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On Solutions and Stability of Stochastic Functional Equations Emerging in Psychological Theory of Learning

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Abstract: We show how to apply the well-known fixed-point approach in the study of the existence, uniqueness, and stability of solutions to some particular types of functional equations. Moreover, we also obtain the Ulam stability result for them. The functional equations that we consider can be used to explain various experiments in mathematical psychology and arise in a natural way in the stochastic approach to the processes of perception, learning, reasoning, and cognition.

Keywords: stochastic functional equations; mathematical psychology; Ulam stability; fixed point

MSC: 39B22; 39B82; 47H10; 92F05

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1. Introduction and Preliminaries

Let X be a nonempty set and $\mathbb{R}$ denote the set of reals. Let $L_1, L_2, L_3, L_4 : X \to X$, $a_1, a_2, a_3, a_4, b : X \to \mathbb{R}$ and $A_1, A_2 \in \mathbb{R}$ be fixed. We present a fixed-point method, which is very effective in the study of the solutions $f : X \to \mathbb{R}$ of the following general stochastic functional equation

$$\begin{aligned} f(x) = a_1(x)f(L_1(x)) + a_2(x)f(L_2(x)) \\ + (A_1 - a_1(x))f(L_3(x)) + (A_2 - a_2(x))f(L_4(x)) + b(x), \end{aligned} \tag{1}$$

for $x \in X$. The equation arises in the stochastic approach in mathematical psychology, which deals with the mathematical modeling of the processes of perception, reasoning, and cognition. We do not make any particular probabilistic assumptions on X, b, a_i and A_i, because the main considerations are valid without them (i.e., in a general situation).

Mathematical psychology is based on the observation that the learning process in an animal or a human being may be seen as a sequence of decisions resulting from a large number of potential feedbacks. These decisions often appear unexpected even in simple repeated tests conducted under well-controlled circumstances, which suggests that they can be considered to be random. Therefore, it seems to make sense to include in our considerations the systemic changes (in the set of possible choices) that represent fluctuations in the probability of responses between individual trials, which means the investigation of a suitable stochastic process.

The idea that a simple learning experiment may behave stochastically is not novel (see, e.g., Refs. [1–3]). It has some drawbacks, but also shows some new relationships. One of the tools applied in the research connected with this idea are functional equations. For instance, in 1967, Epstein [4] proposed the following functional equation (to discuss the learning process of animals in a two-choice situation):

$$f(x) = \left(\frac{e^x}{1+e^x}\right) f(x+a) + \left(1 - \frac{e^x}{1+e^x}\right) f(x-b), \tag{2}$$

where a and b are fixed real constants and $f : \mathbb{R} \to [0,1]$ is an unknown function satisfying the conditions $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} f(x) = 1$. The analytical solution of the above equation was calculated by using the bilateral Laplace transformation.

In 1976, Istrăţescu [5] studied the behavior of predatory animals that prey on two distinct types of prey and used the following functional equation

$$f(x) = x f(\nu_1 + (1-\nu_1)x) + (1-x) f((1-\nu_2)x), \tag{3}$$

where $0 < \nu_1 \leq \nu_2 < 1$ are learning-rate parameters and $f : [0,1] \to \mathbb{R}$ is an unknown function.

Recently, Turab and Sintunavarat [6] introduced the following functional equation

$$f(x) = x f(\nu_1 x + (1-\nu_1)\Theta_1) + (1-x) f(\nu_2 x + (1-\nu_2)\Theta_2), \tag{4}$$

where $f : [0,1] \to \mathbb{R}$ is an unknown function, $0 < \nu_1 \leq \nu_2 < 1$ are learning-rate parameters and Θ_1, Θ_2 are real constants. The functional equation was used to study a specific kind of psychological resistance of dogs enclosed in a small box.

Note that Equations (2)–(4) are particular cases of (1) with $A_2 = 0$, $a_2(X) = \{0\}$ and $b(X) = \{0\}$. For several other studies on human actions in probability-learning scenarios, we refer to [1,7–10] (see also [11–13]).

Further, an apparently two-choice situation regarding the movement of the animals towards food can actually be a four-response situation, if we also take into account the food placement, as did Bush and Wilson [2], dividing the types of responses into four events: right-reward, right-nonreward, left-reward, left-nonreward. They examined the movement of a paradise fish. A very general situation with four different responses is depicted by Equation (1), which additionally include the possibility of the so called 'blank trials'.

The notion of 'blank trials' is motivated by the following very natural question:

> What if an animal or human does not move for any prey or response and sticks to its original position?

Some information on such a situation we can find in the paper of Neimark [14], concerning the human response in the two-choice experiment, in which it should have been foreseen which of two lights would be turned on in every trial, but the case when 'no light was turned on' was possible as well. Such 'blank trials,' as the author called them, established another class of events. Turab and Sintunavarat have also investigated such a situation for a paradise fish [8].

