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Generalization of Fuzzy Connectives

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Abstract: This paper is centered around the creation of new fuzzy connectives using automorphism functions. The fuzzy connectives theory has been implemented in many problems and fields. In particular, the N-negations, t-norms, S-conorms and I-implications concepts played crucial roles in forming the theory and applications of the fuzzy sets. Thus far, there are multiple strategies for producing fuzzy connectives. The purpose of this paper is to provide a new strategy that is more flexible and fast in comparison with the rest. In order to create this method, automorphism and additive generator functions were utilized. The general formulas created with this method can provide new fuzzy connectives. The main conclusion is that new fuzzy connectives can be created faster and with more flexibility with our strategy.

Keywords: fuzzy connectives; fuzzy negations; t-norms; S-conorms; fuzzy implications; automorphisms

MSC: 03B52

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1. Introduction

Fuzzy connectives play a crucial role in many applications of fuzzy logic, such as approximate reasoning, formal methods of proof, inference systems, and decision support systems. Recognizing the above importance, many methods of creating fuzzy connectives have been discovered. Most of them refer to the t-norms and I-implications fuzzy connectives. These methods, as well as the fuzzy connectives they produce, are visible in Figure 1.

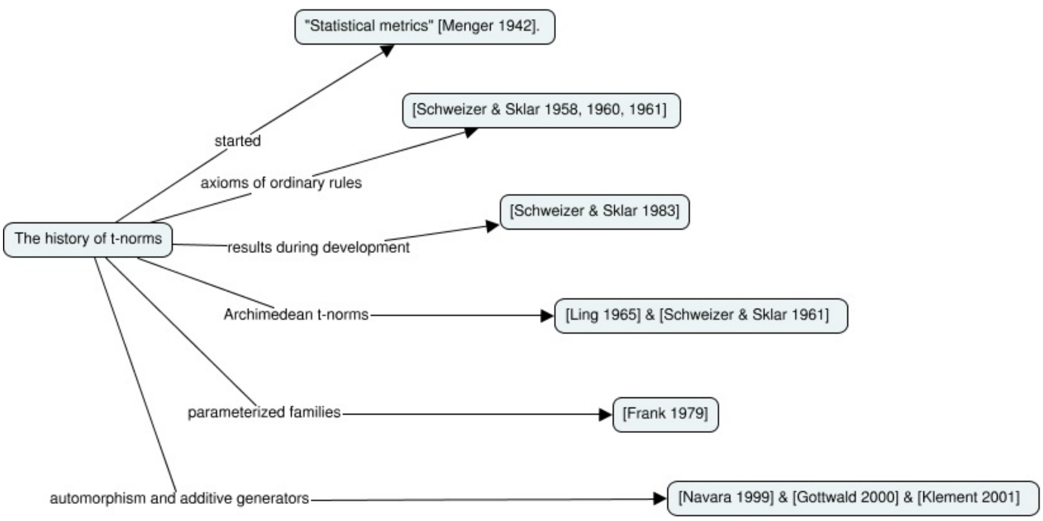

Figure 1. The history and evolution of t-norms.

In 1942, Menger, in his paper "Statistical metrics", was the first to use the concept of t-norms [1]. Schweizer B. and Sklar A., in work published in 1958, 1960, 1961 and

1983 [2], defined the axioms of ordinary rules and presented the results that occurred during development. Then, Ling C.H., in 1965 [3], built upon B. Schweizer's and A. Sklar's work and defined the Archimedean t-norms. Frank M.J., in 1979 [4], defined the parameterized families of t-norms. Finally, Navara M. in 1999 [5], Gottwald S. in 2000 [6] and Klement E.P. in 2001 [7] introduced the method of producing t-norms via automorphism and additive generator functions.

Kerre E., Huang C. and Ruan D. discovered the modus ponens and modus tollens in 2004 [8]; Trillas E., Mas M., Monserrat M. and Torrens J., in 2008, discovered different implications with varying properties [9]. Thereafter, in 2004, Kerre E. and Nachtegael M. formed the fuzzy mathematical morphology [10]. Furthermore, Bustince H. et al., in 2006, discovered fuzzy measures and image processing [11]. Moreover, Baczyński M. and Jayaram B., as well as Mas M., Monserrat M., Torrens J. and Trillas E., in 2007, created the first strategy, which generates (S,N)-implications [12,13]; Fodor J.C. and Roubens M., in 1994, created the second strategy, which generates R-Implications [14]. The third strategy, which generates QL and D-operations, was created by Mas M., Monserrat M. and Torrens J. in 2006 [15]. In 2004, Yager R.R. created the fourth strategy, which generates f- and g-implications [16]. Finally, Bustince H., Burillo P. and Soria F. in 2003 [17], as well as Callejas C., Marcos J. and Bedregal B. in 2012, created the fifth strategy, which generates any fuzzy implication [18].

Since 2012, there has been no further research focused on the fuzzy connectives. Therefore, this paper was created in order to build upon the previous discoveries and improve them by creating a faster and more flexible strategy for producing fuzzy connectives, which, in turn, produces more flexible results.

2. Literature Review

In the Introduction, a review of milestones achieved by other researchers in the field of fuzzy connectives was given. However, this section is dedicated to the presentation of published research of other researchers in the field of the generalization of fuzzy connectives. The goal of this presentation is the exploration of other viewpoints on the subject of this paper. In the following table, the research published for every primary category of fuzzy connectives is presented:

The field of the generalization of fuzzy connectives has been explored by many researchers over the years. As a result, the four main categories of fuzzy connectives have been the subject of many research papers which contributed to the development of the field.

The published research of the negation connectives category (see Table 1) offered many contributions to the field of the generalization of fuzzy connectives. To be more specific, the book *Fuzzy Preference Modelling and Multicriteria Decision Support* (see [14]) and paper "Related Connectives for Fuzzy Logics" (see [19]) contributed by offering definitions, properties and theorems. The paper "A treatise on many-valued logics" (see [6]) contributed by offering a new strategy for generalizing fuzzy connectives via automorphisms.

Similarly, for the conjunction connectives: The paper "A Treatise on Many-Valued Logics" (see [6]) contributed by offering new methods for generalizing conjunction connectives. The paper "Triangular norms" (see [7]) contributed by offering new methods for constructing t-norms as well as t-norm families. The paper "Characterization of Measures Based on Strict Triangular Norms" (see [5]) contributed by offering new strategies for producing t-norms and especially Frank's t-norms. The paper "The best interval representations of t-norms and automorphisms" (see [20]) contributed by offering new methods of producing t-norms, especially interval t-norms and interval automorphisms.

Similarly, for the disjunction connectives: The paper "Connectives in Fuzzy Logic" (see [21]) contributed by offering new triples of t-norms, t-conorms and n-negations, which prove multiple theorems. The book *Fuzzy Implications* (see [22]) contributed by offering a complete presentation of the published research until 2008. The paper "A treatise on many-valued logics" (see [6]) contributed by offering a combination of t-norms and t-conorms, which proves multiple theorems. The paper "Triangular norms" (see [7]) contributed by offering a combination of t-norms and t-conorms, which proves multiple definitions and properties.

Table 1. Published research of every fuzzy connectives category.

