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Products of K-Analytic Sets in Locally Compact Groups and Kuczma–Ger Classes

Taras Banakh [1,2,*], Iryna Banakh [3] and Eliza Jabłońska [4]

1 Katedra Matematyki, Jan Kochanowski University in Kielce, Uniwersytecka 7, 25-406 Kielce, Poland
2 Faculty of Mehcanics and Mathematics, Ivan Franko National University of Lviv, Universytetska 1, 79000 Lviv, Ukraine
3 Ya. Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine, Naukova 3b, 79060 Lviv, Ukraine; ibanakh@yahoo.com
4 Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland; elizajab@agh.edu.pl
* Correspondence: t.o.banakh@gmail.com

Abstract: We prove that for any K-analytic subsets A, B of a locally compact group X if the product AB has empty interior (and is meager) in X, then one of the sets A or B can be covered by countably many closed nowhere dense subsets (of Haar measure zero) in X. This implies that a K-analytic subset A of X can be covered by countably many closed Haar-null sets if the set $AAAA$ has an empty interior in X. It also implies that every non-open K-analytic subgroup of a locally compact group X can be covered by countably many closed Haar-null sets in X (for analytic subgroups of the real line this fact was proved by Laczkovich in 1998). Applying this result to the Kuczma–Ger classes, we prove that an additive function $f : X \to \mathbb{R}$ on a locally compact topological group X is continuous if and only if f is upper bounded on some K-analytic subset $A \subseteq X$ that cannot be covered by countably many closed Haar-null sets.

Keywords: K-analytic space; locally compact group; Haar measure; σ-ideal

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1. Introduction

By the classical Steinhaus Theorem [1], for any Lebesgue measurable subsets A of positive Lebesgue measure on the real line, the set $A - A$ is a neighborhood of zero in $\mathbb{R}$. In [2], Weil extended this result of Steinhaus to all locally compact topological groups proving that for any measurable subset A of positive Haar measure in a locally compact topological group X, the set AA^{-1} is a neighborhood of the identity in X. This result implies that any non-open Haar measurable subgroup of a locally compact topological group X has Haar measure zero.

A Baire category analogue of the Steinhaus–Weil Theorem was obtained by Ostrowski [3], Piccard [4] and Pettis [5], who proved that for any nonmeager subset A with the Baire property in a Baire topological group X, the set AA^{-1} is a neighborhood of the identity. This implies that any non-open subgroup with the Baire property in a Baire topological group X is meager in X. We recall that a subset A of a topological space X has the *Baire property* in X if there exists a Borel subset $B \subseteq X$ such that the symmetric difference $A \triangle B$ is meager in X.

Therefore, any non-open Borel subgroup of a locally compact group X is meager and has Haar measure zero. For analytic subgroups of the real line, this result was essentially improved by Laczkovich [6], who proved that every analytic subgroup of the real line can be covered by countably many closed sets of Lebesgue measure zero. In this paper, we generalize this result of Laczkovich to K-analytic subgroups of locally compact groups.

A Tychonoff space A is called *K-analytic* if A is a continuous image of a Lindelöf Čech-complete space. By a *locally compact group*, we understand a Tychonoff (equivalently, Hausdorff) locally compact topological group. A *Haar measure* on a locally compact group X is any non-trivial left-invariant inner regular Borel σ-additive measure on X such that any compact set in X has finite measure. It is well-known (see [7] or ([8] Chapter 44)) that any locally compact group X possesses a Haar measure, and two Haar measures on X differ by a positive real multiplier. A subset A of a locally compact group X is called *Haar-null* if it is contained in a Borel subset of Haar measure zero in X.

The main result of this paper is the following theorem.

Theorem 1. *Let A, B be two K-analytic subsets of a locally compact group X. If the set AB has empty interior in X, then A or B is contained in the union $\bigcup_{n\in\omega} C_n$ of closed nowhere dense sets in X. If, moreover, AB is meager in X, then the closed nowhere dense sets C_n can be chosen to have Haar measure zero.*

Theorem 1 will be proved in Section 3. Now, we discuss some applications of this theorem.