Our objective is to prove results on the existence, uniqueness and stability of solutions to functional Equation (1) by using the tools afforded to us by fixed-point theory (for details about fixed-point theory we refer to [15–17]).

Finally, let us mention that the standard theory of existence and uniqueness of solutions to the stochastic equations can be found in many books, such as [18]. These books are usually geared towards Polish spaces but methods to extend the standard theory to Tychonoff spaces are now well understood (see [19]). While our spaces are certainly Tychonoff, it is of interest to come up with a simple direct proof of existence and uniqueness without requiring a lot of specialized machinery.

2. Auxiliary Information and Results

In what follows, C^D always denotes the family of all functions that map a set $D \neq \emptyset$ into a set $C \neq \emptyset$.

An extended norm, in a real or complex vector space W, is a function $\|\cdot\| : W \to [0,+\infty]$ (i.e., possibly also taking the value $+\infty$) such that, for each scalar α and every $x, y \in W$ with $\|x\|, \|y\| \in [0,+\infty)$,

$$\|x+y\| \le \|x\| + \|y\|, \qquad \|\alpha x\| = |\alpha|\,\|x\|,$$

and $\|x\| = 0$ if and only if $x = 0$ (the zero vector).

If V is a normed space and $T \neq \emptyset$ is a set, then such an extended norm in V^T can be defined by:

$$\|g\| = \sup_{t\in T} \|g(t)\|, \qquad g \in V^T.$$

An extended metric in a set $B \neq \emptyset$ is a function $d : B^2 \to [0,+\infty]$ fulfilling, for every $x, y, z \in B$, the subsequent three conditions:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \le d(x,y) + d(y,z)$.

If d is an extended metric in a nonempty set B, then we say that the pair (B,d) is an extended metric space.

If $\|\cdot\|$ is an extended norm in a real vector space W, then it is easily seen that the formula $d(x,y) = \|x-y\|$ defines an extended metric in W. For the extended norms and metrics the notions of Cauchy sequence and completeness are the same as for the classical norms and metrics.

In the sequel, given a set $X \neq \emptyset$ and $\mathcal{L} : X \to X$, we sometimes write for simplicity $\mathcal{L}x := \mathcal{L}(x)$ for $x \in X$. Moreover, as usual, $\mathcal{L}^0 x := x$ and $\mathcal{L}^n x := \mathcal{L}(\mathcal{L}^{n-1}x)$ for $x \in X$, $n \in \mathbb{N}$ (positive integers).

Now, we are in a position to recall the Diaz–Margolis fixed-point alternative (see [20]), which will be useful in the proof of our main results ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$).

Theorem 1. *Assume that ρ is an extended complete metric in a set $S \neq \emptyset$ and $\mathcal{L} : S \to S$ is a contraction with the constant $L < 1$ (i.e., $\rho(\mathcal{L}x,\mathcal{L}y) \le L\rho(x,y)$ for $x, y \in S$ with $\rho(x,y) \in (0,+\infty)$). Let $x \in S$ be such that there is $k \in \mathbb{N}$ with $\rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x) < \infty$. Then the sequence $(\mathcal{L}^n x)_{n\in\mathbb{N}}$ converges to a fixed point $x^* \in S$ of $\mathcal{L}$, x^* is the unique fixed point of $\mathcal{L}$ in the set $S^* = \{y \in S : \rho(x^*, y) < \infty\}$ and*

$$\rho(\mathcal{L}^n x, x^*) \le \frac{L^{n-k+1}\rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x)}{1-L}, \qquad n \in \mathbb{N}_0, n \ge k-1. \tag{5}$$

Proof. From [20] (Theorem) we can easily deduce the convergence of $\mathcal{L}^n x$ to a fixed point x^* of $\mathcal{L}$. Further, for each fixed point $u \in S^*$ of $\mathcal{L}$, we have the subsequent simple inequality

$$\rho(u, x^*) = \rho(\mathcal{L}^n u, \mathcal{L}^n x^*) \le L^n \rho(u, x^*), \qquad n \in \mathbb{N},$$

which yields the uniqueness of x^*. For the convenience of readers, we also present below a proof of (5).