Category	Published Research
Negation Connectives	*Fuzzy Preference Modelling and Multicriteria Decision Support* [14] "Nilpotent Minimum and Related Connectives for Fuzzy Logics" [19] "A treatise on many-valued logics" [6]
Conjunction Connectives	"A treatise on many-valued logics" [6] "Triangular norms" [7] "Characterization of Measures Based on Strict Triangular Norms" [5] "The best interval representations of t-norms and automorphisms" [20]
Disjunction Connectives	"Connectives in Fuzzy Logic" [21] *Fuzzy Implications* [22] "A treatise on many-valued logics" [6] "Triangular norms" [7]
Implication Connectives	"Fuzzy Implications" [22] "Automorphisms, negations and implication operators" [17] "Actions of Automorphisms on Some Classes of Fuzzy Bi-implications" [18]

Finally, for the implication connectives: The book *Fuzzy Implications* (see [22]) contributed by offering a complete presentation of the published research until 2008. The paper "Automorphisms, negations and implication operators" (see [17]) contributed by offering a new strategy for constructing implications via automorphisms. The paper "Actions of Automorphisms on Some Classes of Fuzzy Bi-implications" (see [18]) contributed by offering a new class of implications, using automorphisms, the bi-implications class.

3. Preliminaries

In this section, the definitions and basic properties of the negation, conjunction, disjunction and implication operators in fuzzy logic are provided. The concepts of automorphism and conjugate are used throughout the whole paper.

3.1. Fuzzy Negations

Some definitions retrieved from the literature can be found in the following references: (Baczyński M., 1.4.1–1.4.2 Definitions, pp. 13–14, [22]), (Bedregal B.C., p. 1126, [23]), (Fodor J., 1.1–1.2 Definitions, p. 3, [14]), (Gottwald S., 5.2.1 Definition, p. 85, [6]), (Weber S., 3.1 Definition, p. 121, [24]) and (Trillas E., p. 49, [25]).

Definition 1. *A function $N : (0, 1) \to [0, 1]$ is called a Fuzzy negation if*
$(N1)$ $N(0) = 1, \; N(1) = 0;$
$(N2)$ *N is decreasing.*
A fuzzy negation N is called strict if, in addition to the former properties, the following apply:
(N3) N is strictly decreasing;
(N4) N is continuous.
A fuzzy negation N is called strong if the following property is satisfied:
$(N5)$ $N(N(x)) = x, \;\; x \in [0, 1].$

The following table presents two well-known families of fuzzy negations. Those fuzzy negations can be found in the work by Baczyński M., p. 15, [22].

3.2. Triangular Norms (Conjunctions)

The history and evolution of t-norms was already explored in a previous section (see Figure 1). Therefore, in this subsection the definition and properties of t-norms will be provided.

The following definition can be found in: (Klement E.P et al., 1.1 Definition, pp. 4–10, [7]), (Baczyński M., 2.1.1, 2.1.2 Definitions, pp. 41–42, [22]), (Weber S., 2.1 Definition, pp. 116–117, [24]) and (Yun s., p. 16, [26]).

Definition 2. *A function $T : [0,1]^2 \to [0,1]$ is called a triangular norm, shortly, t-norm, if it satisfies, for all $x, y \in [0,1]$, the following conditions:*
$(T1)\quad T(x,y) = T(y,x), \quad (commutativity);$
$(T2)\quad T(x,T(y,z)) = T(T(x,y),z), \quad (associativity);$
$(T3)\quad if\ \ y \le z,\ \ then\ \ T(x,y) \le T(x,z), \quad (monotonicity);$
$(T4)\quad T(x,1) = x, \quad (boundary\ condition).$

In the following table, three well-known t-norms are presented. Those t-norms can be found in: (Baczyński M., p. 42, [22]).

3.3. Triangular Conorms (Disjunctions)

The t-conorm or S-conorm are a dual concept. Both ideas allow for the generalization of the union in a lattice or disjunction in logic. The following definition can be found in: (Klement E.P et al., 1.13 Definition, p. 11, [7]), (Baczyński M., 2.2.1, 2.2.2 Definitions, pp. 45–46, [22]) and (Yun s., p. 22, [26]).

Definition 3. *A function $S : [0,1]^2 \to [0,1]$ is called a triangular conorm (shortly t-conorm) if it satisfies, for all $x, y \in [0,1]$, the following conditions:*
$(S1):\ S(x,\ y)\ =\ S(y,\ x)\ (commutativity);$
$(S2):\ S(x,\ S(y,\ z))\ =\ S(S(x,\ y),\ z)\ (associativity);$
$(S3):$ *If* $y\ \le\ z$, *then* $S(x,\ y)\ \le\ S(x,\ z)\ \ (monotonicity);$
$(S4):\ S(x,\ 0)\ =\ x$ *(neutral element 0).*

In the following Table 2, three well-known t-conorms are presented. Those t-conorms can be found: (Baczyński M., p. 46, [22]).

Table 2. Basic t-conorms.

Designation	Equation
Maximum or Gödel t-conorm	$S_M(x,y) = \max\{x,y\}$
Product t-conorm, probabilistic sum	$S_P(x,y) = x + y - x \cdot y$
Lukasiewicz t-conorm, bounded sum	$S_L(x,y) = \min(x+y,1)$
Drastic Sum	$S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$

3.4. Fuzzy Implications

The fuzzy implication functions are probably some of the main functions in fuzzy logic. They play a similar role to that played by classical implications in crisp logic. The fuzzy implication functions are used to execute any fuzzy "if-then" rule on fuzzy systems. The following definition can be found: (Baczyński M., p. 2, [22]), (Yun s., p. 5, [26]) and (Fodor J., p. 299, [27]).

Definition 4. *A binary operator* $I : [0,1]^2 \to [0,1]$ *is said to be an implication function, or an implication, if, for all* $x, y \in [0,1]$*, it satisfies:*
$(I1) : I(x,\ z) \geq I(y,\ z)\ \ when\ x \leq y,\ \ the\ first\ place\ antitonicity;$
$(I2) : I(x,\ y) \leq I(x,\ z)\ \ when\ y \leq z,\ \ the\ second\ place\ isotonicity;$
$(I3) : I(0,0)\ =\ 1,\ boundary\ condition;$
$(I4) : I(1,1)\ =\ 1,\ boundary\ condition;$
$(I5) : I(1,0)\ =\ 0,\ boundary\ condition.$

A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication only if it satisfies $(I1)$–$(I5)$. The set of all these fuzzy implications will be denoted by FI.

3.5. Automorphism Functions

Automorphism functions play an instrumental role in fuzzy connectives. This is the case because they are necessary for their generalization.

The following definition can be found in: (Bedregal B., p. 1127, [23]), (Bustince H, B., p. 211, [17]) and (Yun s., p. 13, [26]).

Definition 5. *A mapping* φ *:*$[a, b] \to [a, b]$ *(*$[a, b] \subset \mathbb{R}$*) is an automorphism of the interval [a, b] if it is continuous and strictly increasing and satisfies the boundary conditions:* $\varphi(a) = a$ *and* $\varphi(b) = b$*. If* φ *is an automorphism of the unit interval, then* φ^{-1} *is also an automorphism of the unit interval.*

Definition 6. *By* Φ*, we denote the family of all increasing bijections from [0, 1] to [0, 1]. We say that functions* $f,\ g :\ [0,\ 1]^n\ \to\ [0,\ 1]$ *are* Φ*-conjugate if there exists a* $\phi \in \Phi$ *such that* $g\ =\ f_\phi$*, where* $f_\phi(x_1,\ \ldots,\ x_n)\ :=\ \phi^{-1}(f(\phi(x_1),\ \ldots,\ \phi(x_n))),\ x_1,\ \ldots,\ x_n \in [0,\ 1]$*.*

4. Materials and Methods

In this section, the methods used in this paper are presented in detail.

The following theorem presents the general form of fuzzy negations using automorphism functions. The researchers (J.C. Fodor and M. Roubens, Theorem 1.1, p. 4, [14]), (Gottwald S., Theorem 5.2.1 p. 86, [6]) and (Fodor J., p. 2077, [19]) have worked with functions of this type, but they focused mainly on natural negations. The general formula (1) can be used in order to generate new fuzzy negations (see Example 1i.).

