $s \in \{1, \dots, n\}$ such that $k_s > 0$. Since $\boldsymbol{l} \neq \boldsymbol{k}$, there exists $s' \in \{1, \dots, n\}$ such that $m_{s'} \geq 2$. Consequently,

$$\sum_{s=1}^{n} \frac{l_s}{p_s} - \sum_{s=1}^{n} \frac{k_s}{p_s} = \sum_{s=1}^{n} \frac{m_s k_s}{p_s} - \sum_{s=1}^{n} \frac{k_s}{p_s} = \sum_{s=1}^{n} \frac{(m_s - 1)k_s}{p_s} \geq \frac{m_{s'} - 1}{p_{s'}} \geq \frac{1}{p_{s'}} \geq \frac{1}{\max \boldsymbol{p}}.$$

This completes the proof. $\square$

For $N \in \mathbb{N}$ and $J \in \{1, \dots, n\}$, let

$$\begin{aligned} M_N^{(J)} = \big\{ \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\} :\ & l_1/p_1 + \ldots + l_n/p_n < 1,\ |\boldsymbol{l}| \leq N \text{ and } \nu(\boldsymbol{l}) \geq J \big\} \\ & \cup \{ \boldsymbol{l} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\} :\ l_1/p_1 + \ldots + l_n/p_n \geq 1 \text{ and } |\boldsymbol{l}| \leq N \}. \end{aligned} \tag{15}$$

Note that ([46], Theorem 6) implies the following theorem.

Theorem 2. *Let M be a finite non-empty subset of $\mathbb{Z}_+^n \setminus \{(0, \overset{n}{\dots}, 0)\}$. Then,*

(i) there exists $m \in \mathbb{N}$ such that, for every $\xi = (\xi_{\boldsymbol{k}})_{\boldsymbol{k} \in M} \in \mathbb{C}^M$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi) = \xi$; and

(ii) there exists a constant $\rho_M > 0$ such that if $\|\xi\|_\infty < 1$, then $\|x_\xi\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1} < \rho_M$.

By Theorem 2, for $M = M_N^{(1)}$, there exists $\rho = \rho_M > 0$ such that $\pi_M(V'_\rho)$ contains the open unit ball of the space $\mathbb{C}^M$ with the norm $\|\cdot\|_\infty$, where

$$V'_\rho = \big\{ x \in c_{00}(\mathbb{C}^n) :\ \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1} < \rho \big\}.$$

Let

$$V_\rho = \big\{ x \in c_{00}(\mathbb{C}^n) :\ \|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}} < \rho \big\}. \tag{16}$$

Since $\|x\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}} \leq \|x\|_{\ell_1 \times \overset{n}{\dots} \times \ell_1}$ for every $x \in c_{00}(\mathbb{C}^n)$, it follows that $V_\rho \supset V'_\rho$. Consequently, $\pi_M(V_\rho)$ also contains the open unit ball of the space $\mathbb{C}^M$.

Proposition 2. *For $J \in \{1, \dots, n\}$, let $q\Big((\xi_{\boldsymbol{l}})_{\boldsymbol{l} \in M_N^{(J)}} \Big)$ be a polynomial on $\mathbb{C}^{M_N^{(J)}}$. If q is bounded on $\pi_{M_N^{(J)}}(V_\rho)$, then q does not depend on $\xi_{\boldsymbol{k}}$, where $\boldsymbol{k} \in M_N^{(J)}$ is such that $\nu(\boldsymbol{k}) = J$ and $k_1/p_1 + \ldots + k_n/p_n < 1$.*

Proof. Let $\boldsymbol{k} \in \mathbb{Z}_+^n$ be such that $\nu(\boldsymbol{k}) = J$ and $k_1/p_1 + \ldots + k_n/p_n < 1$. Let $K = \pi_{M_N^{(J)}}(V_\rho)$, $K_1 = \pi_{M_N^{(J)} \setminus \{\boldsymbol{k}\}}(V_\rho)$ and $\varkappa : K \to K_1$ be an orthogonal projection, defined by

$$\varkappa : (\xi_{\boldsymbol{l}})_{\boldsymbol{l} \in M_N^{(J)}} \mapsto (\xi_{\boldsymbol{l}})_{\boldsymbol{l} \in M_N^{(J)} \setminus \{\boldsymbol{k}\}}.$$