First note that, for every $m \in \mathbb{N}$, $m \ge k$,

$$\begin{aligned}\rho(\mathcal{L}^{k-1}x, \mathcal{L}^m x) &\le \sum_{i=k}^{m} \rho(\mathcal{L}^{i-1}x, \mathcal{L}^i x) \\ &\le \rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x) \sum_{i=0}^{m-k} L^i \le \frac{\rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x)}{1-L}\end{aligned}$$

whence

$$\rho(\mathcal{L}^{k-1}x, x^*) \le \rho(\mathcal{L}^{k-1}x, \mathcal{L}^m x) + \rho(\mathcal{L}^m x, x^*)$$
$$\le \frac{\rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x)}{1-L} + \rho(\mathcal{L}^m x, x^*).$$

Further, $\lim_{m\to\infty} \rho(\mathcal{L}^m x, x^*) = 0$ and consequently,

$$\rho(\mathcal{L}^{k-1}x, x^*) \le \frac{\rho(\mathcal{L}^{k-1}x, \mathcal{L}^k x)}{1-L}.$$

Finally, it is easily seen that (5) can be deduced from the above inequality and from the fact that, for every $n \in \mathbb{N}_0$ with $n \ge k-1$, we have

$$\rho(\mathcal{L}^n x, x^*) = \rho(\mathcal{L}^{n-k+1}(\mathcal{L}^{k-1}x), \mathcal{L}^{n-k+1}x^*) \le L^{n-k+1}\rho(\mathcal{L}^{k-1}x, x^*).$$

□

Remark 1. *Let $k = 1$ in Theorem 1. Then (5) (with $n = 0$) yields $\rho(x, x^*) < +\infty$, which means that $x \in S^*$.*

Further, for every fixed point $z \in S$ of $\mathcal{L}$ such that $\rho(x, z) < +\infty$, we have $\rho(x^*, z) \le \rho(x^*, x) + \rho(x, z) < +\infty$ and therefore $z \in S^*$. This means that $z = x^*$, as x^* is the unique in S^* fixed point of $\mathcal{L}$.

Consequently, if $\mathcal{L}$ has a fixed point $z \ne x^*$, then necessarily $\rho(x, z) = +\infty$.

3. Some Preliminary Remarks

Later in this article (unless explicitly stated otherwise), (X, d) is a metric space, $x_0 \in X$ and $\eta \in \mathbb{R}$ are fixed, $\mathcal{E} := \{f \in \mathbb{R}^X : f(x_0) = \eta\}$ and we write

$$\|f\|_e := \sup_{x\ne y} \frac{|f(x) - f(y)|}{d(x, y)}, \qquad f \in \mathbb{R}^X.$$

It is easily seen that

$$\|f + g\|_e \le \|f\|_e + \|g\|_e, \qquad f, g \in \mathbb{R}^X, \tag{6}$$

and consequently

$$\big|\|f\|_e - \|g\|_e\big| \le \|f - g\|_e, \qquad f, g \in \mathbb{R}^X, \|f\|_e < \infty, \|g\|_e < \infty. \tag{7}$$

If $\eta = 0$, then $\mathcal{E}$ is a real vector space and $\|f\|_e$ is an extended norm in $\mathcal{E}$. If $\eta \ne 0$, then $\mathcal{E}$ is not a real vector space. However, in either case we can define in $\mathcal{E}$ an extended metric d_e by $d_e(f, g) = \|f - g\|_e$.

We show that d_e is complete. So, take a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $(\mathcal{E}, d_e)$. Then, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d_e(f_n, f_m) \le \epsilon$ for $m, n \in \mathbb{N}$ with $\min\{m, n\} > k$, which means that

$$|f_n(x) - f_m(x) - f_n(y) + f_m(y)| \le \epsilon d(x, y), \qquad x, y \in X, m, n \in \mathbb{N}, \min\{m, n\} > k. \tag{8}$$

This, with $y = x_0$, yields

$$|f_n(x) - f_m(x)| \le \epsilon d(x, y), \qquad x \in X, m, n \in \mathbb{N}, \min\{m, n\} > k.$$

Consequently, for every $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ (with the natural metric) and there exists the limit

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Thus we define a function $f \in \mathcal{E}$. Next, letting $m \to \infty$ in (8) we get

$$|f_n(x) - f(x) - f_n(y) + f(y)| \leq \epsilon d(x,y), \qquad x, y \in X, n > k,$$

whence $d_e(f_n, f) \leq \epsilon$ for $n > k$. In this way we have shown that (in $(\mathcal{E}, d_e)$)

$$f = \lim_{n\to\infty} f_n. \tag{9}$$

Let $\mathcal{F} := \{f \in \mathcal{E} : \|f\|_e < \infty\}$. Then, every function $f \in \mathcal{F}$ is a Lipschitz function with the Lipschitz constant equal to $\|f\|_e$ (i.e., $|f(x) - f(y)| \leq \|f\|_e\, d(x,y)$ for every $x, y \in X$); therefore, it is continuous.