Theorem 1. *Let* $N_\varphi : [0,1] \to [0,1]$ *be a function.* N_φ *is a strong negation if and only if there is another strong negation* N *and an automorphism* φ *such that:*

$$N_\varphi(x) = \varphi^{-1}(N(\varphi(x))),\ \forall x \in [0,1] \tag{1}$$

Proof of Theorem 1. (⇒)

It is easy to see that the function N_φ is defined by (1) and is an involution with the properties $N_\varphi(0) = 1$ and $N_\varphi(1) = 0$. In addition, it is strictly decreasing. Hence, N_φ is a strong negation function (see Bedregal B.C., Proposition 3.2, p. 1127, [23]).

(⇐)

We will prove that a strong negation $N_\varphi(x)$ is written in the form (1).

Let be a function $N_\varphi : [0,1] \to [0,1]$ be a strong negation and satisfy the following:
$N_\varphi \downarrow [0,1],\ is\ strictly\ decreasing,$
$N_\varphi(0) = 1,$
$N_\varphi(1) = 0,$
$N_\varphi,\ is\ \ continuous,$ and
$N_\varphi(N_\varphi(x)) = x.$
Suppose there is a fixed point $x_0 \in (0,1) :\ N_\varphi(x_0) = x_0$.
Additionally assume there is a strictly increasing, bijective function

$$h : [0, x_0] \to [0, \varphi(x_0)], h(0) = 0,\ h(x_0) = \varphi(x_0),\ h(1) = 1.$$

Let a function N be a strong negation in [0, 1] with $N(h(x)) = h(x)$.

We define a function $\varphi : [0,1] \to [0,1]$ with formula

$$\varphi(x) = \begin{cases} h(x), & x \in [0, x_0] \\ N(h(N_\varphi(x))), & x \in (x_0, 1] \end{cases}$$

We will prove that φ is an automorphism function.

Indeed:

If $\varphi(x) = h(x)$, then $h(x)$ is a strictly increasing function.

If $x \in (x_0, 1]$, then N_φ is a strictly decreasing function and h is a strictly increasing function. Then $h(N_\varphi(x))$ is a strictly decreasing function. Thus, $N(h(N_\varphi(x)))$ is a strictly increasing function in [0, 1].

Therefore, φ is a strictly increasing function in [0, 1].

$$\varphi(1) = N(h(N_\varphi(1))) = N(h(0)) = N(0) = 1.$$

$$\varphi(0) = N(h(N_\varphi(0))) = N(h(1)) = N(1) = 0.$$

Therefore, φ is an automorphism function.

We define the inverse function with the formula:

$$\varphi^{-1}(x) = \begin{cases} h^{-1}(x), & x \in [0, \varphi(x_0)] \\ N_\varphi(h^{-1}(N(x))), & x \in (\varphi(x_0), 1] \end{cases}$$

If $x \in [0, \varphi(x_0)]$, then

$$\varphi^{-1}(N(\varphi(x))) = N_\varphi(h^{-1}(N(\varphi(x)))) = N_\varphi(h^{-1}(N(h(x)))) = N_\varphi(h^{-1}(h(x))) = N_\varphi(x)$$

If $x \in (\varphi(x_0), 1]$, then

$$\varphi^{-1}(N(\varphi(x))) = h^{-1}(N(\varphi(x))) = h^{-1}(N(N(h(N_\varphi(x))))) = h^{-1}(h(N_\varphi(x))) = N_\varphi(x)$$

Consequently, Formula (1) applies. □

The following theorem presents the general form of t-norms using an automorphism function. Researchers (see René B. et al., Theorem 2.3, p. 372, [20]) and (Gottwald S., Theorem 5.1.3, p. 82, [6]) worked with such functions, but they focused mainly on the specific forms of t-norms (see Table 3). Formula (2) can be used to generate new t-norms (see Example 1ii).

Table 3. Basic fuzzy negations classes.

Designation	Equation
Sugeno class	$N^\lambda(x) = \frac{1-x}{1+\lambda x}, \ \lambda \in (-1, +\infty)$
Yager class	$N^W(x) = (1-x^w)^{\frac{1}{w}}, \ w \in (0, +\infty)$

Theorem 2. *Let $T_\varphi : [0,1] \to [0,1]$ be a function. T_φ is a strict and Archimedean t-norm if and only if there is another strict and Archimedean t-norm T and an automorphism φ such that:*

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))), \ \forall x, y \in [0,1] \tag{2}$$

Proof of Theorem 2. (⇒)

We will prove that Formula (2) is a strict and Archimedean t-norm.

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))) = \varphi^{-1}(T(\varphi(y), \varphi(x))) = T_\varphi(y,x)$$

Therefore, the function T_φ is commutative.

$$T_\varphi(x, T_\varphi(y,z)) = \varphi^{-1}(T(\varphi(x), \varphi(T_\varphi(y,z)))) = \varphi^{-1}(T(\varphi(x), \varphi(\varphi^{-1}(T(\varphi(y), \varphi(z)))))) = \varphi^{-1}(T(\varphi(x), T(\varphi(y), \varphi(z)))) = \varphi^{-1}(T(T(\varphi(x), \varphi(y)), \varphi(z)))$$
$$T_\varphi(T_\varphi(x,y), z) = \varphi^{-1}(T(\varphi(T_\varphi(x,y)), \varphi(z))) = \varphi^{-1}(T(\varphi(\varphi^{-1}(T(\varphi(x), \varphi(y)), \varphi(z))))) = \varphi^{-1}(T(T(\varphi(x), \varphi(y)), \varphi(z)))$$

Therefore, the function T_φ is associative.

$$\forall\ y \leq z \Leftrightarrow \varphi(y) \leq \varphi(z) \Leftrightarrow T(\varphi(x), \varphi(y)) \leq T(\varphi(x), \varphi(z)) \Leftrightarrow \varphi^{-1}(T(\varphi(x), \varphi(y))) \leq \varphi^{-1}(T(\varphi(x), \varphi(z)))$$

Therefore, the function T_φ is monotonous with respect to the second variable.

$$T_\varphi(x, 1) = \varphi^{-1}(T(\varphi(x), \varphi(1))) = \varphi^{-1}(T(\varphi(x), 1)) = \varphi^{-1}(\varphi(x)) = x.$$

Therefore, the function T_φ satisfies the boundary condition.
The function T_φ is continuous with respect to the two variables.

$$\forall\ x < 1 \Leftrightarrow T_\varphi(x,x) < T_\varphi(x,1) \Leftrightarrow T_\varphi(x,x) < x$$

Therefore, the function T_φ is Archimedean.
Consequently, the function given by Formula (2) is a strict and Archimedean t-norm.
(⇐)
From the theorem of the additive generator, we obtain: $T(x,y) = f^{(-1)}(f(x) + f(y))$, where the function f is a strictly decreasing function, $f(0) = b$, $b \in \mathbb{R}_0$ and $f(1) = 0$ (see Baczyński M., Theorem 2.1.5, p. 43, [22]) and (Gottwald S., Theorem 5.1.2, p. 78, [6]).
We define the function $h : [0,1] \to [0,1]$ with the formula:

$$h(x) = -\frac{e^{-b}}{1 - e^{-b}} + \frac{e^{-f(x)}}{1 - e^{-b}},$$

where h is a strictly increasing function in [0, 1], $h(0) = 0$ and $h(1) = 1$.
The function h is inverted with the inverse:

$$h^{-1}(x) = f^{-1}(-\ln(x(1 - e^{-b}) + e^{-b})) = f^{(-1)}(-\ln(x(1 - e^{-b}) + e^{-b}))$$
$$h(T(x,y)) = h(f^{(-1)}(f(x) + f(y))) = h(f^{(-1)}(-\ln(h(x)(1 - e^{-b}) + e^{-b}) - \ln(h(y)(1 - e^{-b}) + e^{-b}))) = h(h^{-1}(T(h(x), h(y))) = T(h(x), h(y))$$

Consequently, $T_\varphi(x,y) = h^{-1}(T(h(x), h(y)))$. □

Theorems 3–5 produce the same t-conorm. To be more specific, Theorem 3 presents the general form of t-conorms using an automorphism function. Formula (3) can be used to generate new t-conorms (see Example 1iii).