Let us show that, for every ball

$$B(u,r) = \{\xi \in \mathbb{C}^{M_N^{(J)} \setminus \{\boldsymbol{k}\}} : \|\xi - u\|_\infty < r\}$$

with center $u = (u_l)_{l \in M_N^{(J)} \setminus \{\boldsymbol{k}\}} \in \mathbb{C}^{M_N^{(J)} \setminus \{\boldsymbol{k}\}}$ and radius $r > 0$ such that $B(u,r) \subset \pi_{M_N^{(J)} \setminus \{\boldsymbol{k}\}}(V_\rho)$, a set $\varkappa^{-1}(B(u,r))$ is unbounded. Since $u \in \pi_{M_N^{(J)} \setminus \{\boldsymbol{k}\}}(V_\rho)$, there exists $x_u \in V_\rho$ such that $\pi_{M_N^{(J)} \setminus \{\boldsymbol{k}\}}(x_u) = u$. For $m \in \mathbb{N}$, we set

$$x_m = \bigoplus_{j=1}^{m} (h(j,k_1), \dots, h(j,k_n)) \odot a_{\boldsymbol{k}},$$

where $a_{\boldsymbol{k}}$ is defined by (10) and

$$h(j,s) = \left(\frac{1}{j}\right)^{\frac{1}{wp_s}},$$

for $j \in \mathbb{N}$ and $s \in \{1, \dots, n\}$, where

$$w = \frac{k_1}{p_1} + \dots + \frac{k_n}{p_n}.$$

Since $0 < w < 1$, it follows that $1/w > 1$. Consequently, the value $\zeta(1/w)$ is finite, where $\zeta(\cdot)$ is the Riemann zeta function. Choose ε such that

$$0 < \varepsilon < \min\Bigg\{1, \frac{\rho - \|x_u\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}}{\|a_{\boldsymbol{k}}\|_{\ell_1 \times \overset{n}{\ldots} \times \ell_1} (n\zeta(1/w))^{1/\max \boldsymbol{p}}},$$
$$r n^{-1} \left(\max\left\{\|a_{\boldsymbol{k}}\|_{\ell_{p_1} \times \ldots \times \ell_{p_n}}, \|a_{\boldsymbol{k}}\|^N_{\ell_{p_1} \times \ldots \times \ell_{p_n}}\right\}\right)^{-1} \left(\zeta\left(\min\left\{\frac{1}{w}, 1 + \frac{1}{w \max \boldsymbol{p}}\right\}\right)\right)^{-1}\Bigg\}.$$

Let $x_{m,\varepsilon} = (\varepsilon x_m) \oplus x_u$. Let us show that $x_{m,\varepsilon} \in V_\rho$. By (7),

$$\|x_m\|^{\max \boldsymbol{p}}_{\ell_{p_1} \times \ldots \times \ell_{p_n}} = \sum_{j=1}^{m} \|(h(j,1), \dots, h(j,n)) \odot a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_{p_1} \times \ldots \times \ell_{p_n}}.$$

By (14),

$$\|(h(j,1), \dots, h(j,n)) \odot a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_{p_1} \times \ldots \times \ell_{p_n}} \le n \|a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_1 \times \overset{n}{\ldots} \times \ell_1} \left(\max_{s \in \{1 \ldots, n\}} |h(j,s)|\right)^{\max \boldsymbol{p}}.$$

Note that

$$\max_{s \in \{1 \ldots, n\}} |h(j,s)| = \max_{s \in \{1 \ldots, n\}} \left(\frac{1}{j}\right)^{\frac{1}{wp_s}} = \left(\frac{1}{j}\right)^{\frac{1}{w \max \boldsymbol{p}}}.$$