Corollary 1. *If a K-analytic subset A of a locally compact group X cannot be covered by countably many closed Haar-null sets in X, then the product AA is not meager in X and $AAAA$ has a nonempty interior in X.*

Corollary 2. *A K-analytic subsemigroup A of a locally compact group X has an empty interior in X if and only if A can be covered by countably many closed Haar-null sets in X.*

Corollary 3. *Each nonopen K-analytic subgroup of a locally compact group X can be covered by countably many closed Haar-null sets in X.*

Now we shall apply Corollary 1 to the problem of automatic continuity of additive real–valued functions on locally compact groups. A function $f : X \to \mathbb{R}$ on a group X is called *additive* if $f(xy) = f(x) + f(y)$ for any elements $x, y \in X$.

Corollary 4. *An additive function $f : X \to \mathbb{R}$ on a locally compact group X is continuous if and only if $\sup f[A] < \infty$ for some K-analytic set $A \subseteq X$, which cannot be covered by countably many closed Haar-null sets in X.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that $\sup f[A] < \infty$ for some K-analytic set $A \subseteq X$ that cannot be covered by countably many closed Haar-null sets in X. By Corollary 1, the set $AAAA$ contains some nonempty open set U. The additivity of f ensures that $\sup f[U] \le \sup f[AAAA] \le 4 \sup f[A] < \infty$. Choose any element $x \in U$ and any neighborhood $V \subseteq X$ of the identity such that $V = V^{-1}$ and $xV \subseteq U$. For every $v \in V$, we have $f(xv) = f(x) + f(v)$, and hence

$$f(v) = -f(x) + f(xv) \le -f(x) + \sup f[xV] \le -f(x) + \sup f[U] < \infty.$$

Then, $b \overset{\text{def}}{=} \sup f[V] < \infty$. The equality $V = V^{-1}$ implies that the upper bounded set $f[V] = f[V^{-1}] = -f[V]$ is symmetric and hence bounded in the real line.

Now, we are ready to prove that the additive function f is continuous. Given any $\varepsilon > 0$, find $n \in \mathbb{N}$ such that $n\varepsilon > b$. By the continuity of the multiplication in the topological group X, there exists a neighborhood $W = W^{-1}$ of the identity in X such that $W^n \subseteq V$. Observe that for every $w \in W$, we have $|nf(w)| = |f(w^n)| \le b$ and hence $|f(w)| \le \frac{1}{n} b < \varepsilon$, which means that f is continuous at the identity of X and hence f is continuous everywhere by the additivity of f. $\square$

Following Kuczma and Ger [9], for a topological group X, let us consider the class $\mathcal{C}_X$ (resp. $\mathcal{B}_X$) of all sets $T \subseteq X$ such that an additive function $f : X \to \mathbb{R}$ is continuous whenever the restriction $f{\restriction}_T$ is (upper) bounded. It is clear that $\mathcal{B}_X \subseteq \mathcal{C}_X$. On the other hand, $\mathcal{B}_X \neq \mathcal{C}_X$ if X admits a discontinuous additive function $f : X \to \mathbb{R}$. In this case the set $T \overset{\text{def}}{=} \{x \in X : f(x) \leq 0\} \in \mathcal{C}_X \setminus \mathcal{B}_X$ witnesses that $\mathcal{B}_X \neq \mathcal{C}_X$. Repeating the argument of the proof of Corollary 4 (see also the proofs of Lemma 9.2.1 and Theorem 9.2.5 in [10]), we can show that a subset A of a topological group X belongs to the Kuczma–Ger class $\mathcal{B}_X$ if for some $n \in \mathbb{N}$ the product A^n of n copies of the set A in X has nonempty interior in X. Reformulating Corollary 4 in terms of Kuczma–Ger classes, we obtain our last corollary of Theorem 1.