Next, take $f_n \in \mathcal{F}$ for $n \in \mathbb{N}$ and assume that (9) holds with some $f \in \mathcal{E}$, which means that

$$\lim_{n\to\infty} \|f_n - f\|_e = 0. \tag{10}$$

Then, by (6) and (7),

$$\big|\|f_n\|_e - \|f_m\|_e\big| \leq \|f_n - f_m\|_e \leq \|f_n - f\|_e + \|f_m - f\|_e, \qquad n, m \in \mathbb{N},$$

which on account of (10) implies that $\left(\|f_n\|_e\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and consequently there exists finite

$$c_0 := \lim_{n\to\infty} \|f_n\|_e.$$

Due to (6), we also have the inequality

$$\|f\|_e \leq \|f - f_n\|_e + \|f_n\|_e, \qquad n \in \mathbb{N}.$$

Hence, by (10) (with $n \to \infty$), we obtain $\|f\|_e \leq c_0$, whence $f \in \mathcal{F}$. Thus, we have proved that $\mathcal{F}$ is a closed subset of $(\mathcal{E}, d_e)$.

4. Main Results

In this section, $L_1, L_2, L_3, L_4 : X \to X$, $a_1, a_2, b : X \to \mathbb{R}$ and $A_1, A_2 \in \mathbb{R}$ are fixed. We investigate solutions $f \in \mathcal{E}$ to (1), i.e., to the functional equation

$$\begin{aligned} f(x) = {} & a_1(x) f(L_1(x)) + a_2(x) f(L_2(x)) \\ & + (A_1 - a_1(x)) f(L_3(x)) + (A_2 - a_2(x)) f(L_4(x)) + b(x). \end{aligned} \tag{11}$$

We write

$$\widehat{a}_1 := \sup_{z\in X} |a_1(z)| < \infty, \qquad \widehat{a}_2 := \sup_{z\in X} |a_2(z)|,$$
$$\widehat{a}_3 := \sup_{z\in X} |A_1 - a_1(z)|, \qquad \widehat{a}_4 := \sup_{z\in X} |A_2 - a_2(z)|.$$

We also need the following three hypotheses.

Hypothesis 1 (H_1)**.** *For each $i \in \{1,2,3,4\}$, L_i is a Lipschitz mapping with a constant κ_i, i.e.,*

$$d(L_i(x), L_i(y)) \leq \kappa_i\, d(x,y), \qquad x, y \in X.$$

Hypothesis 2 (H_2)**.** *$a_1, a_2 \in \mathcal{F}$, $\widehat{a}_1 < \infty$ and $\widehat{a}_2 < \infty$.*

Hypothesis 3 (H_3)**.** *For every $f \in \mathcal{E}$,*

$$\begin{aligned} \eta = {} & a_1(x_0) f(L_1(x_0)) + a_2(x_0) f(L_2(x_0)) \\ & + (A_1 - a_1(x_0)) f(L_3(x_0)) + (A_2 - a_2(x_0)) f(L_4(x_0)) + b(x_0). \end{aligned} \tag{12}$$

Hypothesis (H_3) may seem quite demanding, but it is easily seen that, in the case $b(x_0) = \eta(1 - A_1 - A_2)$, it is fulfilled, for instance, in the following situations:

- x_0 is a fixed point of L_1 and L_2, $a_1(x_0) = A_1$, $a_2(x_0) = A_2$;
- x_0 is a fixed point of L_3 and L_4, $a_1(x_0) = a_2(x_0) = 0$;
- x_0 is a fixed point of L_1 and L_4, $a_1(x_0) = A_1$, $a_2(x_0) = 0$.

Define $\mathcal{T} : \mathcal{E} \to \mathbb{R}^X$ by:

$$\begin{aligned}(\mathcal{T}f)(x) = a_1(x)f(L_1(x)) + a_2(x)f(L_2(x)) + (A_1 - a_1(x))f(L_3(x)) \\ + (A_2 - a_2(x))f(L_4(x)) + b(x), \qquad f \in \mathcal{E}, x \in X.\end{aligned} \tag{13}$$

Note that if hypothesis (H_3) is valid, then $(\mathcal{T}f)(x_0) = \eta$ for every $f \in \mathcal{E}$ and consequently $\mathcal{T}(\mathcal{E}) \subset \mathcal{E}$.

Now, we are in a position to prove the following main result of this paper.

Theorem 2. *Let hypotheses $(H_1) - (H_3)$ be valid, $\lambda_0 := \widehat{a}_1\kappa_1 + \widehat{a}_2\kappa_2 + \widehat{a}_3\kappa_3 + \widehat{a}_4\kappa_4$ and $\delta(X) := \max_{x,y \in X} d(x,y)$. Suppose that one of the following three conditions is valid.*

(a) *There exist points $u_1, u_2 \in X$ such that*

$$L_1(u_1) = L_3(u_1), \qquad L_2(u_2) = L_4(u_2) \tag{14}$$

and

$$\lambda := \lambda_0 + \delta(X)(\|a_1\|_e(\kappa_1 + \kappa_3) + \|a_2\|_e(\kappa_2 + \kappa_4)) < 1. \tag{15}$$