Theorem 3. *Let $S_\varphi : [0,1] \to [0,1]$ be a function which is a strict and Archimedean t-conorm if and only if there is another strict and Archimedean S t-conorm and an automorphism φ such that:*

$$S_\varphi(x,y) = \varphi^{-1}(S(\varphi(x), \varphi(y))),\ \ \forall x, y \in [0,1] \tag{3}$$

Proof of Theorem 3. **(⇒)**
We will prove that Formula (3) is a strict and Archimedean t-conorm.

$$S_\varphi(x,y) = \varphi^{-1}(S(\varphi(x), \varphi(y))) = \varphi^{-1}(S(\varphi(y), \varphi(x))) = S_\varphi(y,x),\ \ \forall x, y \in [0,1]$$

Therefore, the function S_φ is commutative.

$$S_\varphi(x, S_\varphi(y,z)) = \varphi^{-1}(S(\varphi(x), \varphi(S_\varphi(y,z)))) = \varphi^{-1}(S(\varphi(x), \varphi(\varphi^{-1}(S(\varphi(y), \varphi(z)))))) = \\ \varphi^{-1}(S(\varphi(x), S(\varphi(y), \varphi(z)))) = \varphi^{-1}(S(S(\varphi(x), \varphi(y)), \varphi(z)))$$
$$S_\varphi(S_\varphi(x,y), z) = \varphi^{-1}(S(\varphi(S_\varphi(x,y)), \varphi(z))) = \varphi^{-1}(S(\varphi(\varphi^{-1}(S(\varphi(x), \varphi(y)), \varphi(z))))) = \\ \varphi^{-1}(S(S(\varphi(x), \varphi(y)), \varphi(z)))$$

Therefore, the function S_φ is associative.

$If\ x \le z\ and\ y \le u\ \Rightarrow S_\varphi(\varphi(x), \varphi(y)) \le S_\varphi(\varphi(z), \varphi(u))$, then it is monotonous.

$If\ x \le z \Leftrightarrow \varphi(x) \le \varphi(z)$

$If\ y \le u \Leftrightarrow \varphi(y) \le \varphi(u)$

$$If\ x \le z\ and\ y \le u\ \Leftrightarrow S(\varphi(x), \varphi(y)) \le S(\varphi(z), \varphi(u)) \Leftrightarrow \\ \varphi^{-1}(S(\varphi(x), \varphi(y))) \le \varphi^{-1}(S(\varphi(z), \varphi(u))) \Leftrightarrow S_\varphi(\varphi(x), \varphi(y)) \le S_\varphi(\varphi(z), \varphi(u))$$

Therefore, the function S_φ is monotonous.

The boundary condition applies to the function S_φ.

Consequently, the function S_φ is a t-conorm.

The function S_φ is continuous with respect to the two variables.

For a continuous t-conorm S_φ, the Archimedean property is given by the simpler condition $S_\varphi(x,\ x)\ >\ x\ ,\ x \in (0,\ 1)$.

Indeed,

$$S_\varphi(x,\ x)\ >\ x \Leftrightarrow \varphi^{-1}(S(\varphi(x), \varphi(x))) >\ x \Leftrightarrow \varphi(\varphi^{-1}(S(\varphi(x), \varphi(x)))) >\ \varphi(x) \Leftrightarrow \\ S(\varphi(x), \varphi(x)) >\ \varphi(x)$$

holds because the function S is Archimedean. Therefore, the function S_φ is Archimedean. Consequently, the function S_φ given by Formula (3) is a strict and Archimedean t-conorm.

(⇐)

From the theorem of additive generators, we obtain: $S(x,y) = g^{(-1)}(g(x) + g(y))$, where the function is strictly increasing, $g(0) = 0$, $g(1) = b$ and $b \in \mathbb{R}_0$ (see Baczyński M., Theorem 2.2.6, p. 47, [22]).

We define the function $h : [0,1] \to [0,1]$ with the formula $h(x) = \frac{e^{g(x)}-1}{e^b-1}$, where h is a strictly increasing function in [0, 1], $h(0) = 0$ and $h(1) = 1$.

The function h is inverted with inverse:

$$h^{-1}(x) = g^{-1}(\ln(x(e^b - 1) + 1)) = g^{(-1)}(\ln(x(e^b - 1) + 1))$$

$$h(S(x,y)) = h(g^{(-1)}(g(x) + g(y))) = \\ h(g^{(-1)}(\ln(h(x)(e^b - 1) + 1) + \ln(h(y)(e^b - 1) + 1))) = \\ h(h^{-1}(S(h(x), h(y))) = S(h(x), h(y))$$

Consequently, $S_\varphi(x,y) = h^{-1}(S(h(x), h(y)))$. □

The following theorem presents the general form of t-conorms using an automorphism function according to the equation $S(x,y) = 1 - T(1-x, 1-y)$ (see Klement E.P., Proposition 1.15, p. 11 [7]), (Alsina C., Definition 3.3, p. 2, [21]) and (see Baczyński M., Proposition 2.2.3, p. 46, [22]). Formula (4) can be used to generate new t-conorms (see Example 1iv).

Theorem 4. *If there exists a continuous (Archimedean, strict, nilpotent) t-norm and an automorphism φ such that $S_\varphi : [0,1] \to [0,1]$ is defined by*

$$S_\varphi(x,y) = 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1-y))),\ \ \forall x,y \in [0,1] \tag{4}$$

then S_φ is a continuous (Archimedean, strict, nilpotent) t-conorm.

Proof of Theorem 4. From (Klement E.P., Proposition 1.15 , p. 11 [7]), (Alsina C., Definition 3.3, p. 2, [21]) and (Baczyński M., Proposition 2.2.3, p.46, [22]),

$$S(x,y) = 1 - T(1-x, 1-y) \Leftrightarrow S_\varphi(x,y) = 1 - T_\varphi(1-x, 1-y) \Leftrightarrow \\ S_\varphi(x,y) = 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1-y)))$$

$$S_\varphi(x,y) = 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1-y))) = 1 - \varphi^{-1}(T(\varphi(1-y), \varphi(1-x))) = S_\varphi(y,x).$$

Therefore, the function S_φ satisfies the commutativity property.

$$S_\varphi(x, S_\varphi(y,z)) = 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1 - S_\varphi(y,z)))) = \\ 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1 - 1 + \varphi^{-1}(T(\varphi(1-y), \varphi(1-z)))))) = \\ 1 - \varphi^{-1}(T(\varphi(1-x), T(\varphi(1-y), \varphi(1-z)))) = \\ 1 - \varphi^{-1}(T(T(\varphi(1-x), \varphi(1-y)), \varphi(1-z))) \\ S_\varphi(S_\varphi(x,y), z) = 1 - \varphi^{-1}(T(\varphi(1 - S_\varphi(x,y), \varphi(1-z)))) = \\ 1 - \varphi^{-1}(T(\varphi(1 - 1 + \varphi^{-1}(T(\varphi(1-x), \varphi(1-y), \varphi(1-z)))))) = \\ 1 - \varphi^{-1}(T(T(\varphi(1-x), \varphi(1-y)), \varphi(1-z)))$$