Therefore,

$$\|(h(j,1), \dots, h(j,n)) \odot a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_{p_1} \times \ldots \times \ell_{p_n}} \le n \|a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_1 \times \overset{n}{\ldots} \times \ell_1} \left(\frac{1}{j}\right)^{\frac{1}{w}}.$$

Consequently,

$$\|x_m\|^{\max \boldsymbol{p}}_{\ell_{p_1}\times\ldots\times\ell_{p_n}} \le n\|a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_1\times\overset{n}{\ldots}\times\ell_1} \sum_{j=1}^{m}\left(\frac{1}{j}\right)^{\frac{1}{w}}$$
$$< n\|a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_1\times\overset{n}{\ldots}\times\ell_1} \sum_{j=1}^{\infty}\left(\frac{1}{j}\right)^{\frac{1}{w}} = n\|a_{\boldsymbol{k}}\|^{\max \boldsymbol{p}}_{\ell_1\times\overset{n}{\ldots}\times\ell_1}\zeta(1/w).$$

Therefore,

$$\|x_m\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}} < \|a_{\boldsymbol{k}}\|_{\ell_1\times\overset{n}{\ldots}\times\ell_1}(n\zeta(1/w))^{1/\max \boldsymbol{p}}.$$

By the triangle inequality,

$$\|x_{m,\varepsilon}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}} \le \varepsilon\|x_m\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}} + \|x_u\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}$$
$$< \varepsilon\|a_{\boldsymbol{k}}\|_{\ell_1\times\overset{n}{\ldots}\times\ell_1}(n\zeta(1/w))^{1/\max \boldsymbol{p}} + \|x_u\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}.$$

Since $\varepsilon < \frac{\rho-\|x_u\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}}{\|a_{\boldsymbol{k}}\|_{\ell_1\times\overset{n}{\ldots}\times\ell_1}(n\zeta(1/w))^{1/\max \boldsymbol{p}}}$, it follows that $\|x_{m,\varepsilon}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}} < \rho$. Hence, $x_{m,\varepsilon} \in V_\rho$. Note that, for arbitrary $\boldsymbol{l} \in \mathbb{Z}^n_+ \setminus \{(0,\overset{n}{\ldots},0)\}$, by (8),

$$H_{\boldsymbol{l}}(x_m) = \sum_{j=1}^{m} H_{\boldsymbol{l}}\big((h(j,1),\ldots,h(j,n))\odot a_{\boldsymbol{k}}\big).$$

By (13),

$$H_{\boldsymbol{l}}\big((h(j,1),\ldots,h(j,n))\odot a_{\boldsymbol{k}}\big) = H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\prod_{s=1}^{n} h(j,s)^{l_s}$$
$$= H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\prod_{s=1}^{n}\left(\frac{1}{j}\right)^{\frac{l_s}{wp_s}} = H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\left(\frac{1}{j}\right)^{\sum_{s=1}^{n} l_s/(wp_s)} = H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\left(\frac{1}{j}\right)^{\frac{1}{w}\sum_{s=1}^{n} l_s/p_s}.$$

Therefore,

$$H_{\boldsymbol{l}}(x_m) = H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\sum_{j=1}^{m}\left(\frac{1}{j}\right)^{\frac{1}{w}\sum_{s=1}^{n} l_s/p_s}.$$

Consequently, taking into account (8), we have

$$H_{\boldsymbol{l}}(x_{m,\varepsilon}) = \varepsilon^{|\boldsymbol{l}|}H_{\boldsymbol{l}}(x_m) + H_{\boldsymbol{l}}(x_u) = \varepsilon^{|\boldsymbol{l}|}H_{\boldsymbol{l}}(a_{\boldsymbol{k}})\sum_{j=1}^{m}\left(\frac{1}{j}\right)^{\frac{1}{w}\sum_{s=1}^{n} l_s/p_s} + H_{\boldsymbol{l}}(x_u). \quad (17)$$

