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A Non-Standard Finite Difference Scheme for a Diffusive HIV-1 Infection Model with Immune Response and Intracellular Delay

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Abstract: In this paper, we propose and study a diffusive HIV infection model with infected cells delay, virus mature delay, abstract function incidence rate and a virus diffusion term. By introducing the reproductive numbers for viral infection R_0 and for CTL immune response number R_1 , we show that R_0 and R_1 act as threshold parameter for the existence and stability of equilibria. If $R_0 \leq 1$, the infection-free equilibrium E_0 is globally asymptotically stable, and the viruses are cleared; if $R_1 \leq 1 < R_0$, the CTL-inactivated equilibrium E_1 is globally asymptotically stable, and the infection becomes chronic but without persistent CTL response; if $R_1 > 1$, the CTL-activated equilibrium E_2 is globally asymptotically stable, and the infection is chronic with persistent CTL response. Next, we study the dynamic of the discreted system of our model by using non-standard finite difference scheme. We find that the global stability of the equilibria of the continuous model and the discrete model is not always consistent. That is, if $R_0 \leq 1$, or $R_1 \leq 1 < R_0$, the global stability of the two kinds model is consistent. However, if $R_1 > 1$, the global stability of the two kinds model is not consistent. Finally, numerical simulations are carried out to illustrate the theoretical results and show the effects of diffusion factors on the time-delay virus model.

Keywords: basic reproduction number; equilibrium; global stability; immune response; nonstandard finite difference scheme; numerical simulation

MSC: 35C07; 35K55; 35K57; 92D25

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1. Introduction

In the past few years, host-virus dynamics models have been developed to explain the interactions between virus and target T cells, much attention has been given to the role of the immune response to human immunodeficiency virus (HIV) infection. Many different mechanisms of immune system, defenses against viral infections are of interest because lots of the diseases caused by them, e.g., hepatitis B and AIDS, are chronic and incurable [1,2]. With the new coronavirus epidemic rages around the world [3–5], virus dynamics has become a hot spot again. In the immune response mechanism in vivo for viral infections, the cytotoxic T lymphocyte (CTL) plays a particularly important role, therefore many authors have examined various CTL dynamics.

A virus must take over host cells and use them to replicate because it can not replicate on its own. HIV targets the $CD4^+T$ cells, often referred to as "helper" T cells, when it invades the body. These cells can be considered "messengers", or the command centres of the immune system. They send signal to other immune cells that an invader is to be fought. Once invaded by the viruses, these infected cells will cause a cytotoxic T-lymphocyte (CTL) response from the immune system. The immune response cells, or cytotoxic lymphocytes, respond to this message and set out to eliminate infection by killing infected cells. Through the lysis of the infected cells, the viruses are prevented from further replication [2]. The

CTL response is also striking in that it sometime does damage to the body when it tries to clear the virus. Over half the tissue damage caused by hepatitis is actually caused by the CTL response [1,6].

If the immune system is functioning normally, these components work together efficiently and an infection is eliminated quickly, causing only temporary discomfort to the host. However, over time HIV is able to deplete the population of $CD4^+T$ cells. What remains unknown is the exact mechanism by which this occurs, but several models have been suggested. For a variety of different hypotheses of how this occurs, we refer the reader to papers [7–9]. The natural killer cells may be fit to eliminate infection, but they are never deployed, which is the the impact of the depletion of $CD4^+T$ cells on the host. This then culminates in a clinical problem wherein the patient becomes vulnerable to infections that a healthy immune system would normally handle.

Quite a lot of mathematical models of HIV have been set up. The classical model is a system with three ordinary differential equations [10,11]. To better understanding the dynamics of these infections, many mathematical models have been proposed by using different kinds of differential equations, see [12–16] and references therein. For example, Yang et al. [15] studied the following model

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \lambda - d_1 T(x,t) - \beta_1 T(x,t)V(x,t), \\ \frac{\partial I(x,t)}{\partial t} = \beta_1 T(x,t)V(x,t) - d_2 I(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d\Delta V(x,t) + \gamma I(x,t) - d_3 V(x,t), \end{cases} \tag{1}$$

where $T(x,t), I(x,t)$ and $V(x,t)$ denote the densities of uninfected cells, infected cells and free virus cells at position x at time t, respectively. λ stands for the recruitment rate of the uninfected cells; β_1 is the virus-to-cell infection rate; d_1, d_2 and d_3 represent death rates of uninfected cells, infected cells and free viruses; γ stands for the recruitment rate for free viruses; d stands for the diffusion coefficient and Δ is the Laplacian operator.

To help the body heal, cytotoxic T-lymphocyte effectors ($CTLe$) of the immune system will remove the infected cells to prevent further viral replications. To model these extra dynamics, researchers have studied the model of viral interaction with CTL response [10,17]

$$\begin{cases} \dot{x} = \lambda - dx - \beta xy, \\ \dot{y} = \beta xy - ay - pyz, \\ \dot{z} = cyz - bz, \end{cases} \tag{2}$$

where variables x, y and z denote the density of the healthy cells, the infected cells, and the $CTLs$ populations, respectively. Healthy cells are produced at rate λ and their natural mortality is dx; these cells may come into contact with the virus and become infected cells at rate βxy, infected cells's natural mortality is ay, and they are removed by $CTLs$ at rate pyz; the CTL population increases at the rate cyz and they are removed at the rate bz.

In [18,19], researchers studied a mathematical model for HIV-1 infection with both intracellular delay and cell-mediated immune response:

$$\begin{cases} \frac{dx(t)}{dt} = \lambda - dx(t) - \beta xv, \\ \frac{dy(t)}{dt} = e^{-a\tau}\beta x(t-\tau)v(t-\tau) - ay(t) - py(t)z(t), \\ \frac{dv(t)}{dt} = ky(t) - \mu v(t), \\ \frac{dz(t)}{dt} = cy(t)z(t) - bz(t). \end{cases} \tag{3}$$

Researchers obtain the global stability of the infection-free equilibrium and give many conditions for the local stability of the two infection equilibria: one without $CTLs$ being activated and the other with. There are many references in the dynamics of HIV-1 infection with $CTLs$ response (see, e.g., [17,20–22] and the references therein).

However, there is no diffusion term and only one delay in (3). As we know, the virus is not stationary in space, the movement of the virus in space leads to the spatial spread of the disease, and mostly with general nonlinear incidence rate. Fickian diffusion can reasonably describe the spread of this virus in space and this diffusion process is often represented by the Laplace operator. Inspired by [16,23] , in this paper, we extend the classic model of virus dynamics to a diffusive infection model with intracellular delay and cell-mediated immune response, with two delays and general nonlinear incidence rate, as follows

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \lambda - d_1 T(x,t) - \beta_1 T(x,t) f\Big(V(x,t)\Big) - \beta_2 T(x,t) g\Big(I(x,t)\Big), \\ \frac{\partial I(x,t)}{\partial t} = e^{-\mu_1 \tau_1}\Big(\beta_1 T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) + \beta_2 T(x,t-\tau_1) g(I(x,t-\tau_1))\Big) \\ \qquad -d_2 I(x,t) - p_1 I(x,t) Z(x,t), \\ \frac{\partial V(x,t)}{\partial t} = D\Delta V(x,t) + p_2 e^{-\mu_2 \tau_2} I(x,t-\tau_2) - d_3 V(x,t), \\ \frac{\partial Z(x,t)}{\partial t} = qI(x,t)Z(x,t) - d_4 Z(x,t), \end{cases} \tag{4}$$

here $T(x,t), I(x,t), V(x,t)$ and $Z(x,t)$ stand for the densities of uninfected cells, infected cells, virus cells and $CTLs$ at position x at time t, respectively. λ and d_1 denote the natural produce and mortality rate of uninfected cells, and uninfected cells are infected with a rate β_2; and β_1 is the virus-to-cell infection rate; and β_1 is the virus-to-cell infection rate; the natural mortality rate of the infected cells are d_2 and are killed by CTL with a rate p_1 (Note that d_2 reflects the combined effects of natural death rate of uninfected cells, d_1, and any additional cytotoxic effects the virus may have); μ_1 represents the death rate for infected but not yet virus-producing cells, τ_1 represents the latent delay, i.e., the time period from being infected to becoming productive infected cells. Therefore, the probability of surviving from time $t-\tau_1$ to time t is $e^{-\mu_1\tau_1}$; the probability of survival of immature virus is denoted by $e^{-\mu_2\tau_2}$ and the average life time of an immature virus is $\frac{1}{\mu_2}$; where τ_2 represents the time necessary for the newly produced virus to become mature; D is the diffusion coefficient and Δ is the Laplacian operator; p_2 is the recruitment rate for free viruses. Virus particles are removed from the system at rate d_3; q stands for the CTL responsiveness and d_4 denotes decay rate for $CTLs$ in the absence of stimulation.

Here, the incidences are assumed to be the nonlinear responses to the concentrations of virus particles and infected cells, using the forms $\beta_1 Tf(V)$ and $\beta_2 Tg(I)$, where $f(V)$ and $g(I)$ are the force of infection by virus particles and infected cells and satisfy the following properties [24]:

$$f(0) = g(0) = 0, f'(V) > 0, g'(I) > 0, f''(V) \le 0, g''(I) \le 0. \tag{A_1}$$

It follows from (A_1) and the Mean Value Theorem that

$$f'(V)V \le f(V) \le f'(0)V, g'(I)I \le g(I) \le g'(0)I, for I, V \ge 0. \tag{A_2}$$

Epidemiologically, condition (A_1) implies that : (1) the disease cannot spread if there is no infection; (2) the incidences $\beta_1 Tf(V)$ and $\beta_2 Tg(I)$ become faster when the densities of the virus particles and infected cells increase; (3) the per capita infection rates by virus particles and infected cells will slow down as certain inhibiting effect since (A_2) implies that $(\frac{f(V)}{V})' \le 0$ and $(\frac{g(I)}{I})' \le 0$. The incidence rate with condition (A_1) contains the bilinear and the saturation incidences.

In this paper, we will consider the system (4) with initial conditions

$$\begin{array}{l} T(x,s) = \phi_1(x,s) \ge 0, I(x,s) = \phi_2(x,s) \ge 0, \\ V(x,s) = \phi_3(x,s) \ge 0, Z(x,s) = \phi_4(x,s) \ge 0, (x,s) \in \bar{\Omega} \times [-\tau, 0] \end{array} \tag{5}$$

and homogeneous Neumann boundary conditions

$$\frac{\partial V}{\partial n} = 0, t > 0, x \in \partial\Omega. \tag{6}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and Ω is a bounded domain in R^4 with smooth boundary $\partial\Omega$, and $\frac{\partial V}{\partial n}$ stands for the outward normal derivative on $\partial\Omega$.

Usually, the exact solution for a system as (1) is difficult or even impossible to be determined. Hence, researchers seek numerical ones instead. However, how to choose the proper discrete scheme so that the global dynamics of solutions of the corresponding continuous models can be efficiently preserved is still an open problem [25]. Mickens has made an attempt in this connection, by presenting a robust non-standard finite difference (NSFD)scheme [26], which has been widely employed in the study of different epidemic models [23,25–32]. For example, Yang et al. [30] applied the NSFD scheme to discretize system (1) and found that the dynamical behaviors of the discrete model are consistent with the original system. Motivated by the work of [23,25–32], we apply the NSFD scheme to discretize system (4) and obtain:

$$\begin{cases} \frac{T_{n+1}^m - T_n^m}{\Delta t} = \lambda - d_1 T_{n+1}^m - \beta_1 T_{n+1}^m f(V_n^m) - \beta_2 T_{n+1}^m g(I_n^m), \\ \frac{I_{n+1}^m - I_n^m}{\Delta t} = e^{-\mu_1 \tau_1}\Big(\beta_1 T_{n-m_1+1}^m f(V_{n-m_1}^m) + \beta_2 T_{n-m_1+1}^m g(I_{n-m_1}^m)\Big) \\ \qquad -d_2 I_{n+1}^m - p_1 I_{n+1}^m Z_n^m, \\ \frac{V_{n+1}^m - V_n^m}{\Delta t} = D\frac{V_{n+1}^{m+1} - 2V_{n+1}^m + V_{n+1}^{m-1}}{(\Delta x)^2} + p_2 e^{-\mu_2 \tau_2} I_{n-m_2+1}^m - d_3 V_{n+1}^m, \\ \frac{Z_{n+1}^m - Z_n^m}{\Delta t} = q I_{n+1}^m Z_n^m - d_4 Z_{n+1}^m. \end{cases} \tag{7}$$

Here, we assume that $x \in \Omega = [a, b]$, let $\Delta t > 0$ be the time step size and $\Delta x = \frac{b-a}{N}$ be the space step size with N a positive integer. Suppose that there exist two integers $m_1, m_2 \in \aleph$ with $\tau_1 = m_1\Delta t, \tau_2 = m_2\Delta t$. Denote the mesh grid point as $\{(x_m, t_n), m = 0, 1, 2, \cdots, N, n \in N\}$ with $x_m = a + m\Delta x$ and $t_n = n\Delta t$. At each point, we use approximations of $\Big(T(x_m, t_n), I(x_m, t_n), V(x_m, t_n), Z(x_m, t_n)\Big)$ by $(T_n^m, I_n^m, V_n^m, Z_n^m)$. We set all the approximation solutions at the time t_n by the $N+1$-dimensional vector $U_n = (U_n^0, U_n^1, \cdots, U_n^N)^T$, where $U_n^{(\cdot)} \in \{(T_n, I_n, V_n, Z_n)\}$ and the notation $(\cdot)^T$ is the transposition of a vector. $U_n \geq 0$ means that all components of a vector U_n are nonnegative. The discrete initial conditions of system (7) are given as

$$\begin{array}{l} T_s^m = \phi_1(x_m, t_s) \geq 0, I_s^m = \phi_2(x_m, t_s) \geq 0, \\ V_s^m = \phi_3(x_m, t_s) \geq 0, Z_s^m = \phi_4(x_m, t_s) \geq 0, \end{array} \tag{8}$$

for all $s = -l, -l+1, \cdots, 0,\ \ l = \max\{m_1, m_2\}$, and the discrete boundary conditions are

$$V_n^{-1} = V_n^0, V_n^N = V_n^{N+1}, for\ \ n \in \aleph.$$

The main purpose of this paper is to investigate the asymptotic stability of system (4) and (7). Another purpose of this paper is to discuss , whether the discretized system (7) that derived by using NSFD scheme can efficiently preserves the global asymptotic stability of the equilibria to the original system (4) or not.