Therefore, the function S_φ satisfies the associativity property.

$$\forall\, x,y,z,u \in [0,1] \; with \; x \leq z \; and \; y \leq u \text{ apply:} \\ S_\varphi(\varphi(x), \varphi(y)) \leq S_\varphi(\varphi(z), \varphi(u)) \Leftrightarrow \\ 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(1-y))) \leq 1 - \varphi^{-1}(T(\varphi(1-z), \varphi(1-u))) \Leftrightarrow \\ \varphi^{-1}(T(\varphi(1-x), \varphi(1-y))) \geq \varphi^{-1}(T(\varphi(1-z), \varphi(1-u))) \Leftrightarrow \\ T(\varphi(1-x), \varphi(1-y)) \geq T(\varphi(1-z), \varphi(1-u)) \Leftrightarrow \\ \begin{cases} \varphi(1-x) \geq \varphi(1-z) \\ \varphi(1-y) \geq \varphi(1-u) \end{cases} \Leftrightarrow \begin{cases} 1-x \geq 1-z \\ 1-y \geq 1-u \end{cases} \Leftrightarrow \begin{cases} x \leq z \\ y \leq u \end{cases}$$

Therefore, the function S_φ satisfies the monotonicity property.

$$S_\varphi(x,0) = 1 - \varphi^{-1}(T(\varphi(1-x), \varphi(0))) = 1 - \varphi^{-1}(T(\varphi(1-x), 0)) = \\ 1 - \varphi^{-1}(\varphi(1-x)) = 1 - 1 + x = x$$

Therefore, the function S_φ satisfies the boundary condition.
We observe that the function S_φ is a t-conorm.
In addition, the function S_φ is continuous because it is continuous in both arguments.
The function S_φ is Archimedean if $S_\varphi(x,y) > x$.
Suppose that

$$S_\varphi(x,y) > x \Leftrightarrow \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) > x \Leftrightarrow \\ N(T(N(\varphi(x)), N(\varphi(y)))) > \varphi(x) \Leftrightarrow T(N(\varphi(x)), N(\varphi(y))) < N(\varphi(x)$$

applies because the t-norm T is Archimedean.

The function S_φ is strict because it is continuous and strictly monotonous.

The function S_φ is nilpotent because, if S_φ is continuous and Archimedean, then there exist some $x, y \in (0,1)$ such that $S_\varphi(x,y) = 1$.

Indeed,

$$S_\varphi(x,y) = 1 \Leftrightarrow \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) = 1 \Leftrightarrow \\ N(T(N(\varphi(x)), N(\varphi(y)))) = \varphi(1) \Leftrightarrow N(T(N(\varphi(x)), N(\varphi(y)))) = 1 \Leftrightarrow \\ T(N(\varphi(x)), N(\varphi(y))) = 0$$

applies, because the t-norm T is continuous, strict and Archimedean; therefore, there are $x, y \in (0,1)$ such that $T(x,y) = 0$ (see Klement E.P., Theorem 2.18, p. 33, [7]). □

Theorem 5 presents the general form of t-conorms using an automorphism function, according to the equation $S(x,y) = N(T(N(x), N(y)))$ (see Gottwald S., Proposition 5.3.1, p. 90, [6]). Formula (5) can be used to generate new t-conorms (see Example 1v).

Theorem 5. *If there exists a continuous (Archimedean, strict, nilpotent) t-conorm* $S_\varphi : [0,1] \to [0,1]$*, a (strong negation)* $N_\varphi : [0,1] \to [0,1]$*, a continuous (Archimedean, strict, nilpotent) t-norm* $T_\varphi : [0,1] \to [0,1]$ *and an automorphism* φ *such that it is defined by*

$$S_\varphi(x,\ y) = \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))),\ \ \forall x, y \in [0,1] \tag{5}$$

then S_φ *is a continuous (Archimedean, strict, nilpotent) t-conorm.*

Proof of Theorem 5. From (Gottwald S., Proposition 5.3.1, p. 90, [6]),

$$\begin{gathered} S_\varphi(x,\ y)\ =\ N_\varphi\big(T_\varphi\big(N_\varphi(x), N_\varphi(y)\big)\big),\ \ \forall x, y \in [0,1] = \\ \varphi^{-1}(N(\varphi(T_\varphi\big(N_\varphi(x), N_\varphi(y)\big)))) = \varphi^{-1}(N(\varphi(\varphi^{-1}(T(\varphi(N_\varphi(x), \varphi(N_\varphi(y)))))) = \\ \varphi^{-1}(N(T(\varphi(\varphi^{-1}(N(\varphi(x))), \varphi(\varphi^{-1}(N(\varphi(y)))))) = \\ \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) \end{gathered}$$

$$S_\varphi(x,\ y)\ =\ \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) = \varphi^{-1}(N(T(N(\varphi(y)), N(\varphi(x))))) = S_\varphi(y,\ \mathrm{x})$$

Therefore, the function S_φ satisfies the commutativity property.

$$\begin{gathered} S_\varphi(x, S_\varphi(y,z)) = \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(S_\varphi(y,z)))))) = \\ \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(\varphi^{-1}(N(T(N(\varphi(y)), N(\varphi(z)))))))))) = \\ \varphi^{-1}(N(T(N(\varphi(x)), T(N(\varphi(y)), N(\varphi(z)))))) = \\ \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y)), T(N(\varphi(z)))))) = \\ S_\varphi(S_\varphi(x,y), z) \end{gathered}$$

Therefore, the function S_φ satisfies the associativity property.

$$\begin{gathered} \forall\, x, y, z, u \in [0,1]\ with\ x \le z\ and\ y \le u \text{ apply:} \\ S_\varphi(\varphi(x), \varphi(y)) \le S_\varphi(\varphi(z), \varphi(u)) \Leftrightarrow \\ \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) \le \varphi^{-1}(N(T(N(\varphi(z)), N(\varphi(u))))) \Leftrightarrow \\ N(T(N(\varphi(x)), N(\varphi(y)))) \le N(T(N(\varphi(z)), N(\varphi(u)))) \Leftrightarrow \\ T(N(\varphi(x)), N(\varphi(y))) \ge T(N(\varphi(z)), N(\varphi(u))) \Leftrightarrow \\ \left\{ \begin{array}{c} N(\varphi(x)) \ge N(\varphi(z)) \\ N(\varphi(y)) \ge N(\varphi(u)) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} \varphi(x) \le \varphi(z) \\ \varphi(y) \le \varphi(u) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x \le z \\ y \le u \end{array} \right. \end{gathered}$$

Therefore, the function S_φ satisfies the monotonicity property.

$$\begin{gathered} S_\varphi(x,\ 0) = \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(0))))) = \\ \varphi^{-1}(N(T(N(\varphi(x)), N(0)))) = \varphi^{-1}(N(T(N(\varphi(x)), 1))) = \varphi^{-1}(N(T(N(\varphi(x)), 1))) = \\ \varphi^{-1}(N(N(\varphi(x)))) = \varphi^{-1}(\varphi(x)) = x \end{gathered}$$

Therefore, the function S_φ satisfies the boundary condition.
We observe that the function S_φ is a t-conorm.
In addition, the function S_φ is continuous because it is continuous in both arguments.
The function is Archimedean if $S_\varphi(x,y) > x$ applies.
Suppose that

$$\begin{gathered} S_\varphi(x,y) > x \Leftrightarrow \varphi^{-1}(N(T(N(\varphi(x)), N(\varphi(y))))) > x \Leftrightarrow \\ N(T(N(\varphi(x)), N(\varphi(y)))) > \varphi(x) \Leftrightarrow T(N(\varphi(x)), N(\varphi(y))) < N(\varphi(x) \end{gathered}$$

applies because the t-norm T is Archimedean.