Let us show that $\pi_{M_N^{(J)}\setminus\{\boldsymbol{k}\}}(x_{m,\varepsilon}) \in B(u,r)$. For $\boldsymbol{l} \in M_N^{(J)} \setminus \{\boldsymbol{k}\}$ such that $\boldsymbol{l} \not\succ \boldsymbol{k}$, by (11), $H_{\boldsymbol{l}}(a_{\boldsymbol{k}}) = 0$, therefore, by (17),

$$H_{\boldsymbol{l}}(x_{m,\varepsilon}) = u_{\boldsymbol{l}}.$$

Let $\boldsymbol{l} \in M_N^{(J)} \setminus \{\boldsymbol{k}\}$ be such that $\boldsymbol{l} \succ \boldsymbol{k}$. Consider the case $l_1/p_1 + \ldots + l_n/p_n \ge 1$ and $|\boldsymbol{l}| \le N$. Since $l_1/p_1 + \ldots + l_n/p_n \ge 1$, it follows that

$$\frac{1}{w}\sum_{s=1}^{n}\frac{l_s}{p_s} \ge \frac{1}{w}.$$

Consider the case $l_1/p_1 + \ldots + l_n/p_n < 1$, $|\boldsymbol{l}| \le N$ and $\nu(\boldsymbol{l}) \ge J$. Since $\nu(\boldsymbol{k}) = J$, it follows that $\nu(\boldsymbol{l}) \ge \nu(\boldsymbol{k})$. By Lemma 4, since $\boldsymbol{l} \succ \boldsymbol{k}$ and $\nu(\boldsymbol{l}) \ge \nu(\boldsymbol{k})$,

$$\frac{1}{w}\sum_{s=1}^{n}\frac{l_s}{p_s} \ge 1 + \frac{1}{w\max \boldsymbol{p}}.$$

Thus, for $\boldsymbol{l} \in M_N^{(J)} \setminus \{\boldsymbol{k}\}$ such that $\boldsymbol{l} \succ \boldsymbol{k}$, we have

$$\sum_{s=1}^{n} \frac{l_s}{p_s} \geq \min\left\{\frac{1}{w}, 1 + \frac{1}{w \max \boldsymbol{p}}\right\}. \tag{18}$$

By (17), taking into account the equality $H_{\boldsymbol{l}}(x_u) = u_{\boldsymbol{l}}$, we have

$$|H_{\boldsymbol{l}}(x_{m,\varepsilon}) - u_{\boldsymbol{l}}| \leq \varepsilon^{|\boldsymbol{l}|} |H_{\boldsymbol{l}}(a_{\boldsymbol{k}})| \sum_{j=1}^{m} \left(\frac{1}{j}\right)^{\frac{1}{w}\sum_{s=1}^{n} l_s/p_s}.$$

Since $\varepsilon < 1$, it follows that $\varepsilon^{|\boldsymbol{l}|} \leq \varepsilon$. By (2), taking into account the inequality $\|H_{\boldsymbol{l}}\| \leq n$, we have $|H_{\boldsymbol{l}}(a_{\boldsymbol{k}})| \leq n\|a_{\boldsymbol{k}}\|^{|\boldsymbol{l}|}_{\ell_{p_1}\times\ldots\times\ell_{p_n}}$. Since $1 \leq |\boldsymbol{l}| \leq N$, for every $b > 0$, we have $b^{|\boldsymbol{l}|} \leq \max\{b, b^N\}$. Therefore,

$$\|a_{\boldsymbol{k}}\|^{|\boldsymbol{l}|}_{\ell_{p_1}\times\ldots\times\ell_{p_n}} \leq \max\left\{\|a_{\boldsymbol{k}}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}, \|a_{\boldsymbol{k}}\|^{N}_{\ell_{p_1}\times\ldots\times\ell_{p_n}}\right\}.$$

Thus,

$$|H_{\boldsymbol{l}}(a_{\boldsymbol{k}})| \leq n \max\left\{\|a_{\boldsymbol{k}}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}, \|a_{\boldsymbol{k}}\|^{N}_{\ell_{p_1}\times\ldots\times\ell_{p_n}}\right\}.$$