Corollary 5. *Let X be a locally compact group. Every K-analytic set $A \subseteq X$ that cannot be covered by countably many closed Haar-null sets in X belongs to the Kuczma–Ger classes $\mathcal{B}_X$ and $\mathcal{C}_X$.*

A subset A of a topological group X is called *Haar–null* if there exists a Borel set $B \supseteq A$ in X and a probability Radon measure μ on X such that $\mu(xBy) = 0$ for any $x, y \in X$. Haar-null sets were introduced by Christensen [11] who proved that a subset of a locally compact group is Haar-null if and only if its Haar measure is zero. For more information on Haar-null sets and their generalizations, see [12,13]. By ([12], Example 8.1), the Polish group $\mathbb{Z}^\omega$ contains a non-open Borel subgroup, which cannot be covered by countably many closed Haar-null sets in $\mathbb{Z}^\omega$. By [14], the countable product of lines contains a meager Borel linear subspace $L \subseteq \mathbb{R}^\omega$ that cannot be covered by countably many Haar-null sets in $\mathbb{R}^\omega$. Then, any discontinuous additive function $f : \mathbb{R}^\omega \to \mathbb{R}$ with $f[L] = \{0\}$ witnesses that the linear space L does not belong to the Kuczma–Ger class $\mathcal{C}_X$. Those examples show that Corollaries 1–5 cannot be generalized beyond the class of locally compact groups.

2. Preliminaries

In this section, we collect some known facts on K-analytic spaces and locally compact groups.

A Tychonoff topological space X is called

- *Čech-complete* if X is homeomorphic to a G_δ-subset of some compact Hausdorff space;
- *K-analytic* if $X = f[P]$ for some continuous function $f : P \to X$ defined on a Lindelöf Čech-complete space;
- *analytic* if $X = f[P]$ for some continuous function $f : P \to X$ defined on a Polish space P.

Theorem 2.6.1 in [15] implies that in the class of Tychonoff spaces, our definition of a K-analytic space is equivalent to the original definition (via upper semicontinuous compact-valued maps) given in [15]. In the following lemma, we collect some known properties of K-analytic spaces. The first two statements of this lemma are proved in Theorems 2.5.5 and 5.5.1 of [15], and the last statement follows from Corollary 2.9.4 in [15] and Theorem 12 in [16].

Lemma 1.

1. *The product of two K-analytic spaces is K-analytic.*
2. *Every separable metrizable K-analytic space is analytic.*
3. *Every K-analytic subspace of a Tychonoff space X has the Baire property in X.*

A subset A of a topological space X is *functionally open* (resp. *functionally closed*) if $A = f^{-1}[B]$ for some continuous map $f : X \to Y$ to a metrizable separable space Y and some open (resp. closed) subset B of Y. It is easy to see that a Hausdorff space is Tychonoff if and only if it has a base of the topology consisting of functionally open sets.

Lemma 2. *For any meager subset M of a Lindelöf locally compact group X, there exists a compact normal subgroup K of X such that the quotient group X/K is Polish and for the quotient homomorphism $q : X \to X/K$ the image $q[M]$ is a meager subset of the Polish group X/K.*

Proof. Find a sequence of closed nowhere dense subsets $(M_n)_{n\in\omega}$ in X such that $M \subseteq \bigcup_{n\in\omega} M_n$. Being Lindelöf and locally compact, the topological group X is σ-compact and hence has countable Souslin number, see ([17], 5.4.8). Using Kuratowski–Zorn Lemma, for every $n \in \omega$ choose a maximal family $\mathcal{U}_n$ of pairwise disjoint functionally open subsets of X that are contained in the open set $X \setminus M_n$. The countability of the Souslin number of X implies that the family $\mathcal{U}_n$ is countable, and hence its union $\bigcup \mathcal{U}_n$ is a functionally open set in X. By the maximality of $\mathcal{U}_n$, the set $\bigcup \mathcal{U}_n$ is dense in X, and hence the complement $F_n = X \setminus \bigcup \mathcal{U}_n$ is a nowhere dense functionally closed set in X containing the nowhere dense set M_n. By Theorem 8.1.6 and Lemma 8.1.2 in [17], there exists a normal compact G_δ-subgroup K of X such that $F_n = KF_n$ for all $n \in \omega$. Let X/K be the quotient topological group and $q : X \to X/K$ be the quotient homomorphism. The openness of q and nowhere density of the sets $F_n = KF_n$ in X imply the nowhere density of the sets $q[F_n]$ in the quotient group X/K. Then, $q[M] \subseteq \bigcup_{n\in\omega} q[F_n]$ is a meager subset of X/K.