(b) *There exist $\kappa_5, \kappa_6 \in [0, \infty)$ such that*

$$d(L_1(x), L_3(y)) \le \kappa_5 d(x,y), \qquad d(L_2(x), L_4(y)) \le \kappa_6 d(x,y) \tag{16}$$

for all $x, y \in X$ with $x \neq y$, and

$$\lambda := \lambda_0 + \delta(X)(\|a_1\|_e(\kappa_1 + \kappa_5) + \|a_2\|_e(\kappa_2 + \kappa_6)) < 1. \tag{17}$$

(c) *There exist $\gamma_1, \gamma_2 \in [0, \infty)$ with*

$$d(L_1(x), L_3(x)) \le \gamma_1, \qquad d(L_2(x), L_4(x)) \le \gamma_2, \qquad x \in X, \tag{18}$$

and

$$\lambda := \lambda_0 + \|a_1\|_e \gamma_1 + \|a_2\|_e \gamma_2 < 1. \tag{19}$$

If $f_0 \in \mathcal{E}$ is such that there is $k \in \mathbb{N}$ with $d_e(\mathcal{T}^{k-1}f_0, \mathcal{T}^k f_0) < \infty$, then the sequence $(\mathcal{T}^n f_0)_{n \in \mathbb{N}}$ is convergent to a fixed point $f^* \in \mathcal{E}$ of $\mathcal{T}$, which is the unique solution of Equation (11) in the set

$$\mathcal{E}_{f^*} := \{f \in \mathcal{E} : d_e(f, f^*) < \infty\}.$$

Moreover, the speed of convergence is estimated by the following inequality:

$$d_e(\mathcal{T}^n f_0, f^*) \le \frac{\lambda^{n-k+1} d_e(\mathcal{T}^{k-1}f_0, \mathcal{T}^k f_0)}{1 - \lambda}, \qquad n \in \mathbb{N}_0, n \ge k - 1. \tag{20}$$

Proof. We show that $\mathcal{T}$ is a contraction on $\mathcal{E}$.

First, observe that, for each $x_1, x_2 \in X$,

$$\begin{aligned}\mathcal{T}f(x_1) - \mathcal{T}f(x_2) =& a_1(x_1)f(L_1(x_1)) + a_2(x_1)f(L_2(x_1)) \\ & + (A_1 - a_1(x_1))f(L_3(x_1)) + (A_2 - a_2(x_1))f(L_4(x_1)) \\ & - \big[a_1(x_2)f(L_1(x_2)) + a_2(x_2)f(L_2(x_2)) \\ & + (A_1 - a_1(x_2))f(L_3(x_2)) + (A_2 - a_2(x_2))f(L_4(x_2))\big] \\ =& a_1(x_1)f(L_1(x_1)) - a_1(x_1)f(L_1(x_2)) \\ & + a_2(x_1)f(L_2(x_1)) - a_2(x_1)f(L_2(x_2)) \\ & + (A_1 - a_1(x_1))f(L_3(x_1)) - (A_1 - a_1(x_1))f(L_3(x_2)) \\ & + (A_2 - a_2(x_1))f(L_4(x_1)) - (A_2 - a_2(x_1))f(L_4(x_2)) \\ & + a_1(x_1)f(L_1(x_2)) - a_1(x_2)f(L_1(x_2)) \\ & + a_2(x_1)f(L_2(x_2)) - a_2(x_2)f(L_2(x_2)) \\ & + (A_1 - a_1(x_1))f(L_3(x_2)) - (A_1 - a_1(x_2))f(L_3(x_2)) \\ & + (A_2 - a_2(x_1))f(L_4(x_2)) - (A_2 - a_2(x_2))f(L_4(x_2)),\end{aligned}$$

and consequently

$$\begin{aligned}\mathcal{T}f(x_1) - \mathcal{T}f(x_2) =& a_1(x_1)(f(L_1(x_1)) - f(L_1(x_2))) \\ & + a_2(x_1)(f(L_2(x_1)) - f(L_2(x_2))) \\ & + (A_1 - a_1(x_1))(f(L_3(x_1)) - f(L_3(x_2))) \\ & + (A_2 - a_2(x_1))(f(L_4(x_1)) - f(L_4(x_2))) \\ & + (a_1(x_1) - a_1(x_2))(f(L_1(x_2)) - f(L_3(x_2))) \\ & + (a_2(x_1) - a_2(x_2))(f(L_2(x_2)) - f(L_4(x_2))).\end{aligned} \tag{21}$$