By (18),

$$\sum_{j=1}^{m} \left(\frac{1}{j}\right)^{\frac{1}{w}\sum_{s=1}^{n} l_s/p_s} \leq \sum_{j=1}^{m} \left(\frac{1}{j}\right)^{\min\left\{\frac{1}{w}, 1+\frac{1}{w\max \boldsymbol{p}}\right\}} < \zeta\left(\min\left\{\frac{1}{w}, 1 + \frac{1}{w \max \boldsymbol{p}}\right\}\right).$$

Hence,

$$|H_{\boldsymbol{l}}(x_{m,\varepsilon}) - u_{\boldsymbol{l}}| < \varepsilon n \max\left\{\|a_{\boldsymbol{k}}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}, \|a_{\boldsymbol{k}}\|^{N}_{\ell_{p_1}\times\ldots\times\ell_{p_n}}\right\} \zeta\left(\min\left\{\frac{1}{w}, 1 + \frac{1}{w \max \boldsymbol{p}}\right\}\right).$$

Since

$$\varepsilon < r\left(n \max\left\{\|a_{\boldsymbol{k}}\|_{\ell_{p_1}\times\ldots\times\ell_{p_n}}, \|a_{\boldsymbol{k}}\|^{N}_{\ell_{p_1}\times\ldots\times\ell_{p_n}}\right\} \zeta\left(\min\left\{\frac{1}{w}, 1 + \frac{1}{w \max \boldsymbol{p}}\right\}\right)\right)^{-1},$$

it follows that $|H_{\boldsymbol{l}}(x_{m,\varepsilon}) - u_{\boldsymbol{l}}| < r$, therefore, $\pi_{M_N^{(J)}\setminus\{\boldsymbol{k}\}}(x_{m,\varepsilon}) \in B(u, r)$.

By (12), $H_{\boldsymbol{k}}(a_{\boldsymbol{k}}) = 1$, therefore, by (17),

$$H_{\boldsymbol{k}}(x_{m,\varepsilon}) = \varepsilon^{|\boldsymbol{k}|} \sum_{j=1}^{m} \frac{1}{j} + H_{\boldsymbol{k}}(x_u) \to \infty$$

as $m \to +\infty$. Hence, $\varkappa^{-1}(B(u,r))$ is unbounded. By Lemma 1, q does not depend on $\xi_{\boldsymbol{k}}$. This completes the proof. □

Theorem 3. *Let $P : \ell_{p_1} \times \ldots \times \ell_{p_n} \to \mathbb{C}$ be an N-homogeneous symmetric continuous polynomial. If $N < \min \boldsymbol{p}$, then $P \equiv 0$. Otherwise, there exists the polynomial $\hat{q} : \mathbb{C}^{M_{\boldsymbol{p},N}} \to \mathbb{C}$ such that $P = \hat{q} \circ \pi^{(\boldsymbol{p})}_{M_{\boldsymbol{p},N}}$, where*

$$M_{\boldsymbol{p},N} = \left\{\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\} : \ k_1/p_1 + \ldots + k_n/p_n \geq 1 \text{ and } |\boldsymbol{k}| \leq N\right\}$$

and $\pi^{(\boldsymbol{p})}_{M_{\boldsymbol{p},N}} : \ell_{p_1} \times \ldots \times \ell_{p_n} \to \mathbb{C}^{M_{\boldsymbol{p},N}}$ is defined by $\pi^{(\boldsymbol{p})}_{M_{\boldsymbol{p},N}}(x) = (H_{\boldsymbol{p},\boldsymbol{k}}(x))_{\boldsymbol{k}\in M_{\boldsymbol{p},N}}$.

Proof. Let $\tilde{P}$ be the restriction of P to $c_{00}(\mathbb{C}^n)$. Note that $\tilde{P}$ is a symmetric N-homogeneous polynomial. By Theorem 1, there exists a unique polynomial $q : \mathbb{C}^{M_N} \to \mathbb{C}$ such that

$$\tilde{P} = q \circ \pi_{M_N}. \tag{19}$$

Since P is continuous, P is bounded on V_ρ, defined by (16). Consequently, $\tilde{P}$ is bounded on V_ρ. Therefore, q is bounded on $\pi_{M_N}(V_\rho)$. Note that $M_N = M_N^{(1)}$, where $M_N^{(1)}$ is defined by (15).