The function S_φ is strict because it is continuous and strictly monotonous.

The S_φ function is continuous and Archimedean, so it is nilpotent. Therefore, some $x, y \in (0,1)$ exist such that $S_\varphi(x,y) = 1$.

Indeed,

$$S_\varphi(x,y)=1 \Leftrightarrow \varphi^{-1}(N(T(N(\varphi(x)),N(\varphi(y)))))=1 \Leftrightarrow \\ N(T(N(\varphi(x)),N(\varphi(y))))=\varphi(1) \Leftrightarrow N(T(N(\varphi(x)),N(\varphi(y))))=1 \Leftrightarrow \\ T(N(\varphi(x)),N(\varphi(y)))=0$$

applies because the t-norm T is continuous, strict and Archimedean; therefore, $x,y \in (0,1)$ such that $T(x,y)=0$ exist (see Klement E.P., Theorem 2.18, p. 33, [7]). □

Theorem 6 presents the general form of I-implications using an automorphism function, according to the equation $I(x,y)=N(T(x,N(y)))$ (see Corollary 2.5.31, p. 87, [22]). Formula (6) can be used to generate new I-implications (see Example 1vi).

Theorem 6. *If there exists a function $I_\varphi:[0,1]\to[0,1]$, a strong negation $N_\varphi:[0,1]\to[0,1]$, a t-norm $T_\varphi:[0,1]\to[0,1]$ and an automorphism φ such that the function I_φ is fuzzy implication is defined by:*

$$I_\varphi(x,y)=\varphi^{-1}(N(T(\varphi(x),N(\varphi(y))))) \tag{6}$$

Proof of Theorem 6. Property (I_1):

$$\forall\ x_1,\ x_2\ \in[0,1]\ with\ x_1\le x_2 \Leftrightarrow I_\varphi(x_1,y)\ge I_\varphi(x_2,y)\Leftrightarrow \\ \varphi^{-1}(N(T(\varphi(x_1),N(\varphi(y)))))\ge\varphi^{-1}(N(T(\varphi(x_2),N(\varphi(y)))))\Leftrightarrow \\ N(T(\varphi(x_1),N(\varphi(y))))\ge N(T(\varphi(x_2),N(\varphi(y))))\Leftrightarrow \\ T(\varphi(x_1),N(\varphi(y)))\le T(\varphi(x_2),N(\varphi(y)))\Leftrightarrow \\ \varphi(x_1)\le\varphi(x_2)\Leftrightarrow x_1\le x_2$$

Therefore, the function I_φ satisfies the property (I_1).
Property (I_2):

$$\forall\ y_1,\ y_2\ \in[0,1]\ with\ y_1\le y_2 \Leftrightarrow I_\varphi(x,y_1)\le I_\varphi(x,y_2)\Leftrightarrow \\ \varphi^{-1}(N(T(\varphi(x),N(\varphi(y_1)))))\le\varphi^{-1}(N(T(\varphi(x),N(\varphi(y_2)))))\Leftrightarrow \\ N(T(\varphi(x),N(\varphi(y_1))))\le N(T(\varphi(x),N(\varphi(y_2))))\Leftrightarrow \\ T(\varphi(x),N(\varphi(y_1)))\ge T(\varphi(x),N(\varphi(y_2)))\Leftrightarrow \\ N(\varphi(y_1))\ge N(\varphi(y_2))\Leftrightarrow\varphi(y_1)\le\varphi(y_2)\Leftrightarrow y_1\le y_2$$

Therefore, the function I_φ satisfies the property (I_2).
Property (I_3):

$$I_\varphi(0,0)=\varphi^{-1}(N(T(\varphi(0),N(\varphi(0)))))=\varphi^{-1}(N(T(0,N(0))))= \\ \varphi^{-1}(N(T(0,1)))=\varphi^{-1}(N(0))=\varphi^{-1}(1)=\varphi^{-1}(\varphi(1))=1$$

Therefore, the function I_φ satisfies the property (I_3).
Property (I_4):

$$I_\varphi(1,1)=\varphi^{-1}(N(T(\varphi(1),N(\varphi(1)))))=\varphi^{-1}(N(T(1,N(1))))= \\ \varphi^{-1}(N(T(1,0)))=\varphi^{-1}(N(0))=\varphi^{-1}(1)=\varphi^{-1}(\varphi(1))=1$$

Therefore, the function I_φ satisfies the property (I_4).
Property (I_5):

$$I_\varphi(1,0)=\varphi^{-1}(N(T(\varphi(1),N(\varphi(0)))))=\varphi^{-1}(N(T(1,N(0))))= \\ \varphi^{-1}(N(T(1,1)))=\varphi^{-1}(N(1))=\varphi^{-1}(0)=\varphi^{-1}(\varphi(0))=0$$

Therefore, the function I_φ satisfies the property (I_5).
Consequently, the function I_φ satisfies the properties of the family of fuzzy implications. The set of all fuzzy implications will be denoted by FI. □

Example 1. *Let f be a automorphism function $f(x) = x^n$, $x \in [0,1]$, $n \in N^*$*
The function f is a strictly increasing in $[0,1]$ with $f(0) = 0$, $f(1) = 1$.
(i). Let N be a strong fuzzy negation of the Sugeno class: $N(x) = \frac{1-x}{1+\lambda x}$, $\lambda \in (-1, +\infty)$
From Formula (1) of Theorem 1:

$$N_\varphi(x) = \sqrt[n]{\frac{1-x^n}{1+\lambda x^n}},\ \lambda \in (-1, +\infty),\ n \in N^*. \tag{7}$$

(ii). Let T_M be a strict t-norm $T_M(x,y) = \min\{x,y\}$.
From Formula (2) of Theorem 2:

$$T_\varphi(x,y) = \sqrt[n]{\min\{x^n, y^n\}},\ \forall x,y \in [0,1],\ n \in N^*. \tag{8}$$

(iii). Let S_M be a strict t-conorm $S_M(x,y) = \max\{x,y\}$.
From Formula (3) of Theorem 3:

$$S_\varphi(x,y) = \sqrt[n]{\max\{x^n, y^n\}},\ \forall x,y \in [0,1],\ n \in N^*. \tag{9}$$

(iv). Alternatively, the S-conorm can be defined from Formula (4) of Theorem 4:

$$S_\varphi(x,y) = 1 - \sqrt[n]{\min\{(1-x)^n, (1-y)^n\}},\ \forall x,y \in [0,1],\ n \in N^*. \tag{10}$$

(v). In addition, the S-conorm can be defined from Formula (5) of Theorem 5:

$$S_\varphi(x,y) = \sqrt[n]{\frac{1-\min\{\frac{1-x^n}{1+\lambda x^n}, \frac{1-y^n}{1+\lambda y^n}\}}{1+\lambda \min\{\frac{1-x^n}{1+\lambda x^n}, \frac{1-y^n}{1+\lambda y^n}\}}},\ \forall x,y \in [0,1],\ \lambda \in (-1,+\infty),\ n \in N^*. \tag{11}$$

(vi). Let N be a strong fuzzy negation of the Sugeno class $N(x) = \frac{1-x}{1+\lambda x}$, $\lambda \in (-1, +\infty)$, and T_M be a strict t-norm $T_M(x,y) = \min\{x,y\}$.
From Formula (6) of Theorem 6:

$$I_\varphi(x,y) = \sqrt[n]{\frac{1-\min\{x^n, \frac{1-y^n}{1+\lambda y^n}\}}{1+\lambda \min\{x^n, \frac{1-y^n}{1+\lambda y^n}\}}},\ \forall x,y \in [0,1],\ \lambda \in (-1,+\infty),\ n \in N^*. \tag{12}$$

(i). It is easy to see that a function defined by (7) is an involution with the following properties: $N_\varphi(0) = 1$ and $N_\varphi(1) = 0$. It is also strictly decreasing. Hence, N_φ is a strong negation function.