The paper is organized as follows. In Section 2.1, the model is introduced, and, under some assumptions, positivity and boundedness properties of the solutions are proved by using nonlinear functional analysis methods. In Section 2.2, we consider the existence of infection-free equilibrium, CTL-inactivated equilibrium and infection equilibrium with immunity. In Section 2.3, by introducing the reproductive numbers for viral infection R_0 and for CTL immune response number R_1 , we show that R_0 and R_1 act as threshold parameter for the existence and stability of equilibria. If $R_0 \leq 1$, the infection-free equilibrium E_0 is globally asymptotically stable, and the viruses are cleared; if $R_1 \leq 1 < R_0$, the CTL-inactivated equilibrium E_1 is globally asymptotically stable, and the infection becomes chronic but without persistent CTL response; if $R_1 > 1$, the CTL-activated equilibrium E_2 is globally asymptotically stable, and the infection is chronic with persistent CTL response. In Section 3, we investigate the global dynamics of discrete system (7) correponding to the continuous system (4), by using nonstandard finite difference scheme. We find that the global stability of the equilibria of the continuous model and the discrete model is not always consistent. That is, if $R_0 \leq 1$, or $R_1 \leq 1 < R_0$, the global stability of the two kinds model is consistent. However, if $R_1 > 1$, the global stability of the two kinds model is not consistent. In Section 4, some numerical simulations are given to illustrate the theoretical results and show the effects of diffusion factors on the time-delay virus model. The paper ends with a discussion in Section 5.

2. Dynamical Behaviors of Continuous System

2.1. Positivity and Boundedness of Solutions

In order to study positivity and boundedness of solutions to system (4), we first introduce some notations.

Assume $X = C(\bar{\Omega}, \mathbb{R}^4)$ be the space of continuous functions from the topological space $\bar{\Omega}$ into the space $\mathbb{R}^4$. Let $C = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the usual supremum normal. $\phi \in C$ is defined by

$$\phi(x, s) = \phi(s)(x).$$

Define $x_t(s) = x(t+s), s \in [-\tau, 0]$, where $x(\cdot) : [-\tau, \sigma) \to X$ is a continuous function from $[0, \sigma)$ to C.

Theorem 1. *For any $\phi \in C$,*
(a) system (4)–(6) has a unique solution defined on $[0, +\infty)$; and
(b) the solution of (4)–(6) is nonnegative and bounded for all $t \geq 0$.

Proof. For any $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C$ and $x \in \bar{\Omega}$, assume

$$F = (F_1, F_2, F_3, F_4) : C \to X$$

by

$$\begin{cases} F_1(\phi)(x) = \lambda - d_1\phi_1(x,0) - \beta_1\phi_1(x,0)f\Big(\phi_3(x,0)\Big) - \beta_2\phi_1(x,0)g\Big(\phi_2(x,0)\Big), \\ F_2(\phi)(x) = e^{-\mu_1\tau_1}\Big[\beta_1\phi_1(x,-\tau_1)f\Big(\phi_3(x,-\tau_1)\Big) + \beta_2\phi_1(x,-\tau_1)g\Big(\phi_2(x,-\tau_1)\Big)\Big] \\ \qquad -d_2\phi_2(x,0) - p_1\phi_2(x,0)\phi_4(x,0), \\ F_3(\phi)(x) = p_2e^{-\mu_2\tau_2}k\phi_2(x,-\tau_2) - d_3\phi_3(x,0), \\ F_4(\phi)(x) = q\phi_2(x,0)\phi_4(x,0) - d_4\phi_4(x,0). \end{cases}$$

Then system (4)–(6) can be rewritten as following form

$$\begin{cases} U'(t) = AU + F(U_t), t > 0, \\ U(0) = \phi \in X, \end{cases} \tag{9}$$

where $U = (T, I, V, Z)^T$,$\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $AU = (0, 0, d\Delta v, 0)^T$. It is clear that the operator F is locally Lipschitz in space X. From [27,32–36], we conclude that system (9) has a unique local solution on $t \in [0, T_{max})$, where T_{max} is the maximal existence time for solution of system (4). In addition, it follows from 0 is a sub-solution of each equation of system (4) that $T(x,t) \geq 0, I(x,t) \geq 0, V(x,t) \geq 0, Z(x,t) \geq 0$.

Next, we prove the boundedness of solutions. Let

$$G_1(x,t) = e^{-\mu_1\tau_1}T(x, t-\tau_1) + I(x,t) + \frac{p_1}{q}Z(x,t),$$

then

$$\begin{aligned} \frac{\partial G_1(x,t)}{\partial t} &= \lambda e^{-\mu_1\tau_1} - d_1 e^{-\mu_1\tau_1} T(x,t-\tau_1) - d_2 I(x,t) - \frac{p_1 d_4}{q} Z(x,t) \\ &\leq \lambda - \widetilde{d} G_1(x,t), \end{aligned}$$

where $\widetilde{d} = min\{d_1, d_2, d_4\}$, then

$$G_1(x,t) \leq max\left\{\frac{\lambda}{\widetilde{d}}, max_{x\in\bar{\Omega}}\left\{e^{-\mu_1\tau_1}\phi_1(x,-\tau_1) + \phi_2(x,0) + \frac{p_1}{q}\phi_3(x,0)\right\}\right\} = \xi_1,$$

so $T(x,t)$,$I(x,t)$ and $Z(x,t)$ are bounded.

From the boundedness of $I(x,t)$ and system (4)–(6), $V(x,t)$ satisfies the following system

$$\begin{cases} \frac{\partial V}{\partial t} - D\Delta V \leq p_2 e^{-\mu_2\tau_2}\xi_1 - d_3 V, \\ \frac{\partial V}{\partial n} = 0, \\ V(x,0) = \phi_3(x,0) \geq 0. \end{cases} \tag{2}$$

Assume $V_1(t)$ be a solution to the ordinary differential equation

$$\begin{cases} \frac{dV_1}{dt} = p_2 e^{-\mu_2\tau_2}\xi_1 - d_3 V_1, \\ V_1(0) = max_{x\in\bar{\Omega}}\phi_3(x,0), \end{cases} \tag{3}$$

then

$$V_1(t) \leq max\left\{\frac{p_2 e^{-\mu_2\tau_2}\xi_1}{d_3}, max_{x\in\bar{\Omega}}\phi_3(x,0)\right\}, \quad \forall t \in [0, T_{max}).$$

It follows from the comparison principle [37] that $V(x,t) \leq V_1(t)$. Therefore

$$V(x,t) \leq max\left\{\frac{p_2 e^{-\mu_2\tau_2}\xi_1}{d_3}, max_{x\in\bar{\Omega}}\phi_3(x,0)\right\} = \xi_2, \quad \forall (x,t) \in \bar{\Omega} \times [0, T_{max}).$$

From the above, $T(x,t), I(x,t), V(x,t)$ and $Z(x,t)$ are bounded in $\bar{\Omega} \times [0, T_{max})$. Furthermore, it follows from the standard theory for semilinear parabolic systems [38] that $T_{max} = +\infty$. □

2.2. Existence of Equilibria

It is clear that system (4) always has an infection-free equilibrium

$$E_0 = (T_0, 0, 0, 0),$$

where $T_0 = \frac{\lambda}{d}$, corresponding to the maximal level of healthy CD_4^+T cells. It is the only biologically meaningful equilibrium if

$$R_0 = \frac{\lambda e^{-\mu_1\tau_1}(\beta_1 p_2 f'^{-\mu_2\tau_2} + \beta_2 d_3 g'(0))}{d_1 d_2 d_3} < 1,$$

where R_0 is basic reproduction number.

At an equilibrium of model (4), we have

$$\begin{cases} \lambda = d_1 T + \beta_1 T f(V) + \beta_2 T g(I), \\ e^{-\mu_1\tau_1}\Big(\beta_1 T f(V) + \beta_2 T g(I)\Big) = d_2 I + p_1 I Z, \\ p_2 e^{-\mu_2\tau_2} I = d_3 V, \\ q I Z = d_4 Z, \end{cases} \tag{4}$$

if $Z = 0$, then a short calculation

$$\lambda - d_1 T = \frac{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2} V, I = \frac{d_3 e^{\mu_2\tau_2}}{p_2} V,$$

which implies that in order to have $T \geq 0$ and $V > 0$ at an equilibrium, then $V \in (0, \frac{\lambda p_2}{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}]$. From the second equation of (4), we have

$$T = \frac{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2\Big(\beta_1 f(V) + \beta_2 g(\frac{d_3 e^{\mu_2\tau_2}}{p_2} V)\Big)} V,$$

then substituting T into the first equation of (4)

$$\lambda = \frac{d_1 d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2\Big(\beta_1 f(V) + \beta_2 g(\frac{d_3 e^{\mu_2\tau_2}}{p_2} V)\Big)} V + \frac{d_2 d_3 \mu e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2} V = H(V).$$

According to (A_2), for all $V > 0$, we have

$$\begin{aligned} H'(V) &= \frac{d_1 d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}\Big(\beta_1(f(V) - Vf'(V)) + \beta_2\Big(g(\frac{d_3 e^{\mu_2\tau_2}}{p_2} V) - \frac{d_3 e^{\mu_2\tau_2}}{p_2} V g'(\frac{d_3 e^{\mu_2\tau_2}}{p_2} V)\Big)\Big)}{p_2(\beta_1 f(V) + \beta_2 g(\frac{d_3 e^{\mu_2\tau_2}}{p_2} V))^2} \\ &+ \frac{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2} > 0, \end{aligned}$$

further, from (A_1)

$$\lim_{V\to 0^+} H(V) = \frac{d_1 d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}{p_2\beta_1 f'(0) + d_3\beta_2 e^{\mu_2\tau_2} g'(0)} = \frac{\lambda}{R_0},$$

and

$$H(\frac{\lambda p_2}{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}) = \lambda + \frac{\lambda d_1}{\beta_1 f\Big(\frac{\lambda p_2}{d_2 d_3 e^{\mu_1\tau_1+\mu_2\tau_2}}\Big) + \beta_2 g\Big(\frac{\lambda}{d_2 e^{\mu_1\tau_1}}\Big)} > \lambda,$$

this implies that there exists a CTL-inactivated equilibrium $E_1 = (T_1, I_1, V_1, 0)$ when $R_0 > 1$.

Define

$$R_1 = \frac{\lambda q e^{-\mu_1\tau_1}\Big(\beta_1 f(\frac{d_4 p_2 e^{-\mu_2\tau_2}}{d_3 q}) + \beta_2 g(\frac{d_4}{q})\Big)}{d_2 d_4\Big(d_1 + \beta_1 f(\frac{d_4 p_2 e^{-\mu_2\tau_2}}{d_3 q}) + \beta_2 g(\frac{d_4}{q})\Big)},$$

which stands for the immune response activation number and determines whether a persistent immune response can be established or not. If $Z \neq 0$, then from (4), we have

$$T_2 = \frac{\lambda}{d_1 + \beta_1 f(V_2) + \beta_2 g(I_2)}, \ \ I_2 = \frac{d_4}{q},$$

$$V_2 = \frac{d_4 p_2 e^{-\mu_2\tau_2}}{d_3 q}, \ \ Z_2 = \frac{d_2}{p_1}(R_1 - 1),$$

then, the infection equilibrium with immunity $E_2 = (T_2, I_2, V_2, Z_2)$ exists if $R_1 > 1$. From the above, we have the following result.

Lemma 1. *For system (4),*

(1) *if $R_0 < 1$, then there exists a unique infection-free equilibrium E_0 .*
(2) *if $R_1 \leq 1 < R_0$, then there exists a unique infection equilibrium without immunity E_1 besides E_0.*
(3) *if $R_1 > 1$, then there exists a unique infection equilibrium with immunity E_2 besides E_0 and E_1.*

2.3. Global Asymptotic Stability

In this section, we will investigate the global asymptotic stability of the system (4). Assume $\varphi(u) = u - 1 - \ln u$ for $u \in (0, +\infty)$, then $\varphi(x) \geq \varphi(1) = 0$.