By ([17], 3.1.23), the quotient group X/K is locally compact. Since K is a G_δ-set in X, the locally compact group X/K has countable (pseudo)character and hence is first-countable and metrizable by Birkhoff–Kakutani Theorem ([17], 3.3.12). Since X is Lindelöf, so is the quotient space X/K. Being Lindelöf and metrizable, the locally compact space X/K is separable and Polish. □

Lemma 3. *Let K be a compact normal subgroup of a locally compact group X and $q : X \to X/K$ be the quotient homomorphism. For any Haar-null set A in X/K, the pre-image $q^{-1}[A]$ is Haar-null in X.*

Proof. Let λ be a Haar measure on X. Consider the Borel measure μ on X/K, assigning to each Borel subset $B \subseteq X/K$ the number $\lambda(q^{-1}[B])$. It is easy to see that μ is a Haar measure on X/K. Then, for any Haar-null set $A \subseteq X/K$, the set $q^{-1}[A]$ has Haar measure $\lambda(q^{-1}[A]) = \mu(A) = 0$ and hence $q^{-1}[A]$ is Haar-null in X. □

The following general version of the Steinhaus Theorem VII in [1] was proved in [18].

Lemma 4. *If Borel subsets A, B of a locally compact Polish group X are not Haar-null, then their product AB has a nonempty interior in X.*

3. Proof of Theorem 1

Theorem 1 follows from Lemmas 6 and 7, proved in this section. In the proof of Lemma 6, we shall use the following lemma, whose proof goes along the lines of the proof of Theorem 3.2 in [12].

Lemma 5. *Let A_1, A_2 be analytic sets in a locally compact Polish group X. If A_1A_2 is meager in X, then for some $i \in \{1, 2\}$ the set A_i can be covered by countably many closed Haar-null sets in X.*

Proof. Let $\sigma\overline{\mathcal{N}}_X$ be the family of all subsets of X that can be covered by countably many closed Haar-null sets in X. To derive a contradiction, assume that the set A_1A_2 is meager in X, but for every $i \in \{1, 2\}$ the set A_i does not belong to the σ-ideal $\sigma\overline{\mathcal{N}}_X$. By our assumption, the set A_i is analytic and hence admits a surjective continuous map $f_i : P_i \to A_i$ defined on a Polish space P_i. Let $\mathcal{U}_i$ be the family of all open subsets $U \subseteq P_i$ such that $f_i[U] \in \sigma\overline{\mathcal{N}}_X$. Since the Polish space P_i is hereditarily Lindelöf, the union $\bigcup \mathcal{U}_i$ belongs to the family $\mathcal{U}_i$. By our assumption, $A_i \notin \sigma\overline{\mathcal{N}}_X$, which implies that the complement $F_i \overset{\text{def}}{=} P_i \setminus \bigcup \mathcal{U}_i$ is not empty. Moreover, the maximality of the family $\mathcal{U}_i$ guarantees that for every nonempty open set $U \subseteq F_i$ the image $f_i[U]$ does not belong to the family $\sigma\overline{\mathcal{N}}_X$.

Since the set A_1A_2 is meager in X, there exists an increasing sequence $(M_n)_{n\in\omega}$ of closed nowhere dense sets in X such that $A_1A_2 \subseteq \bigcup_{n\in\omega} M_n$. Consider the continuous map $p\colon X \times X \to X$, $p\colon (x,y) \mapsto xy$. Then, $\big(p^{-1}[M_n]\big)_{n\in\omega}$ is an increasing sequence of closed sets whose union contains the subset $A_1 \times A_2$ of the topological group $X \times X$. By the continuity of the maps f_1, f_2, for every $n \in \omega$ the set $E_n \overset{\text{def}}{=} \{(x,y) \in F_1 \times F_2\colon (f_1(x), f_2(y)) \in p^{-1}[M_n]\}$ is closed in the Polish space $F_1 \times F_2$. Since $F_1 \times F_2 = \bigcup_{n\in\omega} E_n$, we can apply the Baire Theorem and find $n \in \omega$ such that the closed set E_n has nonempty interior in the space $F_1 \times F_2$. Then, there exist nonempty open sets $U_1 \subseteq F_1$ and $U_2 \subseteq F_2$ such that $U_1 \times U_2 \subseteq E_n$. By the maximality of the families $\mathcal{U}_1, \mathcal{U}_2$, the sets $f_1[U_1]$ and $f_2[U_2]$ do not belong to the family $\sigma\overline{\mathcal{N}}_X$ and hence their closures $\overline{f_1[U_1]}$ and $\overline{f_2[U_2]}$ have a positive Haar measure in X. By Lemma 4, the product $\overline{f_1[U_1]} \cdot \overline{f_2[U_2]}$ has a nonempty interior in X. The inclusion $U_1 \times U_2 \subseteq E_n$ implies $f_1[U_1] \cdot f_2[U_2] \subseteq M_n$ and hence $\overline{f_1[U_1]} \cdot \overline{f_2[U_2]} \subseteq \overline{M_n} = M_n$ by the continuity of the map p. Then, the set M_n has a nonempty interior in X and is not nowhere dense in X, which is a desired contradiction completing the proof. □