The Figure 2 is shown below.

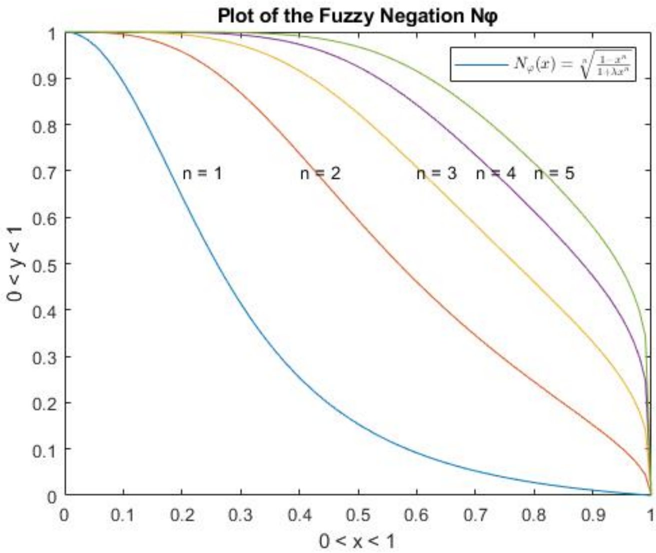

Figure 2. Fuzzy negations generated from Sugeno class using an automorphism function.

(ii). It is easy to see that a function defined by (8) is a strict and Archimedean t-norm. The function T_φ is commutative and associative and it satisfies the boundary condition.

The Figure 3 is shown below.

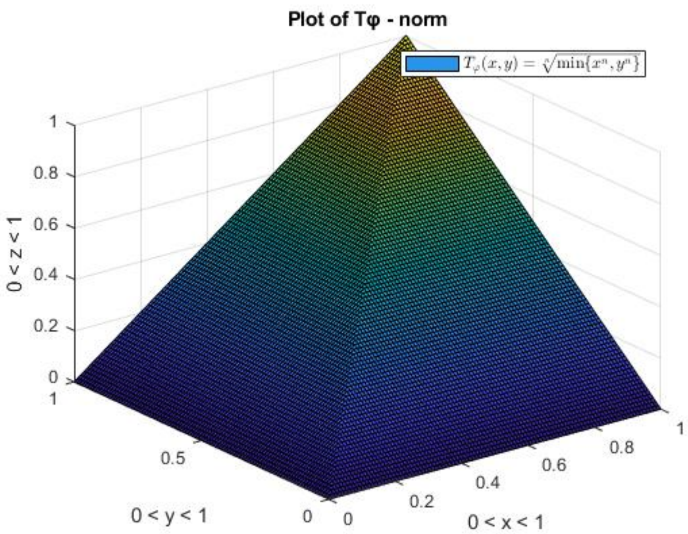

Figure 3. t-norm generated from T_M using an automorphism function.

(iii). It is easy to see that a function defined by (9) is a strict and Archimedean t-conorm. The function is commutative, associative and monotonous and it satisfies the boundary condition.

The graph is shown below.

(iv). It is easy to see that a function defined by (10) is a strict and Archimedean t-conorm. The function S_φ is commutative, associative and monotonous and it satisfies the boundary condition.

The graph is shown below.

(v). It is easy to see that a function defined by (11) is a strict and Archimedean t-conorm.

The function S_φ is commutative, associative and monotonous and it satisfies the boundary condition.

The graph is shown below.

Remark 1. *Figures 4–6 are observed to have the same graph. Therefore, we conclude that the S t-conorms given by Theorems 3–5 express the same S t-conorm.*

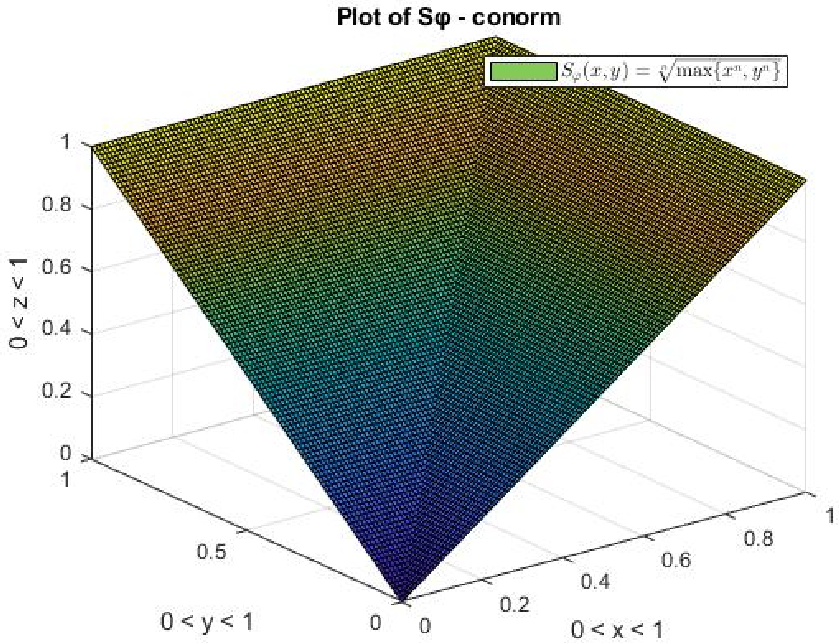

Figure 4. S-conorm generated from S_M using an automorphism function.

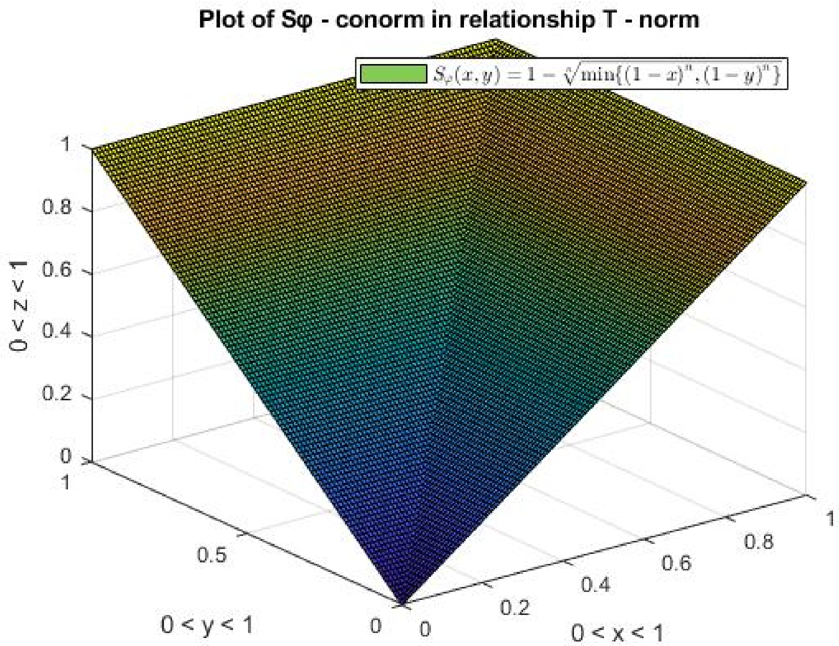

Figure 5. S-conorm generated from $S(x,y) = 1 - T(1-x, 1-y)$ using an automorphism function.