Theorem 2. *For system (4), if $R_0 \leq 1$, the infection-free equilibrium E_0 is globally asymptotically stable.*

Proof. Define the Lyapunov function as follows

$$\begin{aligned} L_1 &= \int_\Omega \left\{ T_0\varphi(\frac{T}{T_0}) + e^{\mu_1\tau_1} I + \frac{\beta_1 p_2 T_0 e^{-\mu_2\tau_2} f'(0)}{R_0 d_3} \int_{t-\tau_2}^{t} I(s)ds + \frac{\beta_1 T_0 f'(0)}{R_0 d_3} V \right. \\ &+ \left. \frac{p_1 e^{\mu_1\tau_1}}{q} Z + \int_{t-\tau_1}^{t} \Big[\beta_1 T(s) f(V(s)) + \beta_2 T(s) g\Big(I(s)\Big)\Big] ds \right\} dx, \end{aligned}$$

then $L_1 \geq 0$, calculating $\frac{dL_1}{dt}$ along the solutions of system (4) and using $\lambda = d_1 T_0$, we have

$$\begin{aligned} \tfrac{dL_1}{dt} &= \int_\Omega \left\{ \Big(1 - \tfrac{T_0}{T(x,t)}\Big)\Big(d_1T_0 - d_1T(x,t) - \beta_1 T(x,t) f\Big(V(x,t)\Big)\Big) \right. \\ &- \beta_2 T(x,t) g\Big(I(x,t)\Big) + \beta_1 T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) \\ &+ \beta_2 T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big) - d_2 e^{\mu_1\tau_1} I(x,t) - p_1 e^{\mu_1\tau_1} I(x,t) Z(x,t) \\ &+ \tfrac{\beta_1 T_0 f'(0)}{R_0 d_3}\Big[D\Delta V(x,t) + p_2 e^{-\mu_2\tau_2} I(x,t-\tau_2) - d_3 V(x,t)\Big] \\ &+ \tfrac{p_1 e^{\mu_1\tau_1}}{q}\Big[qI(x,t)Z(x,t) - d_4 Z(x,t)\Big] + \beta_1 T(x,t) f\Big(V(x,t)\Big) \\ &+ \beta_2 T(x,t) g\Big(I(x,t)\Big) - \beta_1 T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) \\ &- \left. \beta_2 T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big) + \tfrac{\beta_1 p_2 T_0 e^{-\mu_2\tau_2} f'(0)}{R_0 d_3}\Big[I(x,t) - I(x,t-\tau_2)\Big] \right\} dx, \\ &= \int_\Omega \left\{ d_1 T_0 \Big(2 - \tfrac{T_0}{T(x,t)} - \tfrac{T(x,t)}{T_0}\Big) + \Big(\tfrac{T_0}{T(x,t)} - 1\Big)\Big[\beta_1 T(x,t) f\Big(V(x,t)\Big) \right. \\ &+ \beta_2 T(x,t) g\Big(I(x,t)\Big)\Big] - d_2 e^{\mu_1\tau_1} I(x,t) + \tfrac{\beta_1 T_0 f'(0)}{R_0 d_3} D\Delta V(x,t) \\ &+ \tfrac{\beta_1 T_0 f'(0) p_2 e^{-\mu_2\tau_2}}{R_0 d_3} I(x,t-\tau_2) - \tfrac{\beta_1 T_0 f'(0)}{R_0} V(x,t) - \tfrac{p_1 d_4 e^{\mu_1\tau_1}}{q} Z(x,t) \\ &+ \beta_1 T(x,t) f\Big(V(x,t)\Big) + \beta_2 T(x,t) g\Big(I(x,t)\Big) + \tfrac{\beta_1 T_0 f'(0) p_2 e^{-\mu_2\tau_2}}{R_0 d_3} I(x,t) \\ &- \left. \tfrac{\beta_1 T_0 f'(0) p_2 e^{-\mu_2\tau_2}}{R_0 d_3} I(x,t-\tau_2) \right\} dx, \end{aligned}$$

from $\int_\Omega \Delta V(x,t) dx = 0$ and condition (A_2) , we obtain

$$d_2 e^{\mu_1\tau_1} - \frac{\beta_1 p_2 T_0 f'^{-\mu_2\tau_2}}{R_0 d_3} = \frac{\beta_2 T_0 g'(0)}{R_0},$$

therefore

$$
\begin{aligned}
\frac{dL_1}{dt} &= \int_\Omega \left\{ d_1T_0\left[2-\frac{T_0}{T(x,t)}-\frac{T(x,t)}{T_0}\right]+\beta_1T_0f\Big(V(x,t)\Big)+\beta_2T_0g\Big(I(x,t)\Big)\right.\\
&- d_2e^{\mu_1\tau_1}I(x,t)-\frac{\beta_1T_0f'(0)}{R_0}V-\frac{p_1d_4e^{\mu_1\tau_1}}{q}Z(x,t)\\
&+ \left.\frac{\beta_1T_0f'(0)p_2e^{-\mu_2\tau_2}}{R_0d_3}I(x,t)\right\}dx.\\
&\le \int_\Omega \left\{ d_1T_0\left[2-\frac{T_0}{T(x,t)}-\frac{T(x,t)}{T_0}\right]+\frac{\beta_1T_0f'(0)}{R_0}V(x,t)(R_0-1)\right.\\
&+ \left.\frac{\beta_2T_0g'(0)}{R_0}I(x,t)(R_0-1)-\frac{d_4p_1e^{\mu_1\tau_1}}{q}Z(x,t)\right\}dx.
\end{aligned}
$$

It is follows from $R_0 \le 1$ that $\frac{dL_1}{dt} \le 0$. Furthermore, the largest invariant set of $\{\frac{dL_1}{dt}=0\}$ is the singleton $\{E_0\}$. Then, the classical LaSalle's invariance principle implies that E_0 is globally asymptotically stable. This completes the proof. □

Theorem 3. *For system (4), if $R_1 \le 1 < R_0$, the CTL-inactivated infection equilibrium E_1 is globally asymptotically stable.*

Proof. Define the Lyapunov function as follows

$$
\begin{aligned}
L_2 &= \int_\Omega \left\{ T_1\varphi\Big(\frac{T}{T_1}\Big)+e^{\mu_1\tau_1}I_1\varphi\Big(\frac{I}{I_1}\Big)+\frac{\beta_1T_1f(V_1)}{p_2e^{-\mu_2\tau_2}I_1}V_1\varphi\Big(\frac{V}{V_1}\Big)+\frac{p_1e^{\mu_1\tau_1}}{q}Z\right.\\
&+ \beta_1T_1f(V_1)\int_{t-\tau_1}^t\varphi\left(\frac{T(\theta)f\Big(V(\theta)\Big)}{T_1f(V_1)}\right)d\theta+\beta_2T_1g(I_1)\int_{t-\tau_1}^t\varphi\left(\frac{T(\theta)g\Big(I(\theta)\Big)}{T_1g(I_1)}\right)d\theta\\
&+ \left.\beta_1T_1f(V_1)\int_{t-\tau_2}^t\varphi\left(\frac{I(\theta)}{I_1}\right)d\theta\right\}dx.
\end{aligned}
$$

The Lyapunov derivative along system (4) is

$$
\begin{aligned}
\frac{dL_2}{dt} &= \int_\Omega \left\{ \left(1-\frac{T_1}{T(x,t)}\right)\left[\lambda-d_1T(x,t)-\beta_1T(x,t)f\Big(V(x,t)\Big)-\beta_2T(x,t)g\Big(I(x,t)\Big)\right.\right.\\
&+ \left(1-\frac{I_1}{I(x,t)}\right)\left[\beta_1T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)+\beta_2T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)\right]\\
&- d_2e^{\mu_1\tau_1}I(x,t)-p_1e^{\mu_1\tau_1}I(x,t)Z(x,t)]\\
&+ \frac{\beta_1T_1f(V_1)}{p_2e^{-\mu_2\tau_2}I_1}\left(1-\frac{V_1}{V(x,t)}\right)\left[D\Delta V(x,t)+p_2e^{-\mu_2\tau_2}I(x,t-\tau_2)-d_3V(x,t)\right]\\
&+ \frac{p_1e^{\mu_1\tau_1}}{q}\left[qI(x,t)Z(x,t)-d_4Z(x,t)\right]\\
&+ \beta_1T_1f(V_1)\left[\frac{T(x,t)f\Big(V(x,t)\Big)}{T_1f(V_1)}-\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T_1f(V_1)}\right.\\
&+ \left.\ln\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T(x,t)f\Big(V(x,t)\Big)}\right]+\beta_2T_1g(I_1)\left[\frac{T(x,t)g\Big(I(x,t)\Big)}{T_1g(I_1)}\right.\\
&- \left.\frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T_1g(I_1)}+\ln\frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T(x,t)g\Big(I(x,t)\Big)}\right]\\
&+ \left.\beta_1T_1f(V_1)\left[\frac{I(x,t)}{I_1}-\frac{I(x,t-\tau_2)}{I_1}+\ln\frac{I(x,t-\tau_2)}{I_1}\right]\right\}dx.
\end{aligned}
$$

According to the equilibrium conditions of E_1, that

$$\lambda = d_1T_1 + \beta_1T_1f(V_1) + \beta_2T_1g(I_1),$$

$$\beta_1T_1f(V_1) + \beta_2T_1g(I_1) = d_2e^{\mu_1\tau_1}I_1, p_2e^{-\mu_2\tau_2}I_1 = d_3V_1,$$

also recall $\int_\Omega \Delta V(x,t)dx = 0$ and $\int_\Omega \frac{\Delta V(x,t)}{V(x,t)}dx = \int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx$, we have

$$\begin{aligned}
\frac{dL_2}{dt} &= \int_\Omega \Bigg\{ d_1T_1\Big(1-\frac{T_1}{T(x,t)}\Big)\Big(1-\frac{T(x,t)}{T_1}\Big) + \Big(1-\frac{T_1}{T(x,t)}\Big)\Big[\beta_1T_1f(V_1) + \beta_2T_1g(I_1) \\
&- \beta_1T(x,t)f\Big(V(x,t)\Big) - \beta_2T(x,t)g\Big(I(x,t)\Big)\Big] \\
&+ \Big(1-\frac{I_1}{I(x,t)}\Big)\Big[\beta_1T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big) + \beta_2T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)\Big] \\
&- \Big(1-\frac{I_1}{I(x,t)}\Big)\frac{I(x,t)}{I_1}\Big(\beta_1T_1f(V_1) + \beta_2T_1g(I_1)\Big) \\
&- p_1e^{\mu_1\tau_1}I(x,t)Z(x,t) + p_1e^{\mu_1\tau_1}I_1Z(x,t) + \beta_1T_1f(V_1)\Big(1-\frac{V_1}{V(x,t)}\Big)\Big(\frac{I(x,t-\tau_2)}{I_1} \\
&- \frac{V(x,t)}{V_1}\Big) + p_1e^{\mu_1\tau_1}I(x,t)Z(x,t) - \frac{p_1e^{\mu_1\tau_1}d_4}{q}Z(x,t) \\
&+ \beta_1T_1f(V_1)\Big[\frac{T(x,t)f\Big(V(x,t)\Big)}{T_1f(V_1)} - \frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T_1f(V_1)} + \ln\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T(x,t)f\Big(V(x,t)\Big)}\Big] \\
&+ \beta_2T_1g(I_1)\Big[\frac{T(x,t)g\Big(I(x,t)\Big)}{T_1g(I_1)} - \frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T_1g(I_1)} + \ln\frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T(x,t)g\Big(I(x,t)\Big)}\Big] \\
&+ \beta_1T_1f(V_1)\Big[\frac{I(x,t)}{I_1} - \frac{I(x,t-\tau_1)}{I_1} + \ln\frac{I(x,t-\tau_2)}{I(x,t)}\Big]\Bigg\}dx \\
&- \frac{\beta_1T_1f(V_1)DV_1}{p_2I_1e^{-\mu_2\tau_2}}\int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx \\
&= \int_\Omega \Bigg\{ d_1T_1\Big(1-\frac{T_1}{T(x,t)}\Big)\Big(1-\frac{T(x,t)}{T_1}\Big) + \beta_1T_1f(V_1)\Big[3 - \frac{T_1}{T(x,t)} - \frac{V_1I(x,t-\tau_2)}{I_1V(x,t)} \\
&- \frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)I_1}{T_1f(V_1)I(x,t)} + \frac{f(V)}{f(V_1)} - \frac{V}{V_1} + \ln\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)I(x,t-\tau_2)\Big)}{T(x,t)f\Big(V(x,t)\Big)I(x,t)}\Big] \\
&+ \beta_2T_1g(I_1)\Big[2 - \frac{T_1}{T(x,t)} - \frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)I_1}{T_1g(I_1)I(x,t)} + \ln\frac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T(x,t)g\Big(I(x,t)\Big)} \\
&+ \frac{g(I)}{g(I_1)} - \frac{I(x,t)}{I_1}\Big] + p_1I_1e^{\mu_1\tau_1}Z(x,t) - \frac{p_1e^{\mu_1\tau_1}d_4}{q}Z(x,t)\Bigg\}dx \\
&- \frac{\beta_1T_1f(V_1)DV_1}{p_2I_1e^{-\mu_2\tau_2}}\int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx \\
&= \int_\Omega \Bigg\{ d_1T_1\Big(1-\frac{T_1}{T(x,t)}\Big)\Big(1-\frac{T(x,t)}{T_1}\Big) + \beta_1T_1f(V_1)\Big[-\varphi\Big(\frac{T_1}{T(x,t)}\Big) \\
&- \varphi\Big(\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)I_1}{T_1f(V_1)I(x,t)}\Big) - \varphi\Big(\frac{V_1I(x,t-\tau_2)}{I_1V(x,t)}\Big) + \frac{f(V(x,t))}{f(V_1)} - \frac{V(x,t)}{V_1} + \ln\frac{f(V_1)V(x,t)}{f\Big(V(x,t)\Big)V_1}\Big] \\
&+ \beta_2T_1g(I_1)\Big[-\varphi(\frac{T_1}{T(x,t)}) - \varphi\Big(\frac{T(x,t-\tau_1)g(I(x,t-\tau_1))I_1}{T_1g(I_1)I(x,t)}\Big) + \frac{g\Big(I(x,t)\Big)}{g(I_1)} - \frac{I(x,t)}{I_1} \\
&+ \ln\frac{g(I_1)I(x,t)}{g\Big(I(x,t)\Big)I_1}\Big] + p_1e^{\mu_1\tau_1}(I_1 - I_2)Z(x,t)\Bigg\}dx \\
&- \frac{\beta_1T_1f(V_1)DV_1}{p_2I_1e^{-\mu_2\tau_2}}\int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx \\
&= \int_\Omega\{d_1T_1\Big(1-\frac{T_1}{T(x,t)}\Big)(1-\frac{T(x,t)}{T_1}) + \beta_1T_1f(V_1)\Big[-\varphi(\frac{T_1}{T(x,t)}) - \varphi\Big(\frac{V_1I(x,t-\tau_2)}{I_1V(x,t)}\Big) \\
&- \varphi\Big(\frac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)I_1}{T_1f(V_1)I(x,t)}\Big) - \varphi\Big(\frac{f(V_1)V(x,t)}{f\Big(V(x,t)\Big)V_1}\Big) + \Big(\frac{f\Big(V(x,t)\Big)}{f(V_1)} - \frac{V(x,t)}{V_1}\Big)\Big(1-\frac{f(V_1)}{f(V(x,t))}\Big)\Big] \\
&+ \beta_2T_1g(I_1)\Big[-\varphi\Big(\frac{T_1}{T(x,t)}\Big) - \varphi\Big(\frac{T(x,t-\tau_1)g(I(x,t-\tau_1))I_1}{T_1g\Big(I_1I(x,t)\Big)}\Big) - \varphi\Big(\frac{g(I_1)I(x,t)}{g(I(x,t))I_1}\Big) \\
&+ \Big(\frac{g\Big(I(x,t)\Big)}{g(I_1)} - \frac{I(x,t)}{I_1}\Big)\Big(1-\frac{g(I_1)}{g(I(x,t))}\Big)\Big] + p_1e^{\mu_1\tau_1}(I_1 - I_2)Z(x,t)\}dx \\
&- \frac{\beta_1T_1f(V_1)DV_1}{p_2I_1e^{-\mu_2\tau_2}}\int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx.
\end{aligned}$$