Assume that condition (a) is fulfilled. Then by (14) and (21), for every $x_1, x_2 \in X$ with $x_1 \neq x_2$,

$$\begin{aligned}\frac{|\mathcal{T}f(x_1) - \mathcal{T}f(x_2)|}{d(x_1, x_2)} \le& |a_1(x_1)|\, \|f\|_e \frac{d(L_1(x_1), L_1(x_2))}{d(x_1, x_2)} \\ & + |a_2(x_1)|\, \|f\|_e \frac{d(L_2(x_1), L_2(x_2))}{d(x_1, x_2)} \\ & + |A_1 - a_1(x_1)|\, \|f\|_e \frac{d(L_3(x_1), L_3(x_2))}{d(x_1, x_2)} \\ & + |A_2 - a_2(x_1)|\, \|f\|_e \frac{d(L_4(x_1), L_4(x_2))}{d(x_1, x_2)} \\ & + \|a_1\|_e |f(L_1(x_2)) - f(L_1(u_1)) + f(L_3(u_1)) - f(L_3(x_2))| \\ & + \|a_2\|_e |f(L_2(x_2)) - f(L_2(u_2)) + f(L_4(u_2)) - f(L_4(x_2))| \\ \le& \widehat{a}_1 \|f\|_e \kappa_1 + \widehat{a}_2 \|f\|_e \kappa_2 + \widehat{a}_3 \|f\|_e \kappa_3 + \widehat{a}_4 \|f\|_e \kappa_4 \\ & + \|a_1\|_e (\|f\|_e \kappa_1 \delta(X) + \|f\|_e \kappa_3 \delta(X)) \\ & + \|a_2\|_e (\|f\|_e \kappa_2 \delta(X) + \|f\|_e \kappa_4 \delta(X)) = \lambda \|f\|_e,\end{aligned}$$

where

$$\lambda := \lambda_0 + \|a_1\|_e \delta(X)(\kappa_1 + \kappa_3) + \|a_2\|_e \delta(X)(\kappa_2 + \kappa_4).$$

Hence, replacing f by $f_1 - f_2$, we obtain that

$$\begin{aligned}d_e(\mathcal{T}f_1, \mathcal{T}f_2) &= \|\mathcal{T}f_1 - \mathcal{T}f_2\|_e = \|\mathcal{T}(f_1 - f_2)\|_e \\ &\le \lambda \|f_1 - f_2\|_e = \lambda d_e(f_1, f_2), \qquad f_1, f_2 \in \mathcal{E}.\end{aligned} \tag{22}$$

If condition (b) is satisfied, then by (16) and (21), for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have

$$\begin{aligned}\frac{|\mathcal{T}f(x_1) - \mathcal{T}f(x_2)|}{d(x_1,x_2)} &\leq |a_1(x_1)|\,\|f\|_e \frac{d(L_1(x_1),L_1(x_2))}{d(x_1,x_2)} \\ &\quad + |a_2(x_1)|\,\|f\|_e \frac{d(L_2(x_1),L_2(x_2))}{d(x_1,x_2)} \\ &\quad + |A_1 - a_1(x_1)|\,\|f\|_e \frac{d(L_3(x_1),L_3(x_2))}{d(x_1,x_2)} \\ &\quad + |A_2 - a_2(x_1)|\,\|f\|_e \frac{d(L_4(x_1),L_4(x_2))}{d(x_1,x_2)} \\ &\quad + \|a_1\|_e |f(L_1(x_2)) - f(L_1(x_1)) + f(L_1(x_1)) - f(L_3(x_2))| \\ &\quad + \|a_2\|_e |f(L_2(x_2)) - f(L_2(x_1)) + f(L_2(x_1)) - f(L_4(x_2))| \\ &\leq \lambda_0\,\|f\|_e + \|a_1\|_e(\|f\|_e\,\kappa_1\delta(X) + \|f\|_e\,\kappa_5\delta(X)) \\ &\quad + \|a_2\|_e(\|f\|_e\,\kappa_2\delta(X) + \|f\|_e\,\kappa_6\delta(X)) = \lambda\,\|f\|_e,\end{aligned}$$

where

$$\lambda := \lambda_0 + \|a_1\|_e\delta(X)(\kappa_1 + \kappa_5) + \|a_2\|_e\delta(X)(\kappa_2 + \kappa_6).$$

Hence, as in the previous case we obtain (22).

Finally, assume that (c) holds. Then by (18) and (21), for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have

$$\begin{aligned}\frac{|\mathcal{T}f(x_1) - \mathcal{T}f(x_2)|}{d(x_1,x_2)} &\leq |a_1(x_1)|\,\|f\|_e \frac{d(L_1(x_1),L_1(x_2))}{d(x_1,x_2)} \\ &\quad + |a_2(x_1)|\,\|f\|_e \frac{d(L_2(x_1),L_2(x_2))}{d(x_1,x_2)} \\ &\quad + |A_1 - a_1(x_1)|\,\|f\|_e \frac{d(L_3(x_1),L_3(x_2))}{d(x_1,x_2)} \\ &\quad + |A_2 - a_2(x_1)|\,\|f\|_e \frac{d(L_4(x_1),L_4(x_2))}{d(x_1,x_2)} \\ &\quad + \|a_1\|_e |f(L_1(x_2)) - f(L_3(x_2))| + \|a_2\|_e |f(L_2(x_2)) - f(L_4(x_2))| \\ &\leq \lambda_0\,\|f\|_e + \|a_1\|_e\,\|f\|_e\,\gamma_1 + \|a_2\|_e\,\|f\|_e\,\gamma_2 = \lambda\,\|f\|_e,\end{aligned}$$

where

$$\lambda := \lambda_0 + \|a_1\|_e\gamma_1 + \|a_2\|_e\,\gamma_2.$$

Hence, as previously we obtain (22).