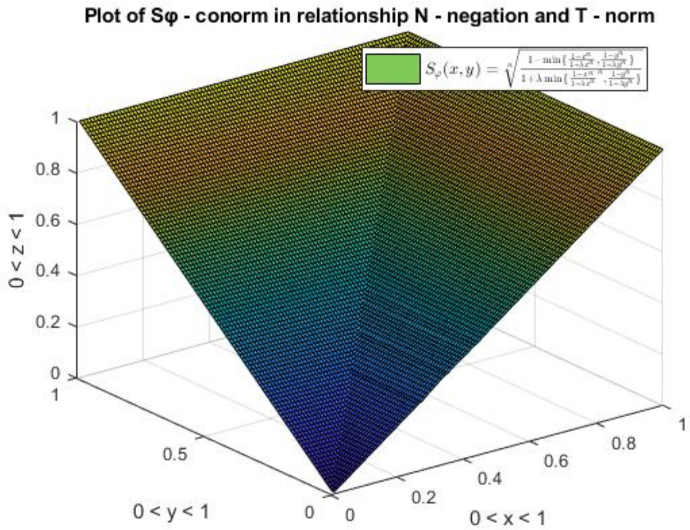

Figure 6. S-conorm generated from $S(x,y) = N(T(N(x), N(y)))$ using an automorphism function.

(vi). The function I_φ satisfies the properties of the family of fuzzy implications. The Figure 7 is shown below.

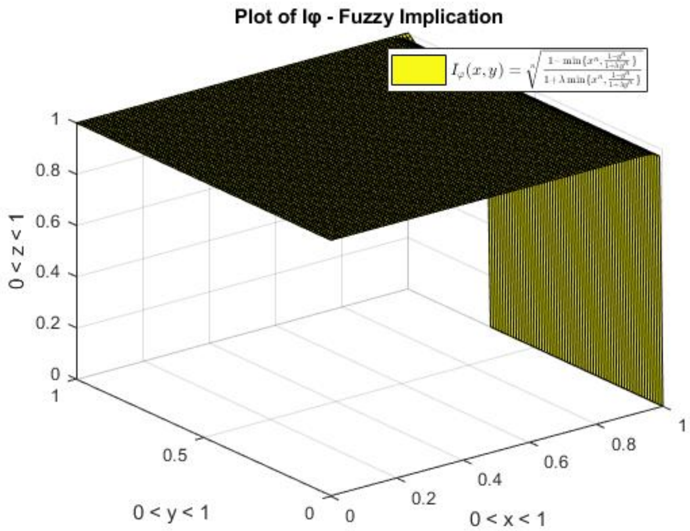

Figure 7. I-implication generated from $I(x,y) = N(T(x, N(y)))$ using an automorphism function.

5. Results

The result of this paper is an improved method of generalizing fuzzy connectives. The way this strategy improves on previous strategies is by being capable of generalizing any fuzzy connective instead of a select few. The conclusion drawn from the creation of this new method is that any fuzzy connective can be generalized (see Equations (1)–(6)).

The motivation behind this paper is the fact that the field of the generalization of fuzzy connectives has been inactive since 2012. Furthermore, the development of the approximate reasoning field, by producing new fuzzy connectives, was another motivation behind our research.

6. Discussion

The field of research of fuzzy connectives has been explored by multiple researchers over the years. As a result, multiple strategies for generalizing fuzzy connectives have been discovered. This paper focused on their limitations and provided solutions, which resulted in the creation of a new strategy. The various applications of this new method, as well a their results, are visible in the following paragraphs.

To be more specific, fuzzy connectives using the natural negation have been generated in the past (see J.C. Fodor and M. Roubens, Theorem 1.1, p. 4, [14]), (Gottwald S., Theorem 5.2.1 p. 86, [6]) and (Fodor J., p. 2077, [19]). However, the limitation is that this strategy involves only the natural negation in the process of generalizing the fuzzy connectives. The strategy presented in this paper, though, is capable of replacing the natural negation with any strong negation. This allows for the creation of new fuzzy connectives capable of involving all negations in the process of generalization.

Furthermore, fuzzy connectives using the T-Minimum, T-Product and T-Lukasiewicz t-norms have been generated in the past (see René B. et al., Theorem 2.3, p. 372, [20]). In addition, Gottwald S., Theorem 5.1.3, p. 82, [6] worked with such functions, but they focused mainly on the specific forms of t-norms (see Table 4). However, the limitation is that this strategy involves only these specific t-norms in the process of generalizing the fuzzy connectives. The strategy presented in this paper, though, is capable of replacing the T-Minimum, T-Product and T-Lukasiewicz t-norms with any t-norm. This allows for the creation of new fuzzy connectives capable of involving all t-norms in the process of generalization.

Table 4. Basic t-norms.

Designation	Equation
Minimum	$T_M(x,y) = \min\{x,y\}$
Algebraic product	$T_p(x,y) = x \cdot y$
Lukasiewicz	$T_{LK}(x,y) = \max(x+y-1,0)$

Moreover, this paper presents the generalization of fuzzy connectives using S-conorms. The prospect of incorporating S-conorms in the process of generalizing fuzzy connectives has not been explored in the past. In order to achieve this, the new strategy is based on the strategies mentioned before.

In addition, a strategy employing S-conorms, t-norms as well N-negations in the process of generalizing fuzzy connectives is explored in this paper. Such a strategy has not been implemented by someone else before.

Finally, a strategy for generalizing the classes of the I-implications was discovered in the past (see Bustince H., Burillo P. and Soria F. in 2003 ([17]). Callejas C., Marcos J. and Bedregal B., in 2012, created the fifth strategy (see Figure 8), which generates any fuzzy implication ([18]). In this paper, however, a new method of generalizing I- implications with a combination of N-negations and t-norms is presented. This method will play a crucial role in future research, as it allows for the generalization of I-implications, which, in conjunction with weather data, can provide a better understanding of climate change.

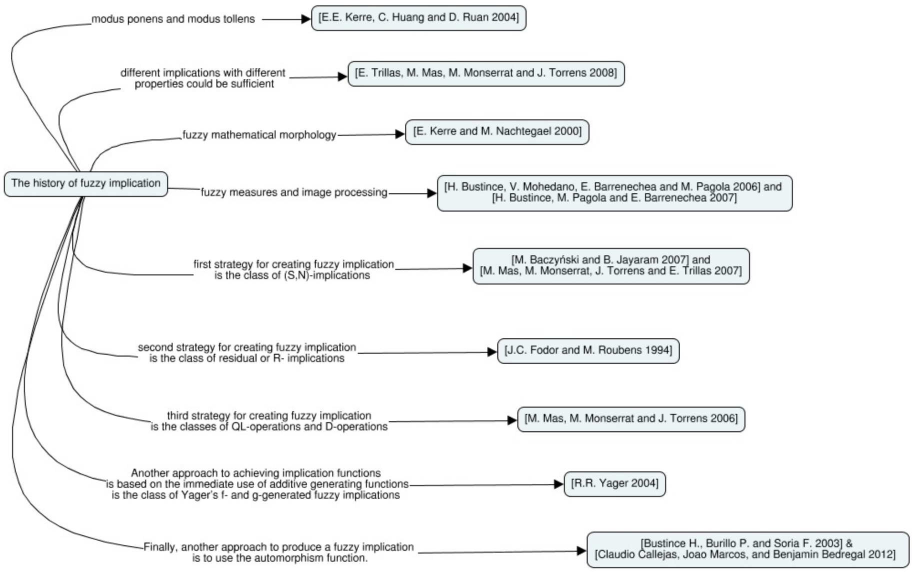

Figure 8. The history and evolution of fuzzy implications.

7. Conclusions

The objective of this paper was to create a new strategy for generalizing fuzzy connectives which is more flexible and faster in comparison with the rest. The way this objective was achieved was by solving the limitations of previous methods. To be more specific, with this new strategy, a wider range of fuzzy connectives and automorphisms is utilized in the process of generalization.

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