Let us prove that q does not depend on arguments $\xi_{\boldsymbol{k}}$ such that $k_1/p_1 + \ldots + k_n/p_n < 1$ by induction on $\nu(\boldsymbol{k})$. By Proposition 2, for $J = 1$, we have that $q\Big((\xi_k)_{k\in M_N}\Big)$ does not depend on arguments $\xi_{\boldsymbol{k}}$ such that $\nu(\boldsymbol{k}) = 1$ and $k_1/p_1 + \ldots + k_n/p_n < 1$. Suppose the statement holds for $\nu(k) \in \{1, \ldots, J-1\}$, where $J \in \{2, \ldots, n\}$, i.e., $q\Big((\xi_{\boldsymbol{k}})_{\boldsymbol{k}\in M_N}\Big)$ does not depend on arguments $\xi_{\boldsymbol{k}}$ such that $1 \le \nu(k) \le J-1$ and $k_1/p_1 + \ldots + k_n/p_n < 1$. Then, the restriction of q to $\mathbb{C}^{M_N^{(J)}}$, by Proposition 2, does not depend on $\xi_{\boldsymbol{k}}$ such that $\nu(\boldsymbol{k}) = J$ and $k_1/p_1 + \ldots + k_n/p_n < 1$. Hence, q does not depend on $\xi_{\boldsymbol{k}}$ such that $k_1/p_1 + \ldots + k_n/p_n < 1$.

Consider the case $N < \min \boldsymbol{p}$. In this case, $k_1/p_1 + \ldots + k_n/p_n < 1$ for every $\boldsymbol{k} \in M_N$. Consequently, q is constant. Therefore, taking into account (19), $\tilde{P}$ is constant. Since $\tilde{P}$ is an N-homogeneous polynomial, where $N > 0$, it follows that $\tilde{P}$ is identically equal to zero. By the continuity of P, taking into account that $\tilde{P}$ is the restriction of P to the dense subspace $c_{00}(\mathbb{C}^n)$ of the space $\ell_{p_1} \times \ldots \times \ell_{p_n}$, the polynomial P is identically equal to zero.

Consider the case $N \ge \min \boldsymbol{p}$. In this case, $M_{\boldsymbol{p},N} \neq \emptyset$. Since q does not depend on $\xi_{\boldsymbol{k}}$ such that $\boldsymbol{k} \in M_N \setminus M_{\boldsymbol{p},N}$, the equality (19) implies the following equality:

$$\tilde{P} = \hat{q} \circ \pi_{M_{\boldsymbol{p},N}}, \tag{20}$$

where $\hat{q}$ is the restriction of q to $\mathbb{C}^{M_{\boldsymbol{p},N}}$, which is the subspace of $\mathbb{C}^{M_N}$. Let us show that $P = \hat{q} \circ \pi^{(\boldsymbol{p})}_{M_{\boldsymbol{p},N}}$. Let $x \in \ell_{p_1} \times \ldots \times \ell_{p_n}$. Since $c_{00}(\mathbb{C}^n)$ is dense in $\ell_{p_1} \times \ldots \times \ell_{p_n}$, there exists the sequence $\{x_m\}_{m=1}^\infty \subset c_{00}(\mathbb{C}^n)$, which is convergent to x. Since $H_{\boldsymbol{p},\boldsymbol{k}}$ is continuous and $H_{\boldsymbol{k}}$ is the restriction of $H_{\boldsymbol{p},\boldsymbol{k}}$, it follows that $\lim_{m\to\infty} H_{\boldsymbol{k}}(x_m) = H_{\boldsymbol{p},\boldsymbol{k}}(x)$ for every $\boldsymbol{k} \in M_{\boldsymbol{p},N}$. Therefore, $\lim_{m\to\infty} \pi_{M_{\boldsymbol{p},N}}(x_m) = \pi^{(\boldsymbol{p})}_{M_{\boldsymbol{p},N}}(x)$. Since $\hat{q}$ is the polynomial on a finite dimensional space, it follows that $\hat{q}$ is continuous. Consequently, $\lim_{m\to\infty}\Big(\hat{q} \circ \pi_{M_{\boldsymbol{p},N}}\Big)(x_m) = \Big(\hat{q} \circ \pi_{M^{(\boldsymbol{p})}_{\boldsymbol{p},N}}\Big)(x)$. On the other hand, since P is continuous, taking into account (20), we have