Thus, we have shown that $\mathcal{T}$ is a contraction in each of the cases (a)–(c). Consequently, Theorem 1 (with $\mathcal{L} = \mathcal{T}$, $L = \lambda$, $S = \mathcal{E}$ and $\rho = d_e$) completes the proof. $\square$

Remark 2. *If one of conditions (a)–(c) of Theorem 2 is fulfilled, then every solution $f \in \mathcal{E}$ of (11) can be obtained in the way depicted in Theorem 2. For, if $f^* \in \mathcal{E}$ fulfils (11), then it is a fixed point of $\mathcal{T}$ and consequently $\mathcal{T}^n f^* = f^*$ for each $n \in \mathbb{N}$, which means that the sequence $(\mathcal{T}^n f^*)_{n\in\mathbb{N}}$ converges to f^*.*

Remark 3. *Observe that if $X = [0,1]$, $d(x,y) = |x-y|$ for $x, y \in X$, $a_1(X), a_2(X) \subset [0,1]$, a_1, a_2 are nonexpansive mappings, $A_1 = A_2 = 1$, and λ is given by (15), then the inequality*

$$2(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4) < 1 \tag{23}$$

implies that $\lambda < 1$.

Remark 4. *Note that a constant function $\omega_\eta \in \mathcal{F}$, $\omega_\eta(x) \equiv \eta$, is a solution to (11) if and only if*

$$\eta(1 - A_1 - A_2) = b(x), \qquad x \in X. \tag{24}$$

Assume that (24) holds. Since $\{f \in \mathcal{E} : d_e(f, \omega_\eta) < \infty\} = \mathcal{F}$, so under the assumptions of Theorem 2 (with $f_0 = \omega_\eta$) ω_η is the only solution of Equation (11) which belongs to $\mathcal{F}$. If x_0 is a fixed point of L_i for $i = 1, 2, 3, 4$, then this is true, e.g., for the equation

$$\begin{aligned} f(x) = {} & \zeta v(x) f(L_1(x)) + (1-\zeta) v(x) f(L_2(x)) \\ & + \zeta(1 - v(x)) f(L_3(x)) + (1-\zeta)(1-v(x)) f(L_4(x)) \end{aligned} \tag{25}$$

with $\eta = 0$ and fixed $\zeta \in \mathbb{R}$ and $v : X \to \mathbb{R}$.

If $\zeta = 1$, then with $v(x) \equiv e^x/(1 - e^x)$ equation (25) can be (2), while for $v(x) \equiv x$ particular cases of (25) are (3) and (4) and the equations considered in [8,21]. This means that with $\eta = 0$ (under suitable assumptions on L_i), for Equations (2)–(4) (and equations considered in [8,21]), ω_η is the only solution that belongs to $\mathcal{F}$.

If (24) does not hold, then clearly every solution of (25) must be a nonconstant function. Therefore, in such a case, the statement of Theorem 2 depicts nonconstant solutions to (25) and moreover, if $k = 1$ and $f_0, \mathcal{T} f_0 \in \mathcal{F}$, then a solution generated in this way belongs to $\mathcal{F}$ (because $\mathcal{F}$ is a closed subset of $(\mathcal{E}, d_e)$, as it has been shown at the end of Section 3).

Remark 5. *Assume that (16) holds. Then, for every $x, z, w \in X$, with $x \neq z \neq w \neq x$,*

$$\begin{aligned} d(L_1(x), L_3(x)) & \le d(L_1(x), L_3(z)) + d(L_3(z), L_1(w)) + d(L_1(w), L_3(x)) \\ & \le \kappa_5 (d(x,z) + d(z,w) + d(w,x)), \end{aligned}$$

$$d(L_2(x), L_4(x)) \le \kappa_6 (d(x,z) + d(z,w) + d(w,x)).$$

So, if $z_0 \in X$ is a limit of a sequence $(z_n)_{n \in \mathbb{N}}$ of points of the set $X \setminus \{z_0\}$ with $z_n \neq z_{n+1}$ for $n \in \mathbb{N}$, then taking in the above inequalities $x = z_0$, $z = z_n$, $w = z_{n+1}$ and letting $n \to \infty$, we get $L_1(x_0) = L_3(x_0)$ and $L_2(x_0) = L_4(x_0)$. This means that condition (14) is valid with $u_1 = u_2 = z_0$. Hence, in the case where (b) holds we obtain anything other than in the case of (a) only when the topology generated in X by d is discrete.