It follows from (A_1) that

$$\Big(\frac{f\Big(V(x,t)\Big)}{f(V_1)}-\frac{V(x,t)}{V_1}\Big)\Big(1-\frac{f(V_1)}{f\Big(V(x,t)\Big)}\Big)\leq 0,$$

$$\Big(\frac{g\Big(I(x,t)\Big)}{g(I_1)}-\frac{I(x,t)}{I_1}\Big)\Big(1-\frac{g(I_1)}{g\Big(I(x,t)\Big)}\Big)\leq 0.$$

As $\varphi(u)\geq 0$ for $u>0$, similar to [23], $sgn(I_1-I_2)=sgn(R_1-1)$, then $\frac{dL_2}{dt}\leq 0$, therefore E_1 is stable, and $\frac{dL_2}{dt}=0$ holds if and only if $T(x,t)=T_1, I(x,t)=I_1, V(x,t)=V_1$ and $Z(x,t)=0$ when $R_1<1$, or $T(x,t)=T_1, I(x,t)=I_1, V(x,t)=V_1$ when $R_1=1$. The largest invariance set of $\{\frac{dL_2}{dt}=0\}$ is the singleton $\{E_1\}$. It follows from the classical LaSalle's invariance principle that E_1 is globally asymptotically stable when $R_1\leq 1<R_0$. This completes the proof. □

Theorem 4. *For system (4), if $R_1>1$, the interior equilibrium E_2 is globally asymptotically stable.*

Proof. Define the Lyapunov function as follows

$$\begin{aligned}
L_3 \;=\; & \int_\Omega\Bigg\{T_2\varphi(\frac{T}{T_2})+e^{\mu_1\tau_1}I_2\varphi(\frac{I}{I_2})+\frac{\beta_1T_2f(V_2)}{p_2I_2e^{-\mu_2\tau_2}}V_2\varphi(\frac{V}{V_2})\\
+\; & \frac{p_1e^{\mu_1\tau_1}}{q}Z_2\varphi(\frac{Z}{Z_2})+\beta_1T_2f(V_2)\int_{t-\tau_1}^{t}\varphi\Big(\frac{T(\theta)f(V(\theta))}{T_2f(V_2)}\Big)d\theta\\
+\; & \beta_2T_2g(I_2)\int_{t-\tau_1}^{t}\varphi\Big(\frac{T(\theta)g\Big(I(\theta)\Big)}{T_2g(I_2)}\Big)d\theta+\beta_1T_2f(V_2)\int_{t-\tau_2}^{t}\varphi\Big(\frac{I(\theta)}{I_2}\Big)d\theta\Bigg\}dx,
\end{aligned}$$

calculating $\frac{dL_3}{dt}$ along the solutions of system (4) , we have

$$\begin{aligned}
\frac{dL_3}{dt} \;=\; & \int_\Omega\Bigg\{\Big(1-\tfrac{T_2}{T(x,t)}\Big)\Big(\lambda-d_1T(x,t)-\beta_1T(x,t)f\Big(V(x,t)\Big)-\beta_2T(x,t)g\Big(I(x,t)\Big)\Big)\\
+\; & \Big(1-\tfrac{I_2}{I(x,t)}\Big)\Big[\beta_1T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)+\beta_2T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)\Big]\\
-\; & e^{\mu_1\tau_1}\Big(1-\tfrac{I_2}{I(x,t)}\Big)\Big[d_2I(x,t)-p_1I(x,t)Z(x,t)\Big]\\
+\; & \tfrac{\beta_1T_2f(V_2)}{p_2I_2e^{-\mu_2\tau_2}}\Big(1-\tfrac{V_2}{V(x,t)}\Big)\Big[\Delta V(x,t)+p_2e^{-\mu_2\tau_2}I(x,t-\tau_2)-d_3V(x,t)\Big]\\
+\; & \tfrac{p_1e^{\mu_1\tau_1}}{q}\Big(1-\tfrac{Z_2}{Z(x,t)}\Big)\Big[qI(x,t)Z(x,t)-d_4Z(x,t)\Big]\\
+\; & \beta_1T_2f(V_2)\Big[\tfrac{T(x,t)f\Big(V(x,t)\Big)}{T_2f(V_2)}-\tfrac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T_2f(V_2)}+\ln\tfrac{T(x,t-\tau_1)f\Big(V(x,t-\tau_1)\Big)}{T(x,t)f\Big(V(x,t)\Big)}\Big]\\
+\; & \beta_2T_2g(I_2)\Big[\tfrac{T(x,t)g\Big(I(x,t)\Big)}{T_2g(I_2)}-\tfrac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T_2g(I_2)}+\ln\tfrac{T(x,t-\tau_1)g\Big(I(x,t-\tau_1)\Big)}{T(x,t)g\Big(I(x,t)\Big)}\Big]\\
+\; & \beta_1T_2f(V_2)\Big[\tfrac{I(x,t)}{I_2}-\tfrac{I(x,t-\tau_2)}{I_2}+\ln\tfrac{I(x,t-\tau_2)}{I(x,t)}\Big]\Bigg\}dx,
\end{aligned}$$

using the equilibrium conditions of E_2, then

$$\lambda=d_1T_2+\beta_1T_2f(V_2)+\beta_2T_2g(I_2),\;\; p_2e^{-\mu_2\tau_2}I_2=d_3V_2,$$

$$\beta_1 T_2 f(V_2) + \beta_2 T_2 g(I_2) = e^{\mu_1\tau_1}(d_2 I_2 + p_1 I_2 Z_2), \ \ I_2 = \frac{d_4}{q},$$

also recall $\int_\Omega \Delta V(x,t)dx = 0$ and $\int_\Omega \frac{\Delta V(x,t)}{V(x,t)}dx = \int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx$, we have

$$\begin{aligned}
\frac{dL_3}{dt} &= \int_\Omega \Bigg\{ d_1 T_2\Big(1-\frac{T_2}{T(x,t)}\Big)\Big(1-\frac{T(x,t)}{T_2}\Big) + \beta_1 T_2 f(V_2)\Big[3 - \frac{T_2}{T(x,t)} - \frac{V_2 I(x,t-\tau_2)}{I_2 V(x,t)} \\
&- \frac{T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) I_2}{T_2 f(V_2) I(x,t)} + \frac{f\Big(V(x,t)\Big)}{f(V_2)} - \frac{V(x,t)}{V_2} + \ln\frac{T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) I(t-\tau_2)}{T(x,t) f(V(x,t)) I(x,t)}\Big] \\
&+ \beta_2 T_2 g(I_2)\Big[2 - \frac{T_2}{T(x,t)} - \frac{T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big) I_2}{T_2 g(I_2) I(x,t)} + \frac{g\Big(I(x,t)\Big)}{g(I_2)} - \frac{I(x,t)}{I_2} \\
&+ \ln\frac{T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big)}{T(x,t) g\Big(I(x,t)\Big)}\Big]\Bigg\} dx - \frac{\beta_1 T_2 f(V_2) D V_2}{p_2 I_2 e^{-\mu_2\tau_2}} \int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx \\
&= \int_\Omega \{ d_1 T_2\Big(1-\frac{T_2}{T(x,t)}\Big)(1-\frac{T(x,t)}{T_2}) + \beta_1 T_2 f(V_2)\Big[-\varphi(\frac{T_2}{T(x,t)}) - \varphi(\frac{V_2 I(x,t-\tau_2)}{I_2 V(x,t)}) \\
&- \varphi\Big(\frac{T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) I_2}{T_2 f(V_2) I(x,t)}\Big) + \frac{f\Big(V(x,t)\Big)}{f(V_2)} - \frac{V(x,t)}{V_2} + \ln\frac{f(V_2)V(x,t)}{V_2 f(V(x,t))}\Big] \\
&+ \beta_2 T_2 g(I_2)\Big[-\varphi(\frac{T_2}{T(x,t)}) - \varphi\Big(\frac{T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big) I_2}{T_2 g(I_2) I(x,t)}\Big) \\
&+ \frac{g\Big(I(x,t)\Big)}{g(I_2)} - \frac{I(x,t)}{I_2} + \ln\frac{g(I_2) I(x,t)}{I_2 g\Big(I(x,t)\Big)}\Big]\} dx - \frac{\beta_1 T_2 f(V_2) D V_2}{p_2 I_2 e^{-\mu_2\tau_2}} \int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx \\
&= \int_\Omega \{ d_1 T_2\Big(1-\frac{T_2}{T(x,t)}\Big)\Big(1-\frac{T(x,t)}{T_2}\Big) + \beta_1 T_2 f(V_2)\Big[-\varphi\Big(\frac{T_2}{T(x,t)}\Big) - \varphi\Big(\frac{V_2 I(x,t-\tau_2)}{I_2 V(x,t)}\Big) - \\
\varphi &\ \Big(\frac{T(x,t-\tau_1) f\Big(V(x,t-\tau_1)\Big) I_2}{T_2 f(V_2) I(x,t)}\Big) - \varphi\Big(\frac{f(V_2)V(x,t)}{V_2 f\Big(V(x,t)\Big)}\Big) + \Big(\frac{f(V(x,t))}{f(V_2)} - \frac{V(x,t)}{V_2}\Big)\Big(1 - \frac{f(V_2)}{f\Big(V(x,t)\Big)}\Big)\Big] \\
&+ \beta_2 T_2 g(I_2)\Big[-\varphi\Big(\frac{T_2}{T(x,t)}\Big) - \varphi\Big(\frac{T(x,t-\tau_1) g\Big(I(x,t-\tau_1)\Big) I_2}{T_2 g(I_2) I(x,t)}\Big) - \varphi\Big(\frac{g(I_2) I(x,t)}{I_2 g\Big(I(x,t)\Big)}\Big) \\
&+ \Big(\frac{g\Big(I(x,t)\Big)}{g(I_2)} - \frac{I(x,t)}{I_2}\Big)\Big(1 - \frac{g(I_2)}{g(I(x,t))}\Big)\Big]\Bigg\} dx - \frac{\beta_1 T_2 f(V_2) D V_2}{k I_2} \int_\Omega \frac{\|\nabla V(x,t)\|^2}{V^2(x,t)}dx,
\end{aligned}$$

from (A_1), it is easy to see that

$$\Big(\frac{f(V(x,t))}{f(V_1)} - \frac{V(x,t)}{V_1}\Big)\Big(1 - \frac{f(V_1)}{f(V(x,t))}\Big) \le 0,$$

$$\Big(\frac{g(I(x,t))}{g(I_1)} - \frac{I(x,t)}{I_1}\Big)\Big(1 - \frac{g(I_1)}{g(I(x,t))}\Big) \le 0.$$

As $\varphi(u) \ge 0$ for $u > 0$, then $\frac{dL_3}{dt} \le 0$. The largest invariant set of $\{\frac{dL_3}{dt} = 0\}$ is the single point $\{E_2\}$, similar to the proof of Theorem 3, E_2 is globally asymptotically stable. This completes the proof. □

3. Dynamical Behaviors of Discrete System

In preceding section, by introducing Lyapunov functions, we have shown by using continuous Lyapunov functionals that the global asymptotic stability of the equilibria of the continuous system (4) is completely determined by the basic reproduction number. R_0 and R_1 act as threshold parameter for the existence and stability of equilibria. This arises a natural question that whether the global asymptotic stability of the equilibria of the discrete system (7) can be preserved. In this section, we will discuss this problem.

Obviously, the discrete system (7) has the same equilibria as system (4). Similarly, $E_0 = (T_0, 0, 0, 0)$ is the infection-free equilibrium, $E_1 = (T_1, I_1, V_1, 0)$ stands for the CTL-inactivated equilibrium and $E_2 = (T_2, I_2, V_2, Z_2)$ is the CTL-activated equilibrium.

Rewriting the discrete system (7) yields

$$\begin{cases} T_{n+1}^m = \frac{\lambda\Delta t + T_n^m}{1+\Delta t(d_1+\beta_1 f(V_n^m + \beta_2 g(I_n^m))}, \\ I_{n+1}^m = \frac{I_n^m + \Delta t e^{-\mu_1\tau_1}(\beta_1 T_{n-m_1+1}^m f(V_{n-m_1}^m) + \beta_2 T_{n-m_1+1}^m g(I_{n-m_1}^m)}{1+\Delta t(d_2+p_1 Z_n^m)}, \\ AV_{n+1} = V_n + \Delta t p_2 e^{-\mu_2\tau_2} I_{n-m_2+1}, \\ Z_{n+1}^m = \frac{(1+\Delta t q I_{n+1}^m)}{1+\Delta t d_4} Z_n^m, \end{cases}, \tag{5}$$

where the square matrix A of dimension $(N+1)\times(N+1)$ is given by

$$\begin{pmatrix} c_1 & c_2 & 0 & \cdots & 0 & 0 & 0 \\ c_2 & c_3 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & c_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_3 & c_2 & 0 \\ 0 & 0 & 0 & \cdots & c_2 & c_3 & c_2 \\ 0 & 0 & 0 & \cdots & 0 & c_2 & c_1 \end{pmatrix}$$

with $c_1 = 1 + D\Delta t/(\Delta x)^2 + d_3\Delta t, c_2 = -D\Delta t/(\Delta x)^2, c_3 = 1 + 2D\Delta t/(\Delta x)^2 + d_3\Delta t$. It is clear that A is strictly diagonally dominant matrix, therefore A is non-singular. From the third equation of the above system, we have

$$V_{n+1} = A^{-1}(V_n + \Delta t p_2 e^{-\mu_2\tau_2} I_{n-m_2+1}).$$

Theorem 5. *For any $\Delta t > 0, \Delta x > 0$, the solutions of the system (7) remain nonnegative and bounded for all $n \in \mathbb{N}$.*

Proof. Since all parameters in (7) are positive, then using the induction , it is easy to deduce from (5) that all solutions of system (7) remain nonnegative provided that the initial value are nonnegative, for all $n \in \aleph$.