$$\lim_{m\to\infty}\Big(\hat{q} \circ \pi_{M_{\boldsymbol{p},N}}\Big)(x_m) = \lim_{m\to\infty} \tilde{P}(x_m) = \lim_{m\to\infty} P(x_m) = P(x).$$

Therefore, $P(x) = \Big(\hat{q} \circ \pi_{M^{(\boldsymbol{p})}_{\boldsymbol{p},N}}\Big)(x)$. Thus, $P = \hat{q} \circ \pi_{M^{(\boldsymbol{p})}_{\boldsymbol{p},N}}$. This completes the proof. □

Proposition 3. *The set of polynomials*

$$\{H_{\boldsymbol{p},\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\} \text{ such that } k_1/p_1 + \ldots + k_n/p_n \ge 1\} \tag{21}$$

is algebraically independent.

Proof. By ([46], Theorem 10), the set of polynomials

$$\Big\{H_{(1,\overset{n}{\ldots},1),\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{(0, \overset{n}{\ldots}, 0)\}\Big\}$$

is an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on $\ell_1 \times \overset{n}{\ldots} \times \ell_1$. Consequently, this set of polynomials is algebraically independent. Since every subset of an algebraically independent set is algebraically independent, the set of polynomials

$$\left\{H_{(1,\overset{n}{\dots},1),\boldsymbol{k}}:\ \boldsymbol{k}\in\mathbb{Z}_+^n\setminus\{(0,\overset{n}{\dots},0)\}\ \textit{such that}\ k_1/p_1+\ldots+k_n/p_n\geq 1\right\}$$

is algebraically independent. Since $H_{(1,\overset{n}{\dots},1),\boldsymbol{k}}$ is the restriction of $H_{\boldsymbol{p},\boldsymbol{k}}$ for every $\boldsymbol{k}\in\mathbb{Z}_+^n\setminus\{(0,\overset{n}{\dots},0)\}$ such that $k_1/p_1+\ldots+k_n/p_n\geq 1$, it follows that the set (21) is algebraically independent. This completes the proof. □

Theorem 4. *The set of polynomials* (21) *is an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on* $\ell_{p_1}\times\ldots\times\ell_{p_n}$.

Proof. Let $P:\ell_{p_1}\times\ldots\times\ell_{p_n}\to\mathbb{C}$ be a symmetric continuous complex-valued polynomial of degree at most N, where $N\in\mathbb{Z}_+$. Then,

$$P=P_0+P_1+\ldots+P_N,$$

where $P_0\in\mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j\in\{1,\ldots,N\}$. By the Cauchy Integral Formula for holomorphic functions on Banach spaces (see, e.g., ([48], Corollary 7.3, p. 47)),

$$P_j(x)=\frac{1}{2\pi i}\int_{|t|=r}\frac{P(tx)}{t^{j+1}}\,dt$$

for every $j\in\{1,\ldots,N\}$, $x\in\ell_{p_1}\times\ldots\times\ell_{p_n}$ and $r>0$, where $t\in\mathbb{C}$. Consequently, P_j is symmetric and continuous for every $j\in\{1,\ldots,N\}$. Therefore, by Theorem 3, P_j can be represented as an algebraic combination of elements of the set (21) for every $j\in\{1,\ldots,N\}$.

Consequently, P can be represented as an algebraic combination of elements of the set (21). Since, by Proposition 3, the set (21) is algebraically independent, the above-mentioned representation of P as an algebraic combination of elements of (21) is unique. Thus, every symmetric continuous complex-valued polynomial on $\ell_{p_1}\times\ldots\times\ell_{p_n}$ can be uniquely represented as an algebraic combination of elements of the set (21). This completes the proof. □

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