5. Remarks on Ulam Stability

In this section we show that Theorem 2 actually also provides the results on Ulam stability. Let us recall that the theory of Ulam stability (often also called the Hyers–Ulam stability) has been motivated by a problem of S. Ulam, concerning approximate homomorphisms of groups, and an answer to it provided by D. Hyers [22] (see [23–28] for more details and references).

To put it very roughly, the main issue of such stability can be expressed as follows: *When a function satisfying an equation approximately (in some sense) must be near an exact solution to the equation?*

The next definition (cf. [25], p. 119, Ch. 5, Definition 8) makes the notion a bit more precise ($\mathbb{R}_+ := [0, \infty)$).

Definition 1. *Let A be a nonempty set, (S, ρ) be a metric space, $\mathcal{E} \subset \mathbb{R}_+{}^A$ be nonempty, $\mathcal{T}$ be an operator mapping $\mathcal{E}$ into $\mathbb{R}_+{}^A$ and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset S^A$ into S^A. We say that the equation*

$$\mathcal{F}_1 \varphi(x) = \mathcal{F}_2 \varphi(x) \tag{26}$$

is $\mathcal{T}-$stable provided for any $\varepsilon \in \mathcal{E}$ and $\varphi_0 \in \mathcal{D}$ with

$$\rho\big((\mathcal{F}_1\varphi_0)(x), (\mathcal{F}_2\varphi_0)(x)\big) \le \varepsilon(x), \qquad x \in A, \tag{27}$$

there exists a solution $\varphi \in \mathcal{D}$ of Equation (26) such that

$$\rho(\varphi(x), \varphi_0(x)) \le (\mathcal{T}\varepsilon)(x), \qquad x \in A. \tag{28}$$

In short, $\mathcal{T}-$stability of (26) means that every approximate (in the sense of (27)) solution of (26) is always close (in the sense of (28)) to an exact solution of (26).

In mathematical modeling, the consistency of solutions to equations applied is critical. Minor changes to the data set, such as those caused by natural measurement errors, should not have a significant impact on the conclusion. Hence, it is also essential to analyze the stability of the suggested functional equation solutions. The next corollary shows that Theorem 2 also yields information on the Ulam stability of Equation (11), which correspond to and complement various earlier stability results for functional equations in single variable (cf., e.g., [29–32]).

Corollary 1. *Let hypotheses (H_1)–(H_3) be valid and let one of conditions (a)–(c) of Theorem 2 be fulfilled.*

If $f_0 \in \mathcal{E}$ is such that $d_e(f_0, \mathcal{T} f_0) < \infty$, then the sequence $(\mathcal{T}^n f_0)_{n\in\mathbb{N}}$ converges to a solution $f^* \in \mathcal{E}$ of Equation (11) and

$$d_e(f^*, f_0) \le \frac{d_e(f_0, \mathcal{T} f_0)}{1-\lambda}. \tag{29}$$

Moreover, f^* is the unique in $\mathcal{E}$ solution to (11) with a finite distance to f_0.

Proof. In view of Theorem 2 (with $k = 1$ and $n = 0$), it is only necessary to show the uniqueness of f^*. So, suppose that $g \in \mathcal{E}$ also is a solution to (11) with $d_e(f_0, g) < \infty$. Then

$$d_e(f^*, g) \le d_e(f^*, f_0) + d_e(f_0, g) \le \frac{d_e(f_0, \mathcal{T} f_0)}{1-\lambda} + d_e(f_0, g).$$

Since f^* and g also are fixed points of $\mathcal{T}$ and from the proof of Theorem 2 it follows that (22) holds with $\lambda < 1$, for each $n \in \mathbb{N}$ we have

$$d_e(f^*, g) = d_e(\mathcal{T}^n f^*, \mathcal{T}^n g) \le \lambda^n d_e(f^*, g) \le \lambda^n \Big(\frac{d_e(f_0, \mathcal{T} f_0)}{1-\lambda} + d_e(f_0, g)\Big),$$

which with $n \to \infty$ yields $f^* = g$. □

6. Conclusions

Mathematical psychology is a branch of psychology that deals with the mathematical modeling of processes studied in theories of cognition and learning. One of its directions is the so-called stochastic approach. From this perspective, most research on learning processes can be reduced to calculating the probability of events occurring in subsequent trials, which leads to the consideration of appropriate stochastic processes.

In this article, we presented some results on Ulam stability and solution (e.g., their existence and uniqueness) of a general functional equation that may be used to study various learning theory experiments on animals and humans. The main tool that we use is the fixed-point alternative of Diaz and Margolis [20].

Unlike the authors of [6,8,21], we do not use any boundary conditions in the proof of our Theorem 2. Therefore our findings are applicable to a broader range of problems.

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