Next, we establish the boundedness of solutions. Define a sequence G_n as follows

$$G_n^m = T_n^m + I_n^m + \frac{p_1}{q} Z_n^m + \Delta t \sum_{j=n-m_1}^{n-1} \Big[\beta_1 f(V_j^m) + \beta_2 g(I_j^m)\Big] T_{j+1}^m e^{-\Delta t\mu_1(n-j)},$$

then

$$\begin{aligned} G_{n+1}^m - G_n^m &= \Delta t\Big(\lambda - d_1 T_{n+1}^m - \beta_1 T_{n+1}^m f(V_n^m) - \beta_2 T_{n+1}^m g(I_n^m)\Big) \\ &+ \Delta t\Big[e^{-\mu_1\tau_1}\Big(\beta_1 T_{n-m_1+1}^m f(V_{n-m_1}^m) + \beta_2 T_{n-m_1+1}^m g(I_{n-m_1}^m)\Big) \\ &- d_2 I_{n+1}^m - p_1 I_{n+1}^m Z_n^m\Big] + \frac{p_1}{q}\Delta t(q I_{n+1}^m Z_n^m - d_4 Z_{n+1}^m) \\ &+ \Delta t \sum_{j=n-m_1+1}^{n} \Big[\beta_1 f(V_j^m) + \beta_2 g(I_j^m)\Big] T_{j+1}^m e^{-\Delta t\mu_1(n-j+1)} \\ &- \Delta t \sum_{j=n-m_1}^{n-1} \Big[\beta_1 f(V_j^m) + \beta_2 g(I_j^m)\Big] T_{j+1}^m e^{-\Delta t\mu_1(n-j)} \\ &= \Delta t\Big\{\lambda - d_1 T_{n+1}^m - d_2 I_{n+1}^m - \frac{p_1 d_4}{q} Z_{n+1}^m \\ &+ \Big(1 - e^{\Delta t\mu_1}\Big) \sum_{j=n-m_1+1}^{n} \Big[\beta_1 f(V_j^m) + \beta_2 g(I_j^m)\Big] T_{j+1}^m e^{-\Delta t\mu_1(n-j+1)}\Big\} \\ &\leq \Delta t(\lambda - \xi G_{n+1}^m), \end{aligned}$$

where $\xi = \min\{d_1, d_2, d_4, \frac{e^{\Delta t\mu_1}-1}{\Delta t}\}$, then we have

$$G_{n+1}^m \le \frac{1}{1+\Delta t\xi} G_n^m + \frac{\Delta t \lambda}{1+\Delta t\xi},$$

it follows from the induction that

$$G_n^m \le (\frac{1}{1+\Delta t\xi})^n G_0^m + \frac{\lambda}{\xi}\Big[1-(\frac{1}{1+\Delta t\xi})^n\Big],$$

therefore

$$\limsup_{n\to\infty} G_n^m \le \frac{\lambda}{\xi}, \quad for \;\; all \;\; m\in\{0,1,\cdots,N\},$$

this implies that $\{G_n\}$ is bounded, then $\{T_n\}$,$\{I_n\}$ and $\{Z_n\}$ are bounded.

From the third equation of system (7)

$$\sum_{m=0}^{N} V_{n+1}^m = \frac{1}{1+d_3\Delta t}\Big(\sum_{m=0}^{N} V_n^m + \Delta t p_2 e^{-\mu_2\tau_2}\sum_{m=0}^{N} I_{n-m_2+1}^m\Big),$$

since $\{I_n\}$ is bounded, then there exists $\eta > 0$ such that $I_n^m \le \eta$ for all $n \in \{-m_2, -m_2+1,\cdots,0,1,\cdots\}$, $m\in\{0,1,\cdots,\aleph\}$, then

$$\sum_{m=0}^{N} V_{n+1}^m \le \frac{1}{1+d_3\Delta t}\Big[\sum_{m=0}^{N} V_n^m + \Delta t p_2 e^{-\mu_2\tau_2}\eta(N+1)\Big],$$

by induction, we have

$$\begin{aligned}\sum_{m=0}^{N} V_n^m &\le \frac{1}{(1+d_3\Delta t)^n}\sum_{m=0}^{N} V_0^m + \frac{p_2 e^{-\mu_2\tau_2}\eta(N+1)}{d_3}\Big[1-\frac{1}{(1+d_3\Delta t)^n}\Big] \\ &\le \sum_{m=0}^{N} V_0^m + \frac{p_2 e^{-\mu_2\tau_2}\eta(N+1)}{d_3},\end{aligned}$$

therefore $\{V_n\}$ is bounded. This completes the proof. □

Global Stability

In this section, we will study the global stability of the equilibria of system (7) .

Theorem 6. *For system (7), if $R_0 \le 1$, the infection-free equilibrium E_0 is globally asymptotically stable.*

Proof. Define the discrete Lyapunov function as follows

$$\begin{aligned} W_n &= \sum_{m=0}^{N}\Bigg\{\frac{1}{\Delta t}\Big[T_0\varphi(\frac{T_n^m}{T_0}) + e^{\mu_1\tau_1}\Big(1+\Delta t\frac{\beta_2 T_0 g'(0)}{R_0 e^{\mu_1\tau_1}}\Big)I_n^m + \frac{\beta_1 T_0 f'(0)}{d_3 R_0}\Big(1+\Delta t d_3\Big)V_n^m \\ &+ \frac{p_1 e^{\mu_1\tau_1}}{q}\Big(1+\Delta t d_4\Big)Z_n^m\Big] + \sum_{j=n-m_1}^{n-1}\Big(\beta_1 f(V_j^m) + \beta_2 g(I_j^m)\Big)T_{j+1}^m \\ &+ \frac{\beta_1 p_2 T_0 f'^{-\mu_2\tau_2}}{R_0 d_3}\sum_{j=n-m_2}^{n-1} I_{j+1}^m\Bigg\}.\end{aligned}$$

It follows from $u-1 \ge \ln u$ for all $u > 0$, that $W_n \ge 0$ for all $n \in \aleph$. Then, along the trajectory of (7)

$$\begin{aligned}
W_{n+1}-W_n &= \sum_{m=0}^{N}\Bigg\{\frac{1}{\Delta t}\Big[T^m_{n+1}-T^m_n+T_0\ln\frac{T^m_n}{T^m_{n+1}}+e^{\mu_1\tau_1}\Big(1+\Delta t\frac{\beta_2T_0g'(0)}{R_0e^{\mu_1\tau_1}}\Big)(I^m_{n+1}-I^m_n)\\
&+ \frac{\beta_1T_0f'(0)}{d_3R_0}\Big(1+\Delta td_3\Big)\Big(V^m_{n+1}-V^m_n\Big)+\frac{p_1e^{\mu_1\tau_1}}{q}\Big(1+\Delta td_4\Big)\Big(Z^m_{n+1}-Z^m_n\Big)\Big]\\
&+ \sum_{j=n-m_1+1}^{n}\Big(\beta_1f(V^m_j)+\beta_2g(I^m_j)\Big)T^m_{j+1}\\
&- \sum_{j=n-m_1}^{n-1}\Big(\beta_1f(V^m_j)+\beta_2g(I^m_j)\Big)T^m_{j+1}\\
&+ \frac{\beta_1p_2T_0f'^{-\mu_2\tau_2}}{R_0d_3}\Big[\sum_{j=n-m_2+1}^{n}I^m_{j+1}-\sum_{j=n-m_2}^{n-1}I^m_{j+1}\Big]\Bigg\},
\end{aligned}$$

using the equilibrium condition of E_0, we have

$$\begin{aligned}
W_{n+1}-W_n &\leq \sum_{m=0}^{N}\Bigg\{\Big(1-\frac{T_0}{T^m_{n+1}}\Big)\Big(d_1T_0-d_1T^m_{n+1}-\Big(\beta_1f(V^m_n)+\beta_2g(I^m_n)\Big)T^m_{n+1}\Big)\\
&+ \Big(1+\Delta t\frac{\beta_2T_0g'(0)}{R_0e^{\mu_1\tau_1}}\Big)\Big(\beta_1T^m_{n-m_1+1}f(V^m_{n-m_1})+\beta_2T^m_{n-m_1+1}g(I^m_{n-m_1})\Big)\\
&+ e^{\mu_1\tau_1}\Big(1+\Delta t\frac{\beta_2T_0g'(0)}{R_0e^{\mu_1\tau_1}}\Big)\Big(-d_2I^m_{n+1}-p_1I^m_{n+1}Z^m_n\Big)\\
&+ \frac{\beta_1T_0f'(0)}{d_3R_0}\Big(1+\Delta td_3\Big)D\frac{V^{m+1}_{n+1}-2V^m_{n+1}+V^{m-1}_{n+1}}{(\Delta x)^2}\\
&+ \frac{\beta_1T_0f'(0)}{d_3R_0}\Big(1+\Delta td_3\Big)\Big(p_2e^{\mu_2\tau_2}I^m_{n-m_2+1}-d_3V^m_{n+1}\Big)\\
&+ \frac{p_1e^{\mu_1\tau_1}}{q}\Big(1+\Delta td_4\Big)\Big(qI^m_{n+1}Z^m_n-d_4Z^m_{n+1}\Big)+\Big(\beta_1f(V^m_n)+\beta_2g(I^m_n)\Big)T^m_{n+1}\\
&- \Big(\beta_1f(V^m_{n-m_1})+\beta_2g(I^m_{n-m_1})\Big)T^m_{n-m_1+1}\\
&- \frac{\beta_1p_2T_0f'^{-\mu_2\tau_2}}{R_0d_3}\Big(I^m_{n+1}-I^m_{n-m_2+1}\Big)\Bigg\}\\
&= \sum_{m=0}^{N}\Bigg\{d_1T_0\Big(2-\frac{T_0}{T^m_{n+1}}-\frac{T^m_{n+1}}{T_0}\Big)+\beta_1T_0f(V^m_n)+\beta_2T_0g(I^m_n)\\
&- \frac{\beta_2T_0g'(0)}{R_0}I^m_n-\frac{\beta_1T_0f'(0)}{R_0}V^m_n-\frac{d_4p_1e^{\mu_1\tau_1}}{q}Z^m_n\Bigg\}\\
&+ \frac{\beta_1T_0f'(0)D}{d_3R_0(\Delta x)^2}\Big(V^{N+1}_{n+1}-V^N_{n+1}+V^{-1}_{n+1}-V^0_{n+1}\Big)\\
&\leq \sum_{m=0}^{N}\Bigg\{d_1T_0\Big(2-\frac{T_0}{T^m_{n+1}}-\frac{T^m_{n+1}}{T_0}\Big)+\frac{\beta_1T_0f'(0)}{R_0}V^m_n\Big(R_0-1\Big)\\
&+ \frac{\beta_2g'(0)T_0}{R_0}I^m_n\Big(R_0-1\Big)-\frac{d_4p_1e^{\mu_1\tau_1}}{q}Z^m_n\Bigg\},
\end{aligned}$$

the last inequality is deduced from condition (A_2), if $R_0\leq 1$, then $W_{n+1}-W_n\leq 0$, for all $n\in\aleph$, therefore, $\{W_n\}$ is monotone decreasing sequence. It follows from $W_n\geq 0$ that $\lim_{n\to\infty}W_n\geq 0$, then

$$\lim_{n\to\infty}(W_{n+1}-W_n)=0,$$

therefore

(1) If $R_0 < 1$, then $\lim_{n\to\infty}(W_{n+1} - W_n) = 0$ implies that $\lim_{n\to\infty} T_n^m = T_0$, $\lim_{n\to\infty} V_n^m = 0$, $\lim_{n\to\infty} Z_n^m = 0$, $\lim_{n\to\infty} I_n^m = 0$.

(2) If $R_0 = 1$, then $\lim_{n\to\infty}(W_{n+1} - W_n) = 0$ implies that $\lim_{n\to\infty} T_n^m = T_0$, $\lim_{n\to\infty} Z_n^m = 0$, from system (7), we obtain $\lim_{n\to\infty} I_n^m = 0$, $\lim_{n\to\infty} V_n^m = 0$.