Lemma 6. *Let A_1, A_2 be K-analytic sets in a locally compact group X. If A_1A_2 is meager in X, then for some $i \in \{1,2\}$ the set A_i can be covered by countably many closed Haar-null sets in X.*

Proof. Assume that the set A_1A_2 is meager in X. By Lemma 1 (1), the product $A_1 \times A_2$ of the K-analytic spaces A_1, A_2 is K-analytic and hence Lindelöf and so is the continuous image A_1A_2 of $A_1 \times A_2$. Then, A_1A_2 is contained in an open Lindelöf subgroup of the locally compact group X. Replacing X by this open subgroup, we can assume that the locally compact group X is Lindelöf. By Lemma 2, there exists a compact normal subgroup K in X such that the quotient group X/K is Polish and locally compact, and for the quotient homomorphism $q : X \to X/K$, the image $q[A_1A_2]$ is meager in X/K. By Lemma 1 (2), for every $i \in \{1,2\}$, the K-analytic subspace $B_i \overset{\text{def}}{=} q[A_i]$ of the Polish group X/K is analytic. Since the product $B_1B_2 = q[A_1A_2]$ is meager in the locally compact Polish group X/K, we can apply Lemma 5 and conclude that for some $i \in \{1,2\}$, the set $B_i = q[A_i]$ is contained in the union $\bigcup_{n\in\omega} H_n$ of closed Haar-null sets H_n in X/K. Then, the set A_i is contained in the countable union $\bigcup_{n\in\omega} q^{-1}[H_n]$ of closed Haar-null sets in the locally compact group X, see Lemma 3. □

Lemma 7. *For any nonmeager K-analytic sets A_1, A_2 in a topological group X, the product A_1A_2 has a nonempty interior in X.*

Proof. By Lemma 1 (3), the K-analytic sets A_1, A_2 have the Baire property in X. Then, for every $i \in \{1,2\}$, there exists a G_δ-set G_i in X such that $G_i \subseteq A_i$ and the complement $A_i \setminus G_i$ is meager in X. Since the set A_i is not meager in X, the set G_i is not meager in X. In particular, G_i is not nowhere dense in X. Then, there exists a nonempty open set $U_i \subseteq X$ such that $U_i \cap G_i$ is dense in U_i. We claim that $U_1U_2 \subseteq G_1G_2$. Indeed, for any $x \in U_1U_2$ the intersection $U = U_1 \cap xU_2^{-1}$ is a nonempty open set in X. It follows that $U \cap G_1$ and $U \cap xG_2^{-1}$ are dense G_δ-sets in U. The topological group X contains a non-meager subset, and hence is not meager. The topological homogeneity of X implies that every nonempty open set of X is not meager, and hence the space X is Baire and so is the open subspace U of X. Then, the intersection $(U \cap G_1) \cap (U \cap xG_2^{-1})$ is not empty and hence contains some point g. Then $g \in G_1 \cap xG_2^{-1}$ and hence $x \in gG_2 \subseteq G_1G_2$. Therefore, $\varnothing \neq U_1U_2 \subseteq G_1G_2 \subseteq A_1A_2$, which means that the set A_1A_2 has a nonempty interior in X. □

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