Hence, E_0 is globally asymptotically stable when $R_0 \leq 1$. This completes the proof. □

Theorem 7. *For system (7), if $R_1 < 1 < R_0$, the CTL-inactivated infection equilibrium E_1 of is globally asymptotically stable .*

Proof. Define the discrete Lyapunov function as follows

$$\begin{aligned}
\widetilde{W}_n &= \sum_{m=0}^{N}\left\{\frac{1}{\Delta t}\Big[T_1\varphi(\frac{T_n^m}{T_1}) + e^{\mu_1\tau_1}I_1\varphi(\frac{I_n^m}{I_1}) + \frac{\beta_1 T_1 f(V_1)}{p_2 e^{-\mu_2\tau_2}I_1}V_1\varphi(\frac{V_n^m}{V_1}) + \frac{p_1 e^{\mu_1\tau_1}}{q}Z_n^m\Big]\right.\\
&+ \beta_1 T_1 f(V_1)\sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_1 f(V_1)}\Big) + \beta_2 T_1 g(I_1)\sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_1 g(I_1)}\Big)\\
&+ \left.\beta_1 T_1 f(V_1)\sum_{j=n-m_2}^{n-1}\varphi\Big(\frac{I_{j+1}^m}{I_1}\Big) + \beta_1 T_1 f(V_1)\varphi\Big(\frac{f(V_n^m)}{f(V_1)}\Big) + \beta_2 T_1 g(I_1)\varphi\Big(\frac{g(I_n^m)}{g(I_1)}\Big)\right\}.
\end{aligned}$$

Since $u - 1 \geq \ln u$ for all $u > 0$, then $\widetilde{W}_n \geq 0$ for all $n \in \aleph$. The Lyapunov derivative along (7) is

$$\begin{aligned}
\widetilde{W}_{n+1} - \widetilde{W}_n &= \sum_{m=0}^{N}\left\{\frac{1}{\Delta t}\Big[T_{n+1}^m - T_n^m + T_1\ln\frac{T_n^m}{T_{n+1}^m} + e^{\mu_1\tau_1}\Big(I_{n+1}^m - I_n^m + I_1\ln\frac{I_n^m}{I_{n+1}^m}\Big)\right.\\
&+ \frac{\beta_1 T_1 V_1}{p_2 e^{-\mu_2\tau_2}I_1}\Big(V_{n+1}^m - V_n^m + V_1\ln\frac{V_n^m}{V_{n+1}^m}\Big) + \frac{p_1 e^{\mu_1\tau_1}}{q}\Big(Z_{n+1}^m - Z_n^m\Big)\Big]\\
&+ \beta_1 T_1 f(V_1)\Big[\sum_{j=n-m_1+1}^{n}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_1 f(V_1)}\Big) - \sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_1 f(V_1)}\Big)\Big]\\
&+ \beta_2 T_1 g(I_1)\Big[\sum_{j=n-m_1+1}^{n}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_1 g(I_1)}\Big) - \sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_1 g(I_1)}\Big)\Big]\\
&+ \beta_1 T_1 f(V_1)\Big[\sum_{j=n-m_2+1}^{n}\varphi\Big(\frac{I_{j+1}^m}{I_1}\Big) - \sum_{j=n-m_2}^{n-1}\varphi(\frac{I_{j+1}^m}{I_1})\Big]\\
&+ \beta_1 T_1 f(V_1)\Big(\frac{f(V_{n+1}^m)}{f(V_1)} - \frac{f(V_n^m)}{f(V_1)} + \ln\frac{f(V_n^m)}{f(V_{n+1}^m)}\Big)\\
&+ \left.\beta_2 T_1 g(I_1)\Big(\frac{g(I_{n+1}^m)}{g(I_1)} - \frac{g(I_n^m)}{g(I_1)} + \ln\frac{g(I_n^m)}{g(I_{n+1}^m)}\Big)\right\}\\
&\leq \sum_{m=0}^{N}\{\frac{1}{\Delta t}\Big[\Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(T_{n+1}^m - T_n^m\Big) + e^{\mu_1\tau_1}\Big(1 - \frac{I_1}{I_{n+1}^m}\Big)\Big(I_{n+1}^m - I_n^m\Big)\\
&+ \frac{\beta_1 T_1 f(V_1)}{p_2 e^{-\mu_2\tau_2}I_1}\Big(1 - \frac{V_1}{V_{n+1}^m}\Big)\Big(V_{n+1}^m - V_n^m\Big) + \frac{p_1 e^{\mu_1\tau_1}}{q}(Z_{n+1}^m - Z_n^m)\Big]\\
&+ \beta_1 T_1 f(V_1)\Big[\varphi\Big(\frac{T_{n+1}^m f(V_n^m)}{T_1 f(V_1)}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m f(V_{n-m_1}^m)}{T_1 f(V_1)}\Big)\Big]\\
&+ \beta_2 T_1 g(I_1)\Big[\varphi(\frac{T_{n+1}^m g(I_n^m)}{T_1 g(I_1)}) - \varphi(\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m)}{T_1 g(I_1)})\Big]\\
&+ \beta_1 T_1 f(V_1)\Big[\varphi(\frac{I_{n+1}^m}{I_1}) - \varphi(\frac{I_{n-m_2+1}^m}{I_1})\Big]\\
&+ \beta_1 T_1 f(V_1)\Big[\frac{f(V_{n+1}^m)}{f(V_1)} - \frac{f(V_n^m)}{f(V_1)} + \ln\frac{f(V_n^m)}{f(V_{n+1}^m)}\Big]\\
&+ \beta_2 T_1 g(I_1)\Big[\frac{g(I_{n+1}^m)}{g(I_1)} - \frac{g(I_n^m)}{g(I_1)} + \ln\frac{g(I_n^m)}{g(I_{n+1}^m)}\Big]\Big\}.
\end{aligned}$$

As E_1 satisfies

$$\lambda = d_1 T_1 + \beta_1 T_1 f(V_1) + \beta_2 T_1 g(I_1),$$
$$\beta_1 T_1 f(V_1) + \beta_2 T_1 g(I_1) = e^{-\mu_1\tau_1} d_2 I_1, p_2 e^{-\mu_1\tau_1} I_1 = d_3 V_1,$$

then

$$\begin{aligned}
\widetilde{W}_{n+1} - \widetilde{W}_n \quad \leq \quad & \sum_{m=0}^{N} \Big\{ d_1 T_1 \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_1}\Big) + \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(\beta_1 T_1 f(V_1) + \beta_2 T_1 g(I_1)\Big) \\
- \quad & \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(\beta_1 T_{n+1}^m f(V_n^m) + \beta_2 T_{n+1}^m g(I_n^m)\Big) \\
+ \quad & \Big(1 - \frac{I_1}{I_{n+1}^m}\Big)\Big(\beta_1 T_{n-m_1+1}^m f(V_{n-m_1}^m) + \beta_2 T_{n-m_1+1}^m g(I_{n-m_1}^m)\Big) \\
- \quad & \beta_1 T_1 f(V_1) \frac{I_{n+1}^m}{I_1}\Big(1 - \frac{I_1}{I_{n+1}^m}\Big) - \beta_2 T_1 g(I_1) \frac{I_{n+1}^m}{I_1}\Big(1 - \frac{I_1}{I_{n+1}^m}\Big) \\
+ \quad & \beta_1 T_1 f(V_1)\Big(1 - \frac{V_1}{V_{n+1}^m}\Big)\Big(\frac{I_{n-m_2+1}^m}{I_1} - \frac{V_{n+1}^m}{V_1}\Big) - \frac{p_1 d_4 e^{\mu_1\tau_1}}{q} Z_{n+1}^m \\
+ \quad & \beta_1 T_1 f(V_1)\Big[\varphi\Big(\frac{T_{n+1}^m f(V_n^m)}{T_1 f(V_1)}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m f(V_{n-m_1}^m)}{T_1 f(V_1)}\Big)\Big] \\
+ \quad & \beta_2 T_1 g(I_1)\Big[\varphi\Big(\frac{T_{n+1}^m g(I_n^m)}{T_1 g(I_1)}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m)}{T_1 g(I_1)}\Big)\Big] \\
+ \quad & \beta_1 T_1 f(V_1)\Big[\varphi\Big(\frac{I_{n+1}^m}{I_1}\Big) - \varphi\Big(\frac{I_{n-m_2+1}^m}{I_1}\Big)\Big] \\
+ \quad & \beta_1 T_1 f(V_1)\Big(\frac{f(V_{n+1}^m)}{f(V_1)} - \frac{f(V_n^m)}{f(V_1)} + \ln\frac{f(V_n^m)}{f(V_{n+1}^m)}\Big) \\
+ \quad & \beta_2 T_1 g(I_1)\Big(\frac{g(I_{n+1}^m)}{g(I_1)} - \frac{g(I_n^m)}{g(I_1)} + \ln\frac{g(I_n^m)}{g(I_{n+1}^m)}\Big)\Big\} \\
- \quad & \sum_{m=0}^{N} \frac{\beta_1 T_1 f(V_1) D}{p_2 I_1 e^{-\mu_2\tau_2}(\Delta x)^2}\Big(1 - \frac{V_1}{V_{n+1}^m}\Big)\Big(V_{n+1}^{m+1} - 2V_{n+1}^m + V_{n+1}^{m-1}\Big) \\
= \quad & \sum_{m=0}^{N} \Big\{ d_1 T_1 \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_1}\Big) + \beta_1 T_1 f(V_1)\Big[3 - \frac{T_1}{T_{n+1}^m} - \frac{V_1 I_{n-m_2+1}^m}{V_{n+1}^m I_1} \\
- \quad & \frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_1}{T_1 f(V_1) I_{n+1}^m} + \frac{f(V_{n+1}^m)}{f(V_1)} - \frac{V_{n+1}^m}{V_1} + \ln\frac{T_{n-m_1+1} f(V_{n-m_1}^m) I_{n-m_2+1}^m}{T_{n+1}^m f(V_{n+1}^m) I_{n+1}^m}\Big] \\
+ \quad & \beta_2 T_1 g(I_1)\Big[2 - \frac{T_1}{T_{n+1}^m} - \frac{T_{n-m_1+1}^m g(I_{n-m_1}^m) I_1}{T_1 g(I_1) I_{n+1}^m} + \frac{g(I_{n+1}^m)}{g(I_1)} \\
- \quad & \frac{I_{n+1}^m}{I_1} + \ln\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m)}{T_{n+1}^m g(I_{n+1}^m)}\Big] \\
- \quad & \frac{p_1 d_4 e^{\mu_1\tau_1}}{q} Z_{n+1}^m\Big\} - \frac{\beta_1 T_1 f(V_1) D}{p_2 I_1 e^{-\mu_2\tau_2}(\Delta x)^2} V_1 \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m} \\
= \quad & \sum_{m=0}^{N} \Big\{ d_1 T_1 \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_1}\Big) + \beta_1 T_1 f(V_1)\Big[- \varphi(\frac{T_1}{T_{n+1}^m}) \\
- \quad & \varphi\Big(\frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_1}{T_1 f(V_1) I_{n+1}^m}\Big) - \varphi\Big(\frac{V_1 I_{n-m_2+1}^m}{V_{n+1}^m I_1}\Big) + \frac{f(V_{n+1}^m)}{f(V_1)} - \frac{V_{n+1}^m}{V_1} \\
+ \quad & \ln\frac{f(V_1^m) V_{n+1}^m}{f(V_{n+1}^m) V_1}\Big] + \beta_2 T_1 g(I_1)\Big[- \varphi\Big(\frac{T_1}{T_{n+1}^m}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m) I_1}{T_1 g(I_1) I_{n+1}^m}\Big) \\
+ \quad & \frac{g(I_{n+1}^m)}{g(I_1)} - \frac{I_{n_1}^m}{I_1} + \ln\frac{g(I_1^m) I_{n+1}^m}{g(I_{n+1}^m) I_1}\Big] \\
- \quad & \frac{p_1 d_4 e^{\mu_1\tau_1}}{q} Z_{n+1}^m\Big\} - \frac{\beta_1 T_1 f(V_1) D}{p_2 I_1 e^{-\mu_2\tau_2}(\Delta x)^2} V_1 \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m} \\
= \quad & \sum_{m=0}^{N} \Big\{ d_1 T_1 \Big(1 - \frac{T_1}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_1}\Big) + \beta_1 T_1 f(V_1)\Big[- \varphi(\frac{T_1}{T_{n+1}^m}) \\
- \quad & \varphi\Big(\frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_1}{T_1 f(V_1) I_{n+1}^m}\Big) - \varphi\Big(\frac{V_1 I_{n-m_2+1}^m}{V_{n+1}^m I_1}\Big) \\
- \quad & \varphi\Big(\frac{f(V_1^m) V_{n+1}^m}{f(V_{n+1}^m) V_1}\Big) + \Big(\frac{f(V_{n+1}^m)}{f(V_1)} - \frac{V_{n+1}^m}{V_1}\Big)\Big(1 - \frac{f(V_{n+1}^m) V_1}{f(V_1^m) V_{n+1}^m}\Big)\Big] \\
+ \quad & \beta_2 T_1 g(I_1)\Big[- \varphi\Big(\frac{T_1}{T_{n+1}^m}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m) I_1}{T_1 g(I_1) I_{n+1}^m}\Big) - \varphi\Big(\frac{g(I_1^m) I_{n+1}^m}{g(I_{n+1}^m) I_1}\Big) \\
+ \quad & \Big(\frac{g(I_{n+1}^m)}{g(I_1)} - \frac{I_{n+1}^m}{I_1}\Big)\Big(1 - \frac{g(I_1)}{g(I_{n+1}^m)}\Big)\Big] - \frac{p_1 d_4 e^{\mu_1\tau_1}}{q} Z_{n+1}^m\Big\} \\
- \quad & - \frac{\beta_1 T_1 f(V_1) D}{p_2 I_1 e^{-\mu_2\tau_2}(\Delta x)^2} V_1 \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m}.
\end{aligned}$$

Similar to the proof of Theorem 3, we have

$$\Big(\frac{f(V(x,t))}{f(V_1)} - \frac{V(x,t)}{V_1}\Big)\Big(1 - \frac{f(V_1)}{f(V(x,t))}\Big) \le 0,$$

$$\Big(\frac{g(I(x,t))}{g(I_1)} - \frac{I(x,t)}{I_1}\Big)\Big(1 - \frac{g(I_1)}{g(I(x,t))}\Big) \le 0.$$

It follows from $\varphi(u) \ge 0$ that $(\widetilde{W}_{n+1} - \widetilde{W}_n) \le 0$, for all $n \in \aleph$, this implies that $\{\widetilde{W}_n\}$ is monotone decreasing sequence. As $\widetilde{W}_n \ge 0$, then $\lim_{n\to\infty} \widetilde{W}_n \ge 0$, $\lim_{n\to\infty}(\widetilde{W}_{n+1} - \widetilde{W}_n) = 0$, so that $\lim_{n\to\infty} T_n^m = T_1$. Combined with system (7), we obtain $\lim_{n\to\infty} I_n^m = I_1$, $\lim_{n\to\infty} V_n^m = V_1$ and $\lim_{n\to\infty} Z_n^m = 0$, for all $m \in \{0, 1, \cdots, N\}$, then E_1 of system (7) is globally asymptotically stable. This completes the proof. □

Theorem 8. *For system (7), if $R_1 > 1$, the interior equilibrium E_2 is not globally asymptotically stable .*

Proof. Define the discrete Lyapunov function as follows

$$\begin{aligned}
\overline{W}_n \;=\; & \sum_{m=0}^{N}\Bigg\{\frac{1}{\Delta t}\Big[T_2\varphi\Big(\frac{T_n^m}{T_2}\Big) + e^{\mu_1\tau_1} I_2\varphi\Big(\frac{I_n^m}{I_2}\Big) + \frac{\beta_1 T_2 f(V_2)}{p_2 e^{-\mu_2\tau_2} I_2} V_2\varphi\Big(\frac{V_n^m}{V_2}\Big) \\
+\; & \frac{p_1 e^{\mu_1\tau_1}}{q} Z_2\varphi\Big(\frac{Z_n^m}{Z_2}\Big) + \Delta t p_1 e^{\mu_1\tau_1} I_2 Z_n^m\Big] + \beta_1 T_2 f(V_2)\sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_2 f(V_2)}\Big) \\
+\; & \beta_2 T_2 g(I_2)\sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_2 g(I_2)}\Big) + \beta_1 T_2 f(V_2)\sum_{j=n-m_2}^{n-1}\varphi\Big(\frac{I_{j+1}^m}{I_2}\Big) \\
+\; & \beta_1 T_2 f(V_2)\varphi\Big(\frac{f(V_n^m)}{f(V_2)}\Big) + \beta_2 T_2 g(I_2)\varphi\Big(\frac{g(I_n^m)}{g(I_2)}\Big)\Bigg\},
\end{aligned}$$

it follows from $u - 1 \ge \ln u$ that $\overline{W}_n \ge 0$ for all $n \in \aleph$. Then, along the trajectory of (7)

$$\begin{aligned}
\overline{W}_{n+1} - \overline{W}_n \;=\; & \sum_{m=0}^{N}\Bigg\{\frac{1}{\Delta t}\Big[T_{n+1}^m - T_n^m + T_2\ln\frac{T_n^m}{T_{n+1}^m} + e^{\mu_1\tau_1}\Big(I_{n+1}^m - I_n^m + I_2\ln\frac{I_n^m}{I_{n+1}^m}\Big) \\
+\; & \frac{\beta_1 T_2 f(V_2)}{p_2 e^{-\mu_2\tau_2} I_2}\Big(V_{n+1}^m - V_n^m + V_2\ln\frac{V_n^m}{V_{n+1}^m}\Big) + \frac{p_1 e^{\mu_1\tau_1}}{q}(Z_{n+1}^m - Z_n^m \\
+\; & Z_2\ln\frac{Z_n^m}{Z_{n+1}^m}) + \Delta t p_1 e^{\mu_1\tau_1} I_2\Big(Z_{n+1}^m - Z_n^m\Big)\Big] \\
+\; & \beta_1 T_2 f(V_2)\Big[\sum_{j=n-m_1+1}^{n}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_2 f(V_2)}\Big) - \sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m f(V_j^m)}{T_2 f(V_2)}\Big)\Big] \\
+\; & \beta_2 T_2 g(I_2)\Big[\sum_{j=n-m_1+1}^{n}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_2 g(I_2)}\Big) - \sum_{j=n-m_1}^{n-1}\varphi\Big(\frac{T_{j+1}^m g(I_j^m)}{T_2 g(I_2)}\Big)\Big] \\
+\; & \beta_1 T_2 f(V_2)\Big[\sum_{j=n-m_2+1}^{n}\varphi\Big(\frac{I_{j+1}^m}{I_2}\Big) - \sum_{j=n-m_2}^{n-1}\varphi\Big(\frac{I_{j+1}^m}{I_2}\Big)\Big] \\
+\; & \beta_1 T_2 f(V_2)\Big(\frac{f(V_{n+1}^m)}{f(V_2)} - \frac{f(V_n^m)}{f(V_2)} + \ln\frac{f(V_n^m)}{f(V_{n+1}^m)}\Big) \\
+\; & \beta_2 T_2 g(I_2)\Big(\frac{g(I_{n+1}^m)}{g(I_2)} - \frac{g(I_n^m)}{g(I_2)} + \ln\frac{g(I_n^m)}{g(I_{n+1}^m)}\Big)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \sum_{m=0}^{N}\Bigg\{\frac{1}{\Delta t}\Big[\Big(1-\frac{T_2}{T^m_{n+1}}\Big)\Big(T^m_{n+1}-T^m_n\Big)+e^{\mu_1\tau_1}\Big(1-\frac{I_2}{I^m_{n+1}}\Big)\Big(I^m_{n+1}-I^m_n\Big)\\
+\ & \frac{\beta_1T_2f(V_2)}{p_2e^{-\mu_2\tau_2}I_2}\Big(1-\frac{V_2}{V^m_{n+1}}\Big)\Big(V^m_{n+1}-V^m_n\Big)\\
+\ & \frac{p_1e^{\mu_1\tau_1}}{q}\Big(1-\frac{Z_2}{Z^m_{n+1}}\Big)\Big(Z^m_{n+1}-Z^m_n\Big)+\Delta t p_1e^{\mu_1\tau_1}I_2\Big(Z^m_{n+1}-Z^m_n\Big)\Big]\\
+\ & \beta_1T_2f(V_2)\Big[\varphi\Big(\frac{T^m_{n+1}f(V^m_n)}{T_2f(V_2)}\Big)-\varphi\Big(\frac{T^m_{n-m_1+1}f(V^m_{n-m_1})}{T_2f(V_2)}\Big)\Big]\\
+\ & \beta_2T_2g(I_2)\Big[\varphi\Big(\frac{T^m_{n+1}g(I^m_n)}{T_2g(I_2)}\Big)-\varphi\Big(\frac{T^m_{n-m_1+1}g(I^m_{n-m_1})}{T_2g(I_2)}\Big)\Big]\\
+\ & \beta_1T_2f(V_2)\Big[\varphi\Big(\frac{I^m_{n+1}}{I_2}\Big)-\varphi\Big(\frac{I^m_{n-m_2+1}}{I_2}\Big)\Big]\\
+\ & \beta_1T_2f(V_2)\Big(\frac{f(V^m_{n+1})}{f(V_2)}-\frac{f(V^m_n)}{f(V_2)}+\ln\frac{f(V^m_n)}{f(V^m_{n+1})}\Big)\\
+\ & \beta_2T_2g(I_2)\Big(\frac{g(I^m_{n+1})}{g(I_2)}-\frac{g(I^m_n)}{g(I_2)}+\ln\frac{g(I^m_n)}{g(I^m_{n+1})}\Big)\Bigg\}.
\end{aligned}
$$

From the equilibrium condition of E_2, we have

$$
\lambda=d_1T_2+\beta_1T_2f(V_2)+\beta_2T_2g(I_2),\ \ p_2e^{-\mu_2\tau_2}I_2=d_3V_2,
$$

$$
\beta_1T_2f(V_2)+\beta_2T_2g(I_2)=e^{\mu_1\tau_1}(d_2I_2+p_1I_2Z_2),\ \ I_2=\frac{d_4}{q},
$$

then

$$
\begin{aligned}
\overline{W}_{n+1}-\overline{W}_n\ \leq\ & \sum_{m=0}^{N}\Bigg\{d_2T_2\Big(1-\frac{T_2}{T^m_{n+1}}\Big)\Big(1-\frac{T^m_{n+1}}{T_2}\Big)+\Big(1-\frac{T_2}{T^m_{n+1}}\Big)\Big(\beta_1T_2f(V_2)\\
+\ & \beta_2T_2g(I_2)\Big)-\Big(1-\frac{T_2}{T^m_{n+1}}\Big)\Big(\beta_1T^m_{n+1}f(V^m_n)+\beta_2T^m_{n+1}g(I^m_n)\Big)\\
+\ & \Big(1-\frac{I_2}{I^m_{n+1}}\Big)\Big(\beta_1T^m_{n-m_1+1}f(V^m_{n-m_1})+\beta_2T^m_{n-m_1+1}g(I^m_{n-m_1})\Big)\\
-\ & d_2e^{\mu_1\tau_1}I^m_{n+1}-p_1e^{\mu_1\tau_1}I^m_{n+1}Z^m_n+\beta_1T_2f(V_2)+\beta_2T_2g(I_2)\\
-\ & p_1e^{\mu_1\tau_1}I_2Z_2+p_1e^{\mu_1\tau_1}I_2Z^m_n+p_1e^{\mu_1\tau_1}I_2(Z^m_{n+1}-Z^m_n)\\
+\ & \beta_1T_2f(V_2)\Big(1-\frac{V_2}{V^m_{n+1}}\Big)\Big(\frac{I^m_{n-m_2+1}}{I_2}-\frac{V^m_{n+1}}{V_2}\Big)\\
+\ & p_1e^{\mu_1\tau_1}I^m_{n+1}Z^m_n-\frac{p_1d_4e^{\mu_1\tau_1}}{q}Z^m_{n+1}-p_1e^{\mu_1\tau_1}Z_2\frac{I^m_{n+1}Z^m_n}{Z^m_{n+1}}\\
+\ & p_1e^{\mu_1\tau_1}I_2Z_2+\beta_1T_2f(V_2)\Big[\varphi\Big(\frac{T^m_{n+1}f(V^m_n)}{T_2f(V_2)}\Big)-\varphi\Big(\frac{T^m_{n-m_1+1}f(V^m_{n-m_1})}{T_2f(V_2)}\Big)\Big]\\
+\ & \beta_2T_2g(I_2)\Big[\varphi\Big(\frac{T^m_{n+1}g(I^m_n)}{T_2g(I_2)}\Big)-\varphi\Big(\frac{T^m_{n-m_1+1}g(I^m_{n-m_1})}{T_2g(I_2)}\Big)\Big]\\
+\ & \beta_1T_2f(V_2)\Big[\varphi\Big(\frac{I^m_{n+1}}{I_2}\Big)-\varphi\Big(\frac{I^m_{n-m_2+1}}{I_2}\Big)\Big]\\
+\ & \beta_1T_2f(V_2)\Big(\frac{f(V^m_{n+1})}{f(V_2)}-\frac{f(V^m_n)}{f(V_2)}+\ln\frac{f(V^m_n)}{f(V^m_{n+1})}\Big)\\
+\ & \beta_2T_2g(I_2)\Big(\frac{g(I^m_{n+1})}{g(I_2)}-\frac{g(I^m_n)}{g(I_2)}+\ln\frac{g(I^m_n)}{g(I^m_{n+1})}\Big)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{DV_2\beta_1 T_2 f(V_2)}{p_2 e^{-\mu_2\tau_2} I_2 (\Delta x)^2} \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m} \\
&= \sum_{m=0}^{N} \Big\{ d_1 T_2 \Big(1 - \frac{T_2}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_2}\Big) + \beta_1 T_2 f(V_2)\Big[3 - \frac{T_2}{T_{n+1}^m} - \frac{V_2 I_{n-m_2+1}^m}{V_{n+1}^m I_2} \\
&- \frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_2}{T_2 f(V_2) I_{n+1}^m} + \frac{f(V_{n+1}^m)}{f(V_2)} - \frac{V_{n+1}^m}{V_2} + \ln \frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_{n-m_2+1}}{T_{n+1}^m f(V_{n+1}^m) I_{n+1}}\Big] \\
&+ \beta_2 T_2 g(I_2)\Big[2 - \frac{T_2}{T_{n+1}^m} - \frac{T_{n-m_1+1}^m g(I_{n-m_1}^m) I_2}{T_2 g(I_2) I_{n+1}^m} + \frac{g(I_{n+1}^m)}{g(I_2)} \\
&- \frac{I_{n+1}^m}{I_2} + \ln \frac{T_{n-m_1+1}^m g(I_{n-m_1}^m)}{T_{n+1}^m g(I_{n+1}^m)}\Big] - d_2 e^{\mu_1\tau_1} I_{n+1}^m - p_1 e^{\mu_1\tau_1} Z_2 \frac{I_{n+1}^m Z_n^m}{Z_{n+1}^m} \Big\} \\
&- \frac{DV_2\beta_1 T_2 f(V_2)}{p_2 e^{-\mu_2\tau_2} I_2 (\Delta x)^2} \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m} \\
&= \sum_{m=0}^{N} \Big\{ d_1 T_2 \Big(1 - \frac{T_2}{T_{n+1}^m}\Big)\Big(1 - \frac{T_{n+1}^m}{T_2}\Big) + \beta_1 T_2 f(V_2)\Big[- \varphi\Big(\frac{T_2}{T_{n+1}^m}\Big) \\
&- \varphi\Big(\frac{V_2 I_{n-m_2+1}^m}{V_{n+1}^m I_2}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m f(V_{n-m_1}^m) I_2}{T_2 f(V_2) I_{n+1}^m}\Big) \\
&+ \Big(\frac{f(V_{n+1})}{f(V_2)} - \frac{V_{n+1}}{V_2}\Big)\Big(1 - \frac{f(V_2)}{f(V_{n+1})}\Big)\Big] \\
&+ \beta_2 T_2 g(I_2)\Big[- \varphi\Big(\frac{T_2}{T_{n+1}^m}\Big) - \varphi\Big(\frac{T_{n-m_1+1}^m g(I_{n-m_1}^m) I_2}{T_2 g(I_2) I_{n+1}^m}\Big) + \Big(\frac{g(I_{n+1})}{g(I_2)} \\
&- \frac{I_{n+1}^m}{I_2}\Big)\Big(1 - \frac{g(I_2)}{g(I_{n+1})}\Big)\Big] - d_2 e^{\mu_1\tau_1} I_{n+1}^m - p_1 e^{\mu_1\tau_1} Z_2 \frac{I_{n+1}^m Z_n^m}{Z_{n+1}^m} \Big\} \\
&- \frac{DV_2\beta_1 T_2 f(V_2)}{p_2 e^{-\mu_2\tau_2} I_2 (\Delta x)^2} \sum_{m=0}^{N-1} \frac{(V_{n+1}^{m+1} - V_{n+1}^m)^2}{V_{n+1}^{m+1} V_{n+1}^m}.
\end{aligned}
$$

Similar to the proof of Theorem 3, we have

$$
\Big(\frac{f(V(x,t))}{f(V_1)} - \frac{V}{V_1}\Big)\Big(1 - \frac{f(V_1)}{f(V(x,t))}\Big) \le 0,
$$

$$
\Big(\frac{g(I(x,t))}{g(I_1)} - \frac{I(x,t)}{I_1}\Big)\Big(1 - \frac{g(I_1)}{g(I(x,t))}\Big) \le 0,
$$

this implies that $\{\overline{W}_n\}$ is a monotone decreasing sequence, then $\overline{W}_n \ge 0$, $\lim_{n\to\infty} \overline{W}_n \ge 0$, therefore

$$
\lim_{n\to\infty} (\overline{W}_{n+1} - \overline{W}_n) = 0.
$$

According to the system (7), we claim that the CTL-activated equilibrium E_2 is not globally asymptotically stable. In fact, if the CTL-activated equilibrium E_2 is globally asymptotically stable, from the above inequality, we have

$$
0 \le -d_2 e^{\mu_1\tau_1} I_2 - p_1 e^{\mu_1\tau_1} Z_2 I_2 < 0,
$$

this is a contradiction. This completes the proof. □

4. Numerical Simulation

In this section, we choose $f(V) = V, g(I) = I$, some numerical results of system (4) are presented for supporting our analytic results. Based on biological meanings of virus dynamics model from papers [39,40], we have estimated the values of our model parameters as follows:

If we choose $D = 3$, then we can give a numerical simulation of the stability of system (4). Using the data in Table 1, we first show in a simulation that the interior equilibrium is stable (see Figure 1).

Table 1. State variables and parameters of HIV-1 infection model.

Parameter	Description	
λ	0.9	References [40]
d_1	0.03	Reference [39]
d_2	0.5	Reference [39]
d_3	0.1	Reference [40]
d_4	0.3	Reference [40]
β_1	0.3	Reference [40]
β_2	0.4	Reference [40]
p_1	$0.08\,\text{day}^{-1}$	Estimate
p_2	$0.5\,\text{day}^{-1}$	Reference [40]
q	0.4	Estimate

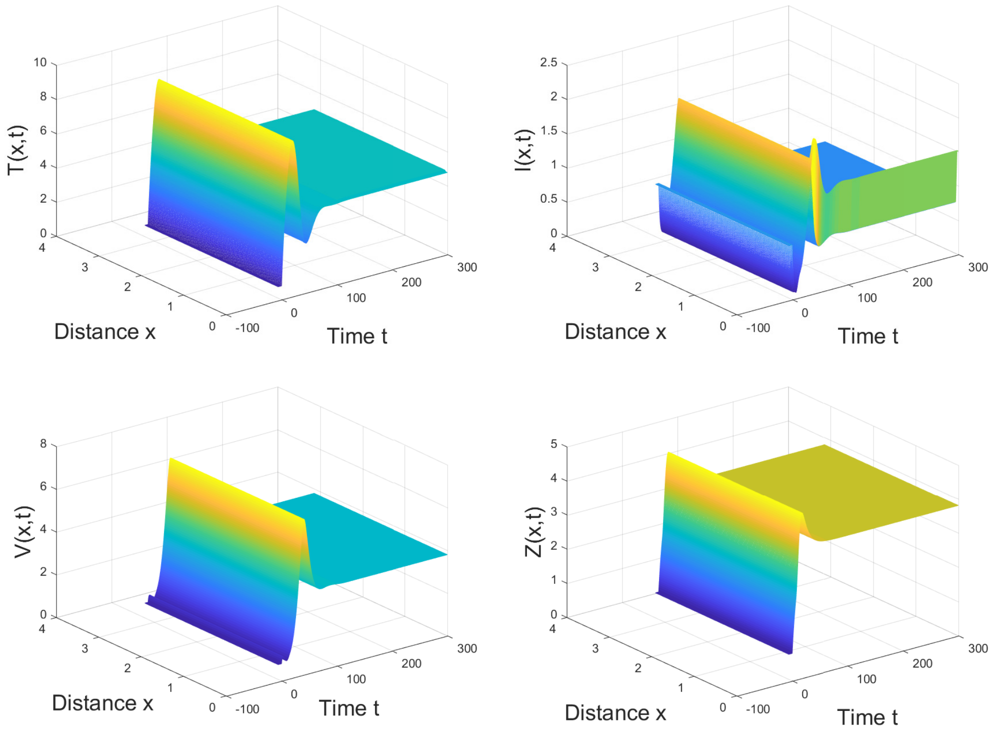

Figure 1. When $D = 3, R_1 > 1$, the interior equilibrium E_2 is globally asymptotically stable.

From Figure 1, we can see that the population has gradually stabilized after a sharp fluctuation.

If we choose $\beta_1 = 0.0003$ and $\beta_2 = 0.004$, then $R_0 < 1$. We can simulate that the infection-free equilibrium is globally asymptotically stable (see Figure 2).

From Figure 2 , we can see that the number of infected cells, virus and CTLs tends to zero, except uninfected cells.

If we choose $q = 0.000004$ and $p_2 = 0.9$, then $R_1 \leq 1 < R_0$. This moment the CTL-inactivated equilibrium is globally asymptotically stable (see Figure 3).

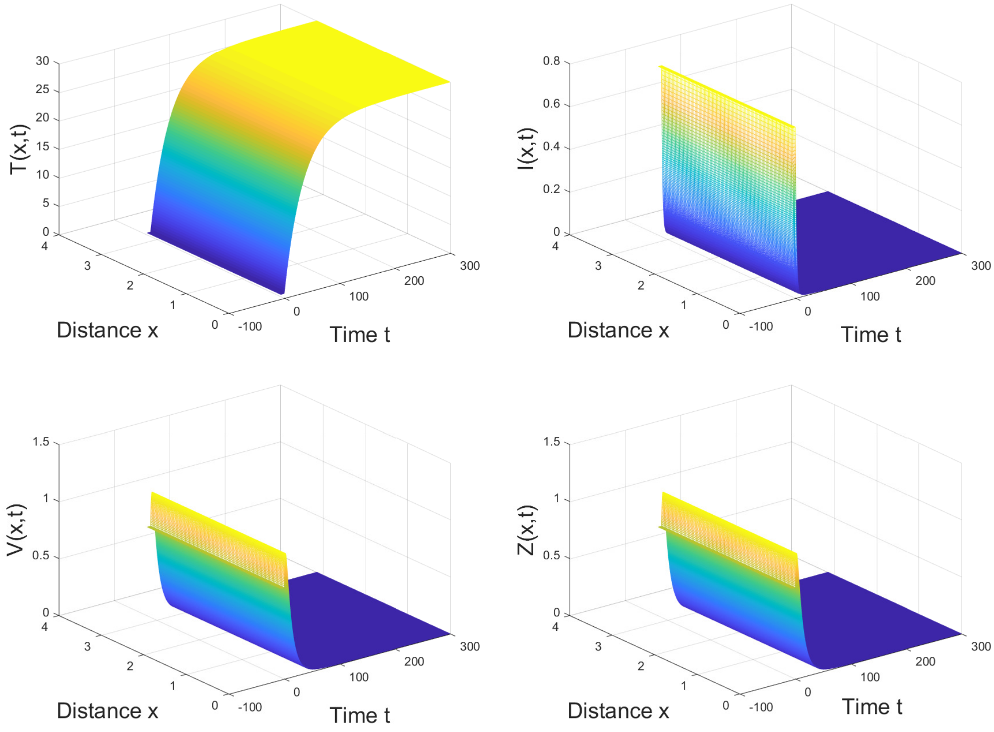

Figure 2. When $D = 3, R_0 < 1$, the infection-free equilibrium E_0 is globally asymptotically stable.

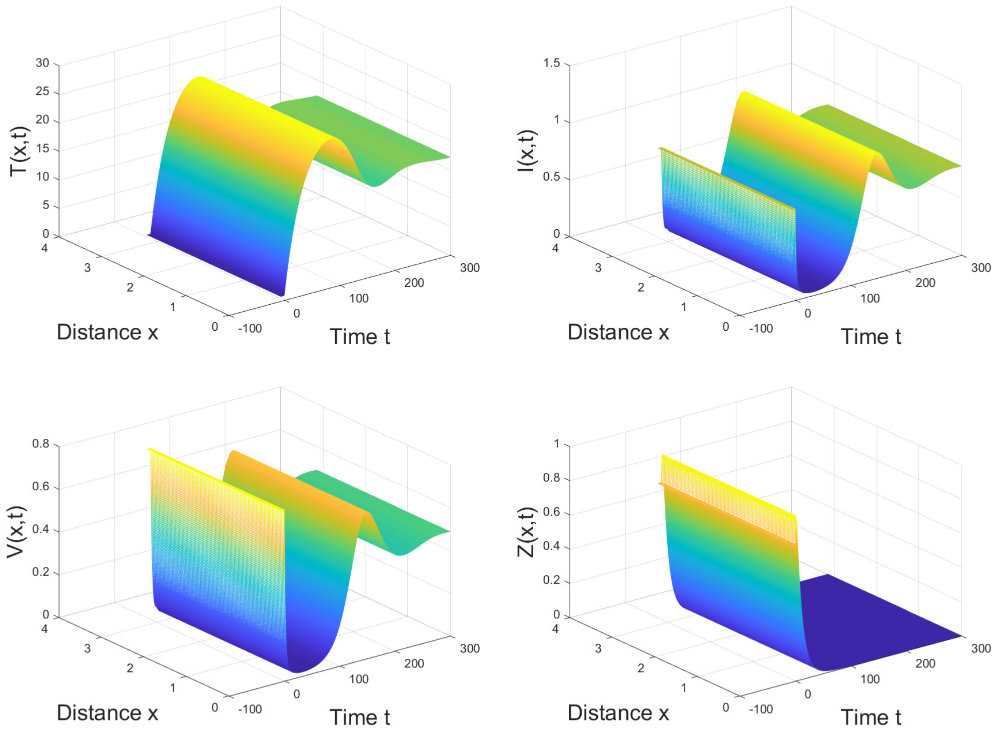

Figure 3. When $D = 3, R_1 \leq 1 < R_0$, the CTL-inactivated equilibrium E_1 is globally asymptotically stable.

From Figure 3, we can see that the population in the compartment $CTLs$ tends to 0. In addition, except for $CTLs$, the number of uninfected cells, infected cells, virus tends to certain constants.

The novelty of this paper is that we consider the effects of diffusion, time delay, and abstract functions on the spread of viruses. In order to see the impact of proliferation on the spread of the virus more intuitively, we first choose $q = 0.04$. Next, we select $D = 0$

and $D = 300$ decibels to simulate the image of I while other parameters keep the values in the Table 1.

The left image in Figure 4 is an image without time delay, and the right image is an image with time delay equal to 300. Since we are simulating long-term dynamic behavior, from the overall image of the two figures, there is no obvious difference in either the stable position or the growth rate. So where is the effect of diffusion reflected? We believe that the effect of diffusion should be reflected in the growth of I. Therefore, we project the two graphs in Figure 4 on the time-quantity axis (Figure 5).

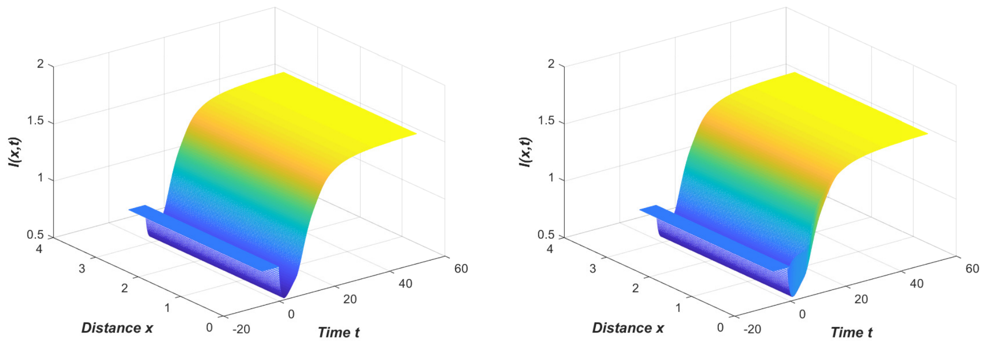

Figure 4. Comparison of compartment I at $D = 0$ and $D = 300$.

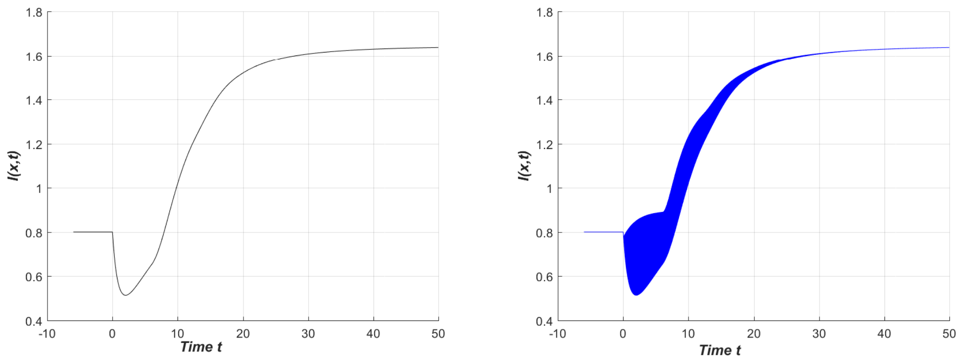

Figure 5. Comparison projection of compartment I when $D = 0$ and $D = 300$.

From the left image of Figure 5, we can clearly see that when there is no time delay, the image rises smoothly and the curve is smooth. When the time lag is equal to 300, the image is not a smooth curve, which shows that the proliferation brings about the proliferation of infected cells and the uneven fluctuation.

5. Conclusions and Discussion

It is necessary to understand the dynamics model for HIV infection since these infected cells usually cause a CTL response from the immune system. In this paper, we first developed a diffusive infection model (4) with general nonlinear incidence rate and two delays on the base of model (3), we show that the global stability of equilibria is completely determined by the reproductive numbers for viral infection R_0 and for CTL immune response R_1. Second, we considered the corresponding discretization of the continuous model by using nonstandard finite difference scheme, and then studied the global stability of the discrete system. Some numerical simulations were also presented to support our analytic results. In general, systems of PDE cannot be solved explicitly, and numerical solutions have to be studied instead. By using the NSFD scheme, we showed that the proposed discrete model partly preserves the global stability of equilibria of the corresponding continuous model. We plan to address how other diffusive terms (for infected and uninfected cells) affect the model in future work.

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